

# Gauss's Law

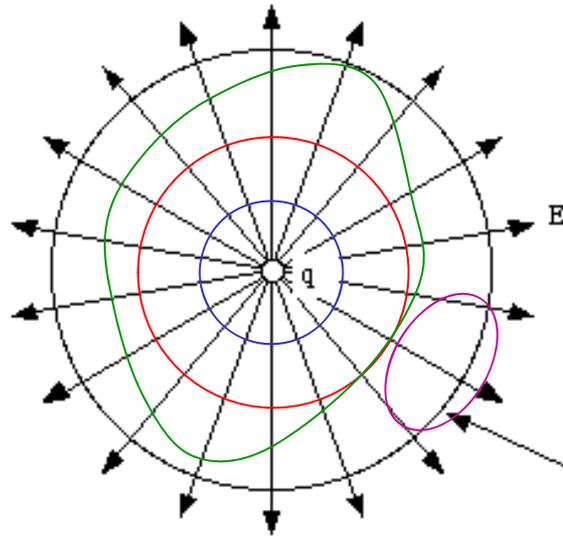
The beauty of  $1/R^2$

For a single point charge:

The density of field lines signifies field strength,  $\propto 1/R^2$ .

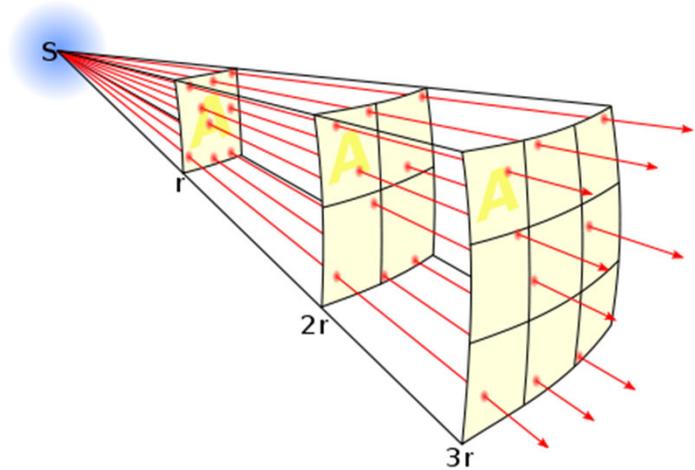
Surface area  $\propto 1/R^2$ .

Constant **flux** of field lines through spheres, regardless of  $R$ .



$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 R^2} \hat{\mathbf{R}}$$

Gaussian Surface



For sphere,  $\oint \mathbf{E} \cdot d\mathbf{s} = \oint \frac{q}{4\pi\epsilon_0 R^2} \frac{1}{R^2} \hat{\mathbf{R}} \cdot d\mathbf{s}$

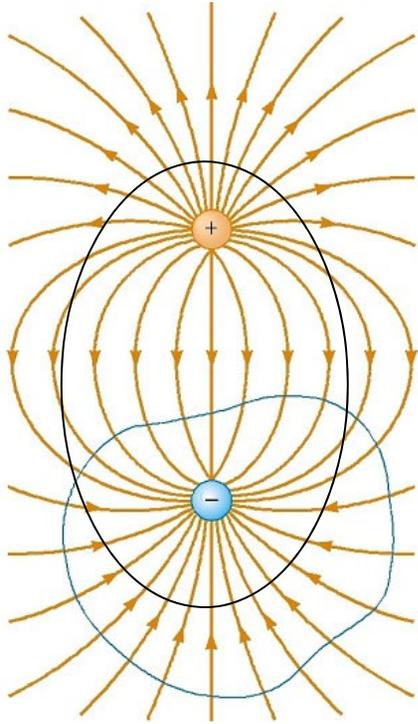
This is why we put a factor  $4\pi$  in Coulomb's law.

$$= \frac{q}{4\pi\epsilon_0 R^2} 4\pi R^2 = \frac{q}{\epsilon_0}$$

Since  $\mathbf{R} \perp d\mathbf{s}$ ,  $\hat{\mathbf{R}} \cdot d\mathbf{s} = ds$ .

The **Gaussian surfaces** do not have to be spheres. **Constant flux** through any **closed surface**.

“From our derivation you see that Gauss' law follows from the fact that the exponent in Coulomb's law is exactly two. A  $1/r^3$  field, or any  $1/r^n$  field with  $n \neq 2$ , would not give Gauss' law. So Gauss' law is just an expression, in a different form, of the Coulomb law...” -- Richard Feynman

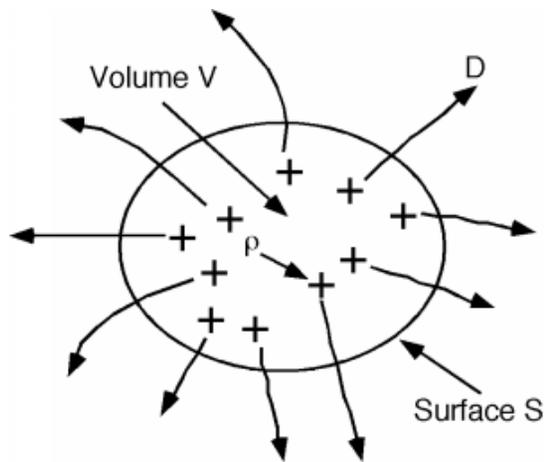


For multiple point charges:  
 The flux of field lines is proportional to the **net charge enclosed** by a Gaussian surface, due to superposition.

A field line comes out of a positive charge, and go into a negative charge.

For a continuous chunk of charge,

$$\oint \mathbf{E} \cdot d\mathbf{s} = \int \frac{\rho}{\epsilon_0} dV = \frac{Q}{\epsilon_0}$$

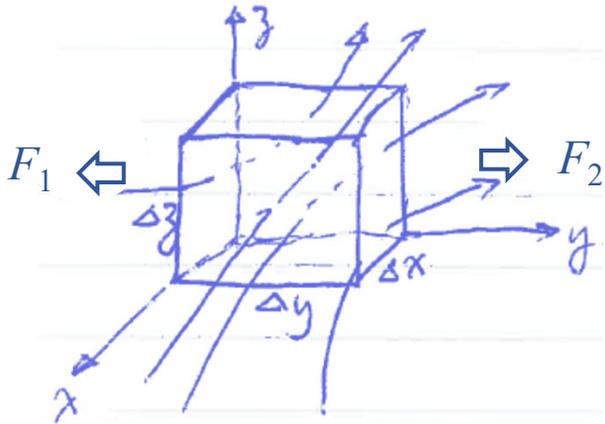


This is the **integral form** of Gauss's law.  
 We may call it the “big picture” of Gauss's law.

Next, we look at the **differential form** of Gauss's law.  
 We may call that the “small picture” of the law.

## The differential form (or “small picture”) of Gauss’s law

To understand the physics (Gauss’s law), we first talk about the math (Gauss’s theorem).



Flux out of the cube through the left face

$$F_1 = -E_y \Delta x \Delta z$$

(Flux out of a closed surface is defined as positive)

Flux out of the cube through the right face

$$F_2 = \left( E_y + \frac{\partial E_y}{\partial y} \Delta y \right) \Delta x \Delta z$$

The net flux of these two faces:  $F_1 + F_2 = \frac{\partial E_y}{\partial y} \Delta y \Delta x \Delta z$

Considering the other two pairs of faces, the net flux out of the cube:

$$\left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

defined as the divergence of  $\mathbf{E}$ , indicating how much flux comes out of a small volume  $\Delta V$  around a point

Notice that the divergence is a scalar.

$$\Delta x \Delta y \Delta z = \Delta V$$

$$\left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

$$\Delta x \Delta y \Delta z = \Delta V$$

defined as the **divergence** of  $\mathbf{E}$ , indicating how much flux comes out of a small volume  $\Delta V$  around a point

Define **vector operator**  $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$

thus 
$$\begin{aligned} \nabla \cdot \vec{E} &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (\hat{x} E_x + \hat{y} E_y + \hat{z} E_z) \\ &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \end{aligned}$$

This is just notation.

For a small volume  $\Delta V$

$$\begin{aligned} \oint_{\Delta S} \vec{E} \cdot d\vec{S} &= \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta V \\ &= (\nabla \cdot \vec{E}) \Delta V \end{aligned}$$

$\Delta S$  is the closed surface enclosing  $\Delta V$  of any arbitrary shape.  $\Delta V$  could be a small cube and  $\Delta S$  then includes its 6 faces.

$$\therefore \nabla \cdot \vec{E} = \lim_{\Delta V \rightarrow 0} \frac{\oint_{\Delta S} \vec{E} \cdot d\vec{S}}{\Delta V}$$

Up to here, just math. "Gauss's theorem." The  $\mathbf{E}$  here does not have to be an electric field.

$$\oint_{\Delta S} \vec{E} \cdot d\vec{S} = \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \Delta V$$

$$= (\nabla \cdot \vec{E}) \Delta V$$

Gauss's theorem in math.

It relates the integral form (“big picture”) and the differential form (“small picture”) of Gauss’s law in physics.

$$\therefore \nabla \cdot \vec{E} = \lim_{\Delta V \rightarrow 0} \frac{\oint_{\Delta S} \vec{E} \cdot d\vec{S}}{\Delta V}$$

Equivalently,

$$\oint_S \vec{E} \cdot d\vec{S} = \int_V (\nabla \cdot \vec{E}) dV = \frac{1}{\epsilon_0} \int_V \rho dV$$

(by recalling that  $\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = \int_V \rho dV = Q$ )

holds for any arbitrary  $S$

Here the physics (Gauss’s law) kicks in.

$$\therefore \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Differential form (“small picture”) of Gauss’s law:

The divergence of electric field at each point is proportional to the local charge density.

$\vec{D} \equiv \epsilon_0 \vec{E}$ , thus

$$\nabla \cdot \vec{D} = \rho$$

for any point in space.

$$\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho dV = Q$$

Integral form (“big picture”) of Gauss’s law:

The flux of electric field out of a closed surface is proportional to the charge it encloses.

for any arbitrary closed surface  $S$  enclosing volume  $V$ .

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{D} \equiv \epsilon_0 \vec{E}, \text{ thus } \nabla \cdot \vec{D} = \rho$$

Differential form (“small picture”) of Gauss’s law:

The divergence of electric field at each point is proportional to the local charge density.

for any point in space.

$$\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = \oint_S \vec{D} \cdot d\vec{S} = \int_V \rho dV = Q$$

Integral form (“big picture”) of Gauss’s law:

The flux of electric field out of a closed surface is proportional to the charge it encloses.

for any **arbitrary** closed surface  $S$  enclosing volume  $V$ .

The above is Gauss’s law in **free space** (vacuum).

For a dielectric, just replace  $\epsilon_0$  with  $\epsilon = \epsilon_r \epsilon_0$ , for now.

We will talk about what the **dielectric constant**  $\epsilon_r$  really means.

Before that, let's look at some examples in free space (vacuum).

Finish Homework 7. Read Section 4-4 and 3-5 of textbook.

Continue working on Chapter 3.

### Example 1: find the field of an infinitely large charge plane

Find the electric field due to an infinitely large sheet of charge with an areal charge density  $\rho_s$ . It is a 2D sheet, with a zero thickness.

By symmetry, the  $\mathbf{E}$  fields on the two sides of the sheet must be equal & opposite, and must be perpendicular to the sheet.

Imagine a cylinder (pie) with area  $A$  and zero height (thickness).

If the cylinder is at the sheet,

$$2\epsilon_0 EA = \rho_s A \quad \Rightarrow \quad E = \frac{\rho_s}{2\epsilon_0}$$

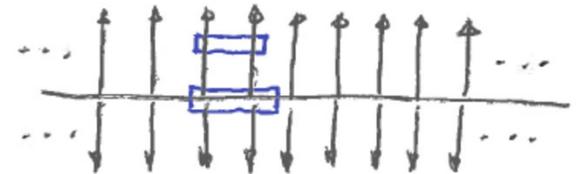
Treat  $\mathbf{E}$  as a scalar, since we already know the direction.

Recall our result for the charged disk:  $E(z \rightarrow 0) = \frac{1}{2\epsilon_0} \frac{Q}{\pi a^2} = \frac{\rho_s}{2\epsilon_0}$

Actually,  $z \rightarrow 0 \Leftrightarrow a \rightarrow \infty \quad \left(\frac{z}{a}\right) \rightarrow 0$

If the cylinder is elsewhere, the net flux is 0:

What goes in comes out; no charge inside the cylinder.



## The field of a uniformly charged finite disk

Recall that we “did not pass the sanity test” for  $E(z \rightarrow 0)$  along  $z$  axis:

$$E = \frac{1}{2\epsilon_0} \frac{Q}{\pi a^2} \left( 1 - \frac{z}{\sqrt{a^2 + z^2}} \right)$$

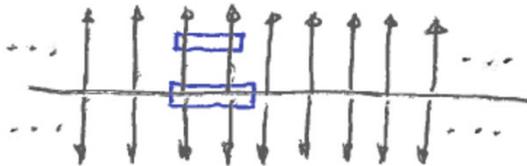
$$E(z \rightarrow 0) = \frac{1}{2\epsilon_0} \cdot \frac{Q}{\pi a^2} \neq 0 !$$

Then we said **what's important is the ratio  $z/a$** .  
Everything is relative.

$$z \rightarrow 0 \Leftrightarrow a \rightarrow \infty \quad \left( \frac{z}{a} \right) \rightarrow 0$$

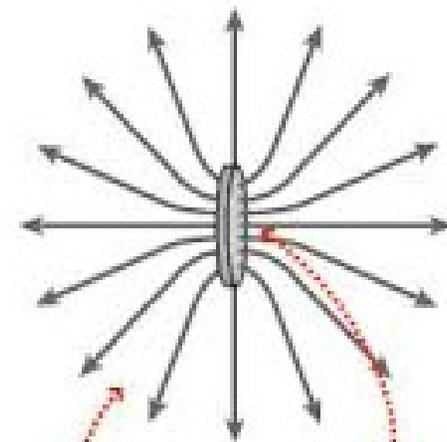
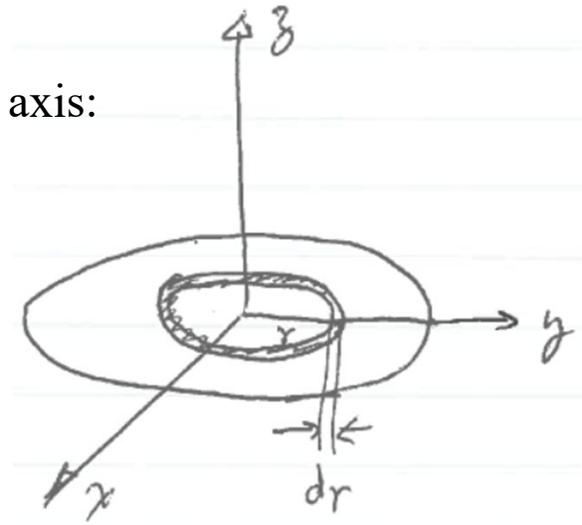
$$E \left( \frac{z}{a} \rightarrow 0 \right) = \frac{1}{2\epsilon_0} \cdot \frac{Q}{\pi a^2} = \frac{\rho_s}{2\epsilon_0}$$

This is the field of an infinitely large sheet of charge.



From another point of view, this **is** the field **at the center** of a **finite** sheet. The donut is different from the pie no matter how small the hole is!

**Visualize the field of the donut.**

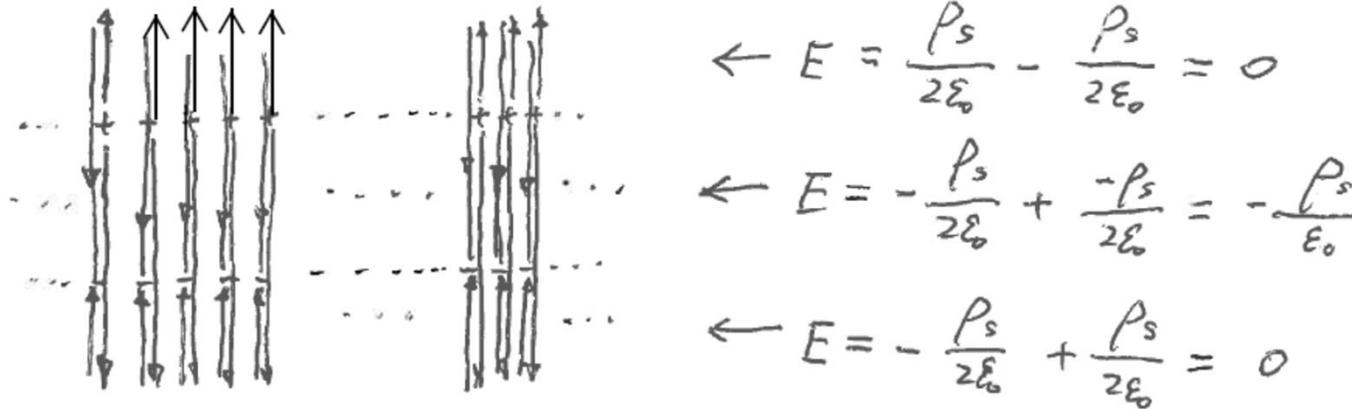


Field out here is essentially that of a point charge.

Field in close is essentially that of an infinite plane of charge.

**Example 2:** field of two infinitely large sheets with equal and opposite charge densities

What if there are two infinitely large sheets, one charged with a surface density  $+\rho_S$ , and the other  $-\rho_S$ . Assume  $\rho_S$  is positive for convenience.

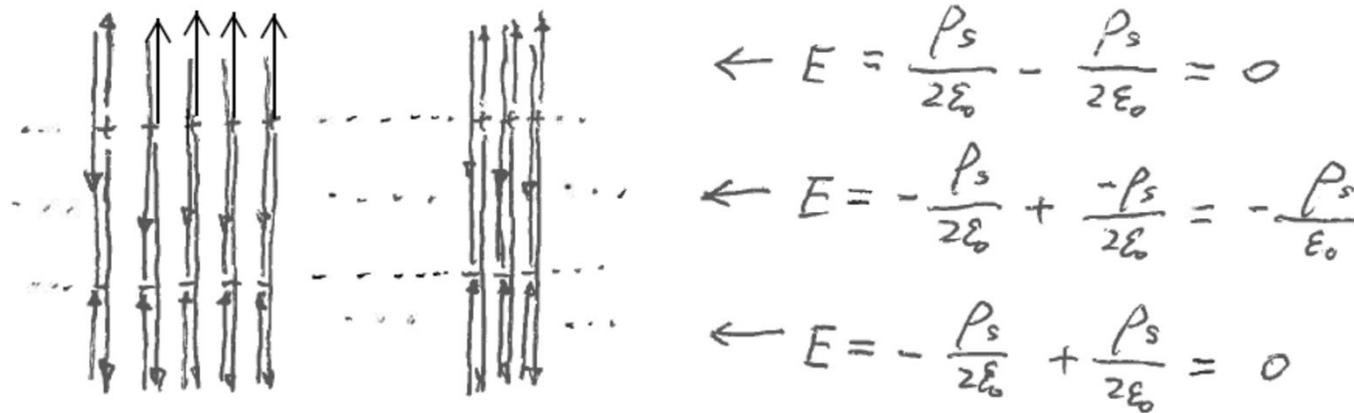


Is this a familiar picture?

What circuit element is this picture a model of?

## Example 2: field of two infinitely large sheets with equal and opposite charge densities

What if there are two infinitely large sheets, one charged with a surface density  $+\rho_S$ , and the other  $-\rho_S$ . Assume  $\rho_S$  is positive for convenience.

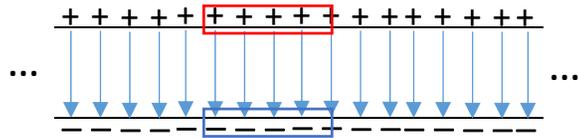


This is an infinitely large parallel-plate capacitor.

It is the simplest model of the capacitor, ignoring the fringe effect.

by assuming infinite lateral size

Another look at the parallel-plate capacitor  
 (Two infinitely large sheets of opposite charges)



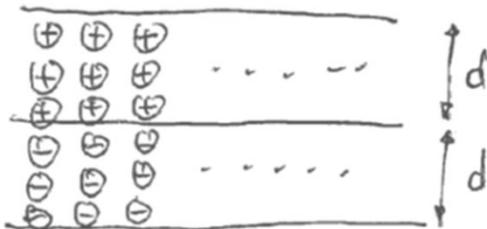
The electric field lines starts from a positive charge and ends at a negative charge.

Gauss's law leads to  $E = \frac{\rho_s}{\epsilon_0}$

You may use a negative sign to signify the “downward” direction.  $E = -\frac{\rho_s}{\epsilon_0}$

Sign conventions are kind of arbitrary. We just need to be self-consistent within the context.

An example for you to work out on your own:



Two charged slabs, one with a volume charge density  $+\rho$ , the other  $-\rho$ , where  $\rho > 0$ . Each slab has a thickness  $d$  and infinite area.

Find the electric field distribution.

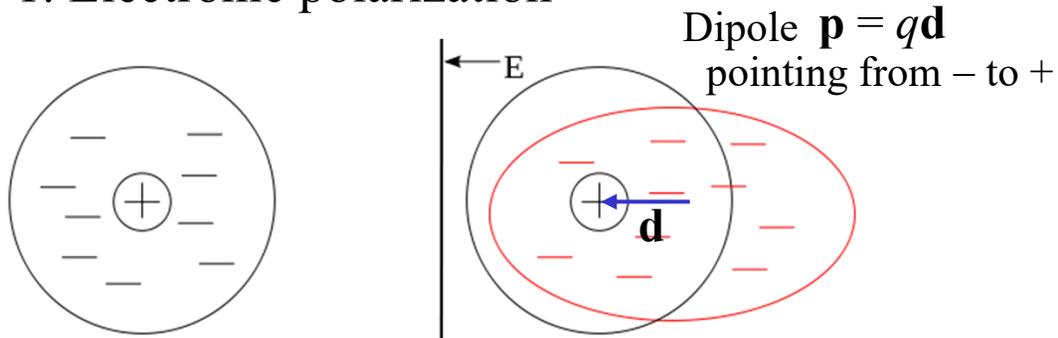
You may define the direction perpendicular to the slabs  $x$ , and set  $x = 0$  for the interface between them.

So far we have limited our discussions to free space.  
 Now, let's talk about dielectrics (insulators).

# Electric Fields in Insulators (Dielectrics)

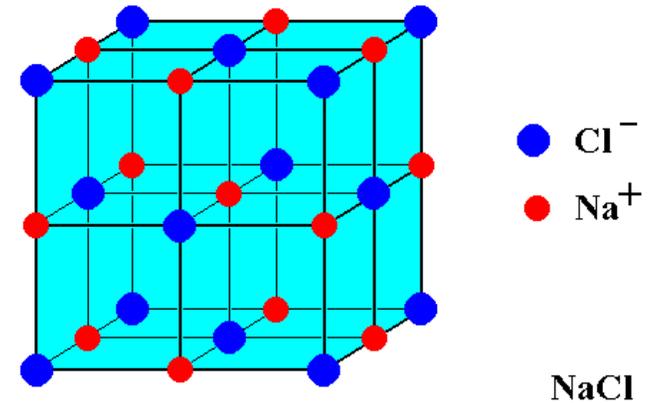
## Polarization

### 1. Electronic polarization



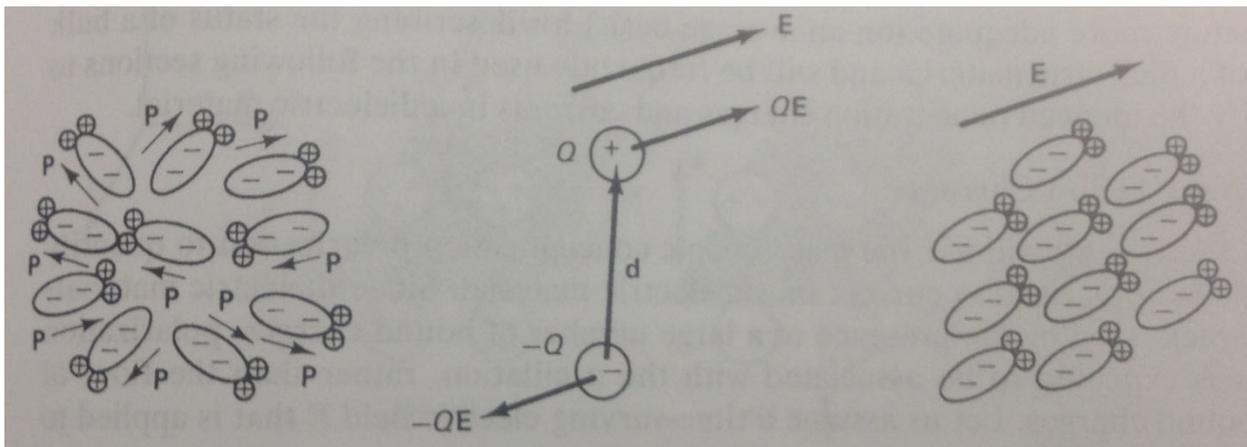
When  $\mathbf{E} = 0$ ,  $\mathbf{p} = 0$ .

### 2. Ionic polarization



When  $\mathbf{E} = 0$ , net dipole is 0.

### 3. Orientational polarization



Again, no net dipole when  $\mathbf{E} = 0$ .

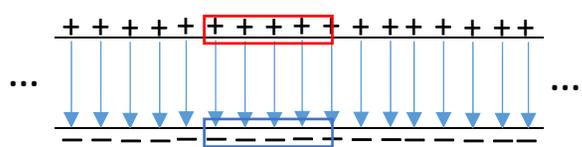
Define polarization

$$\mathbf{P} = \lim_{\Delta V \rightarrow 0} \frac{\sum \mathbf{p}_i}{\Delta V}$$

(net dipole per volume)

For all three cases, when  $\mathbf{E} = 0$ , net dipole is 0, therefore  $\mathbf{P} = 0$ .  
A finite  $\mathbf{E}$  will induce a net polarization  $\mathbf{P}$ .

Let's digress back to the parallel-plate capacitor, for the manifestation of Gauss's law at a conductor surface



Gauss's law leads to  $E = \frac{\rho_s}{\epsilon_0}$

Recall that the field of a parallel-plate capacitor is the consequence of Gauss's law applied to the surface charge densities:

$$\oint \mathbf{E} \cdot d\mathbf{s} = \int \frac{\rho}{\epsilon_0} dV$$

or, in the differential form (the "small picture"),

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \longleftrightarrow \quad E = \frac{\rho_s}{\epsilon_0}$$

Volume density Surface density

Let's define the "electric displacement"  $\mathbf{D} = \epsilon_0 \mathbf{E}$ . Then,

$$\oint \vec{D} \cdot d\vec{S} = \int \rho dV \quad \nabla \cdot \mathbf{D} = \rho \quad D = |\mathbf{D}| = \rho_s$$

At a perfect conductor surface, we can write  $D = |\mathbf{D}| = \rho_s$  as:  $\rho_s = \mathbf{D} \cdot \hat{\mathbf{n}}$

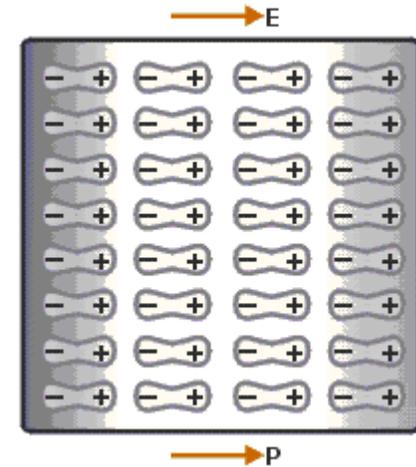
For the perfect conductor plate,  $D = \rho_s$ , as a result of  $\nabla \cdot \mathbf{D} = \rho$

Surface normal pointing out (i.e. into the vacuum or dielectric); check signs for both plate surfaces

Regardless of the mechanism (electronic, ionic, orientational),  $\mathbf{E}$  induces  $\mathbf{P}$ .

For materials [without spontaneous polarization](#) and for not too strong  $\mathbf{E}$ ,  $\mathbf{P} \propto \mathbf{E}$ .

$$\mathbf{P} = \chi\epsilon_0\mathbf{E}$$



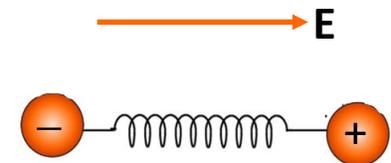
### Side notes:

[Spontaneous polarization](#): Some materials exhibit finite  $\mathbf{P}$  even when  $\mathbf{E} = 0$ , due to low symmetry of their structures. Although not covered in this course, this phenomenon ([pyroelectricity](#)) is important. [GaN](#) and related semiconductors (AlGaN, InGaN) are such materials. If the spontaneous polarization can be switched by an external electric field, such a material is [ferroelectric](#).

Related to this, we can mechanically strain some material to break/lower its symmetry thus induce finite  $\mathbf{P}$  at  $\mathbf{E} = 0$ . This is called [piezoelectricity](#).

Spontaneous and piezoelectric polarizations are exploited in [GaN-based power electronics devices](#) (to obtain carriers without doping the semiconductors).

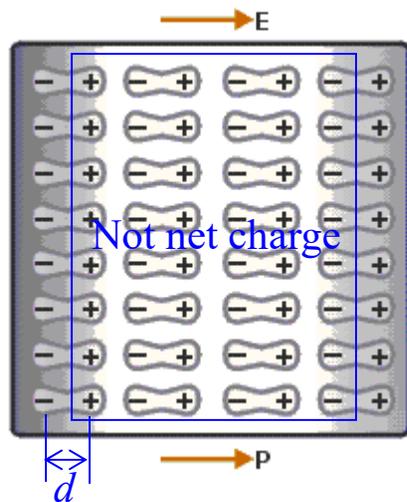
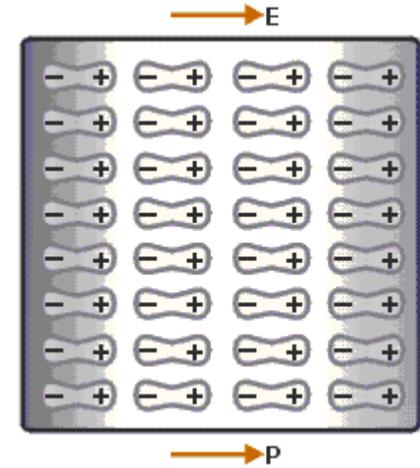
$\mathbf{P} \propto \mathbf{E}$ : For a dielectric without spontaneous polarization, each dipole can be [modeled](#) as the positive and negative charges connected by a Hookean spring, near their equilibrium positions. Electric force  $\mathbf{F} \propto \mathbf{E}$  results in displacement  $\mathbf{d}$  from equilibrium for each dipole, thus  $\mathbf{p} \propto \mathbf{d}$  and the total dipole moment per volume  $\mathbf{P} \propto \mathbf{d}$ . At steady state, the Hookean force  $-K\mathbf{d}$  is balanced by the  $\mathbf{F}$ , thus  $\mathbf{F} = K\mathbf{d}$ . Since  $\mathbf{F} \propto \mathbf{E}$ ,  $\mathbf{F} \propto \mathbf{d}$ , and  $\mathbf{P} \propto \mathbf{d}$ , we have  $\mathbf{P} \propto \mathbf{E}$ .



Consider a dielectric slab of infinite lateral size (cross section shown in figure).

Regardless of the mechanism (electronic, ionic, orientational),  $\mathbf{E}$  induces  $\mathbf{P}$ .

No **spontaneous polarization** and not too strong  $\mathbf{E}$ ,  $\mathbf{P} \propto \mathbf{E}$ .



$$\mathbf{P} = \chi \epsilon_0 \mathbf{E}$$

No net charge in the interior.  
Two sheet charges at surfaces:

$$(|\rho_{sP}|A)d = P(Ad) \implies |\rho_{sP}| = P$$

The **polarization charge density**, or “internal” charge density.

as opposed to external; not interior

As any  $P$  is from  $-$  to  $+$  while any  $\mathbf{D}$  is from  $+$  to  $-$ , we write:

$$\rho_{sP} = -P$$

Actually,  $\rho_{sP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$

Surface normal **pointing into the dielectric**;  
check signs for both plate surfaces

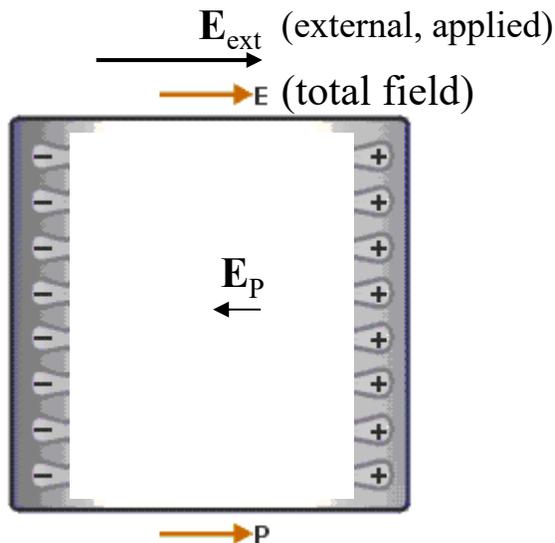
The **polarization** (or “internal”) charge leads to a **polarization** (or “internal”) field

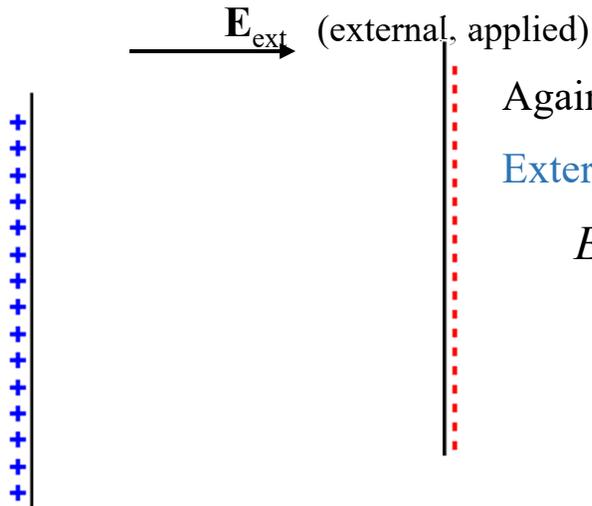
$$E_P = \rho_{sP} / \epsilon_0 \implies \epsilon_0 E_P = \rho_{sP} = -P \implies E_P = -P / \epsilon_0$$

More generally, in the vector form:

$$\epsilon_0 \mathbf{E}_P = -\mathbf{P}$$

Notice that the polarization (internal) field is always in opposite direction to the external, applied field  $\mathbf{E}_{\text{ext}}$ .





Again, consider a parallel capacitor with vacuum/air between the two plates.

External surface charge density  $\rho_s$  induces external field

$$E_{\text{ext}} = \rho_s / \epsilon_0. \quad \text{Actually, } \rho_s = \epsilon_0 \mathbf{E}_{\text{ext}} \cdot \hat{\mathbf{n}}$$

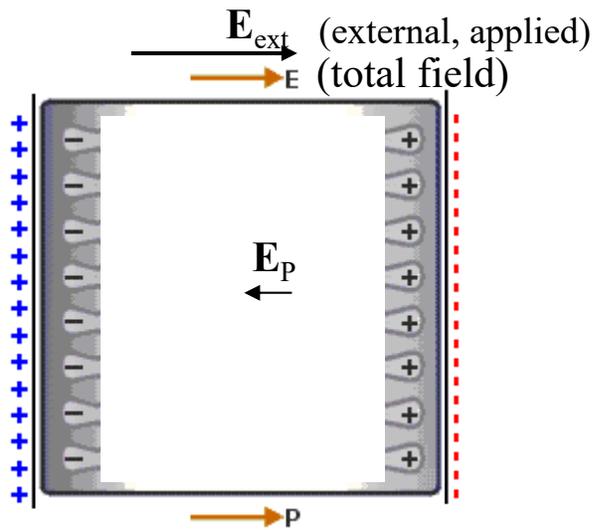
Surface normal **pointing into the vacuum**;  
check signs for both plate surfaces

Now, keep the capacitor isolated (so that  $\rho_s$  cannot change), and push a slab of dielectric into the space between the two plates.

Recall that the polarization (or internal) field is

$$E_P = \rho_{sP} / \epsilon_0 = -P / \epsilon_0 \quad \text{Actually, } \rho_{sP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$$

Surface normal **pointing into the dielectric**;  
check signs for both plate surfaces



The total field  $E = E_{\text{ext}} + E_P = \rho_s / \epsilon_0 - P / \epsilon_0 = \rho_s / \epsilon_0 + \rho_{sP} / \epsilon_0$   
(for the left side.) For both sides, more generally:

$$\mathbf{E} \cdot \hat{\mathbf{n}} = \mathbf{E}_{\text{ext}} \cdot \hat{\mathbf{n}} + \mathbf{E}_P \cdot \hat{\mathbf{n}} = \rho_s / \epsilon_0 - \mathbf{P} \cdot \hat{\mathbf{n}} / \epsilon_0 = \rho_s / \epsilon_0 + \rho_{sP} / \epsilon_0$$

Recall that

$$\mathbf{P} = \chi \epsilon_0 \mathbf{E}$$

Notice this is the total field

Therefore,  $E = \rho_s / \epsilon_0 - \chi E$

$$\mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s / \epsilon_0 - \chi \mathbf{E} \cdot \hat{\mathbf{n}}$$

$$\implies (1 + \chi)E = \rho_s / \epsilon_0$$

$$(1 + \chi)\mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s / \epsilon_0$$

Define  $\epsilon_r = 1 + \chi$ , and we have:  $\epsilon_r E = \rho_s / \epsilon_0$

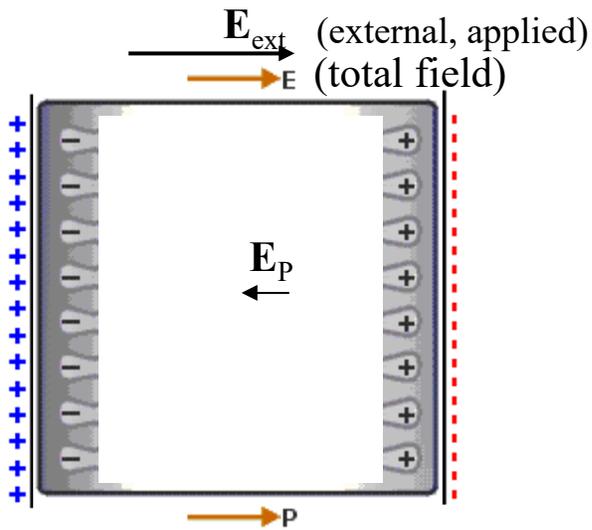
$$\epsilon_r \epsilon_0 E = \rho_s$$

$$\epsilon_r \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s / \epsilon_0 \implies \epsilon_r \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$

Define  $\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 (1 + \chi)$ , and we have:

$$\epsilon E = \rho_s$$

$$\epsilon \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$



$$P = \chi \epsilon_0 E$$

$$\mathbf{P} = \chi \epsilon_0 \mathbf{E}$$

Bulk property

We lump the polarization effect of a dielectric material into a parameter  $\epsilon$ , and substitute  $\epsilon_0$  (for free space) with  $\epsilon$  (for the dielectric) in equations. The Equations will remain the same.

We often write  $\chi$  as  $\chi_e$ , thus 
$$\epsilon \equiv \epsilon_0 (1 + \chi_e) \equiv \epsilon_0 \epsilon_r$$

The field distribution of an infinitely large parallel-plate capacitor with a vacuum gap is the manifestation of Gauss's law. 
$$\epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$

The above relations for the capacitor filled with a dielectric result from Gauss's law and the properties of the dielectric. 
$$\epsilon \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$

Generalization of these relations leads to Gauss's law in dielectrics.

For this simple geometry, we have shown:

$$(1 + \chi)E = \rho_s / \epsilon_0$$

Define  $\epsilon_r = 1 + \chi$ , and we have:

$$\epsilon_r E = \rho_s / \epsilon_0 \quad \Rightarrow \quad \epsilon_r \epsilon_0 E = \rho_s$$

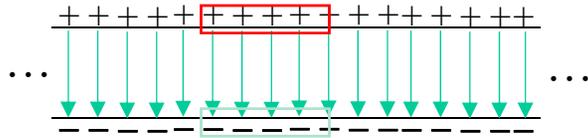
Define  $\epsilon = \epsilon_0 \epsilon_r = \epsilon_0 (1 + \chi)$ , and we have: 
$$\epsilon E = \rho_s$$

$$\epsilon \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$

Surface normal pointing into the dielectric; check signs for both plate surfaces

## Gauss's law in free space

Recall that the field of a parallel-plate capacitor is the consequence of Gauss's law applied to the surface charge densities:

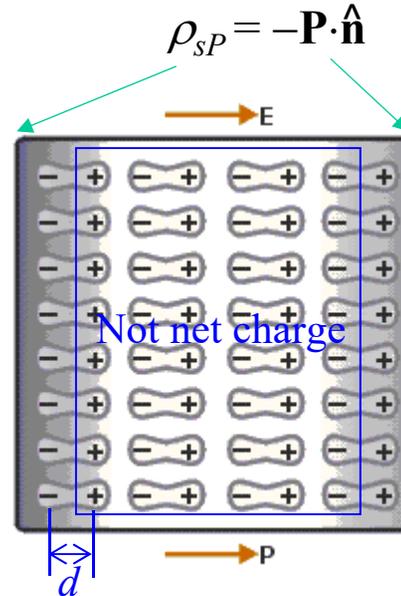


$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \longleftrightarrow \quad \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \rho_s$$

$$\nabla \cdot \mathbf{D} = \rho \quad \longleftrightarrow \quad \mathbf{D} \cdot \hat{\mathbf{n}} = \rho_s$$

## Gauss's law in dielectrics

On both surfaces of a dielectric slab



By analogy:

$$\rho_P = -\nabla \cdot \mathbf{P} \quad \longleftrightarrow \quad \rho_{sP} = -\mathbf{P} \cdot \hat{\mathbf{n}}$$

Notice the sign

## Generalize to other geometries

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho_{\text{total}} = \frac{1}{\epsilon_0} (\rho + \rho_P) \quad \left. \begin{array}{l} \\ \boxed{\rho_P = -\nabla \cdot \mathbf{P}} \end{array} \right\} \Rightarrow \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{1}{\epsilon_0} \nabla \cdot \mathbf{P}$$

$$\Rightarrow \left. \begin{array}{l} \epsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho \\ \mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \end{array} \right\} \Rightarrow$$

$$\nabla \cdot (\epsilon_0 + \chi_e \epsilon_0) \mathbf{E} \equiv \nabla \cdot \mathbf{D} = \rho,$$

where  $\epsilon \equiv \epsilon_0 (1 + \chi_e) \equiv \epsilon_0 \epsilon_r$ ,

$$\mathbf{D} \equiv \epsilon_0 \epsilon_r \mathbf{E} \equiv \epsilon \mathbf{E}$$

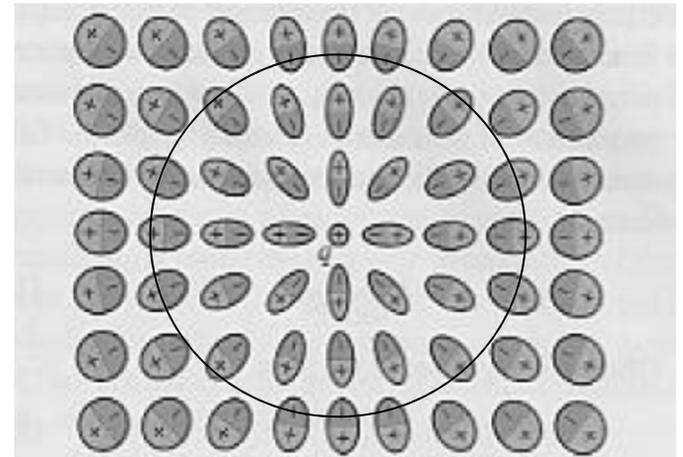


Figure 1-6: Polarization of the atoms of a dielectric material by a positive charge  $q$ .

## Gauss's law in dielectrics

$$\nabla \cdot (\epsilon_0 + \chi_e \epsilon_0) \mathbf{E} \equiv \nabla \cdot \mathbf{D} = \rho,$$

where  $\epsilon \equiv \epsilon_0 (1 + \chi_e) \equiv \epsilon_0 \epsilon_r,$

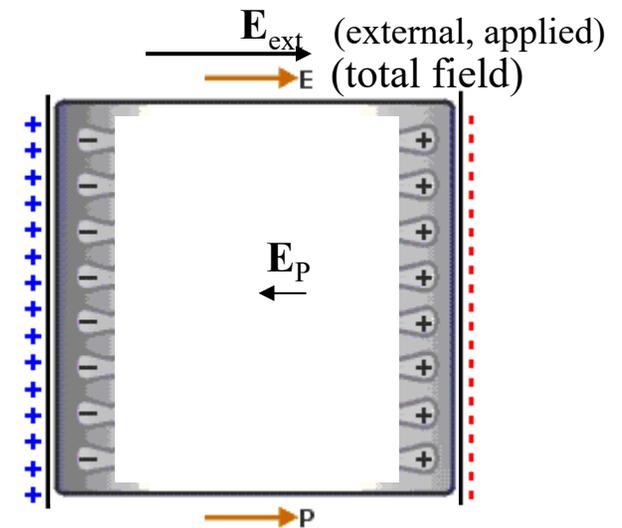
$$\mathbf{D} \equiv \epsilon_0 \epsilon_r \mathbf{E} \equiv \epsilon \mathbf{E}$$

Take-home messages:

We lump the polarization effect of a dielectric material into a parameter  $\epsilon$ , and substitute  $\epsilon_0$  (for free space) with  $\epsilon$  (for the dielectric) in equations.

The polarization charge  $\rho_{sP}$  (or  $\rho_s$  in general) works against the external charge  $\rho_s$  (or  $\rho$  in general). The polarization field  $\mathbf{E}_p$  is always against the external field  $\mathbf{E}_{ext}$ . Therefore the name **dielectric**.  
 $\epsilon_r = 1 + \chi > 1$ , meaning larger  $D$  therefore more charge i.e. larger  $\rho$  needed to get to the same total field  $E$ .

$$\epsilon > \epsilon_0$$



Limitations of our discussion:

- No spontaneous polarization or piezoelectric polarization: whenever  $\mathbf{E} = 0, \mathbf{P} = 0$
- Linearity:  $\mathbf{P} = \chi \epsilon_0 \mathbf{E}, \mathbf{D} = \epsilon \mathbf{E}$
- Isotropy: The proportional constants are the same in all directions, thus  $\mathbf{P} \parallel \mathbf{E}, \mathbf{D} \parallel \mathbf{E}$

### Example 3: $\mathbf{E}$ and $\mathbf{D}$ of a uniformly charged sphere

For a charged dielectric sphere with charge density  $\rho$ , dielectric constant  $\epsilon_r$  (thus  $\epsilon = \epsilon_0 \epsilon_r$ ), and radius  $R$ , find  $\mathbf{E}(\mathbf{r})$  and  $\mathbf{D}(\mathbf{r})$  for all  $\mathbf{r}$ .

The system is spherically symmetric, therefore  $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$  and  $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$ .

For  $r \leq R$ ,  $\cancel{4\pi} r^2 \epsilon E = \frac{\cancel{4}}{3} \pi r^3 \rho$

$$\therefore E = \frac{1}{3\epsilon} r \rho$$

$$E(R) = \frac{R\rho}{3\epsilon}$$

$\Rightarrow$

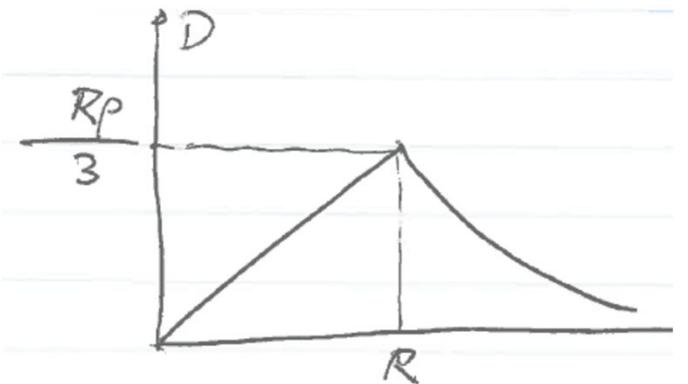
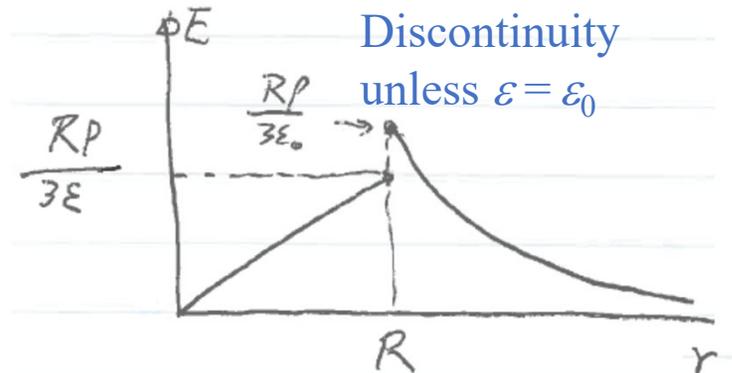
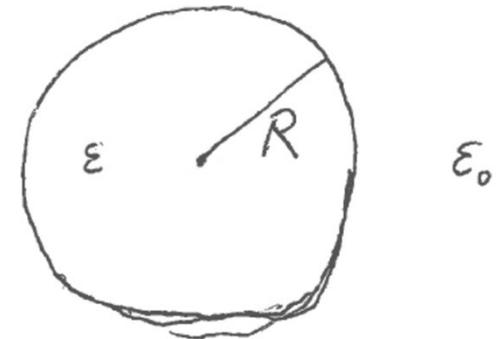
$$D = \frac{1}{3} r \rho$$

$$D(R) = \frac{R\rho}{3}$$

For  $r > R$ ,  $4\pi r^2 \epsilon_0 E = \frac{4}{3} \pi R^3 \rho = Q$

$$\therefore \left( E = \frac{Q}{4\pi r^2 \epsilon_0} \right) = \frac{R^3 \rho}{3\epsilon_0 r^2}$$

Same as point charge



Why is  $E$  discontinuous and  $D$  continuous?

Read textbook Section 4-7 overview & Subsection 4-7.1.