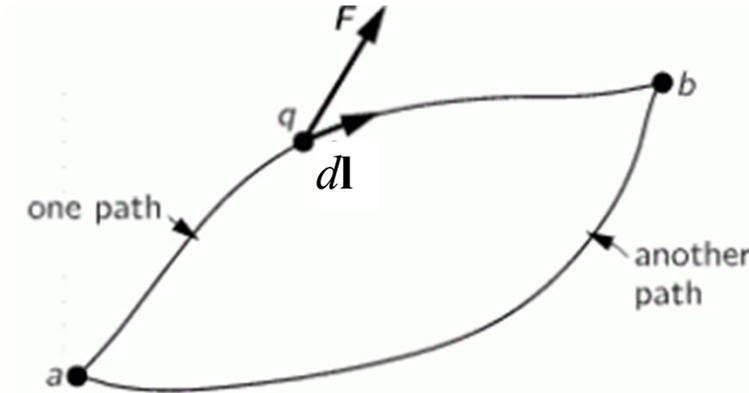


# Potential

The electrostatic field is **conservative** (just like gravity)



The (minimum) work done to move  $q$  from  $a$  to  $b$ :

$$W = -\int_a^b \mathbf{F} \cdot d\mathbf{l}$$

$$= -q \int_a^b \mathbf{E} \cdot d\mathbf{l}$$

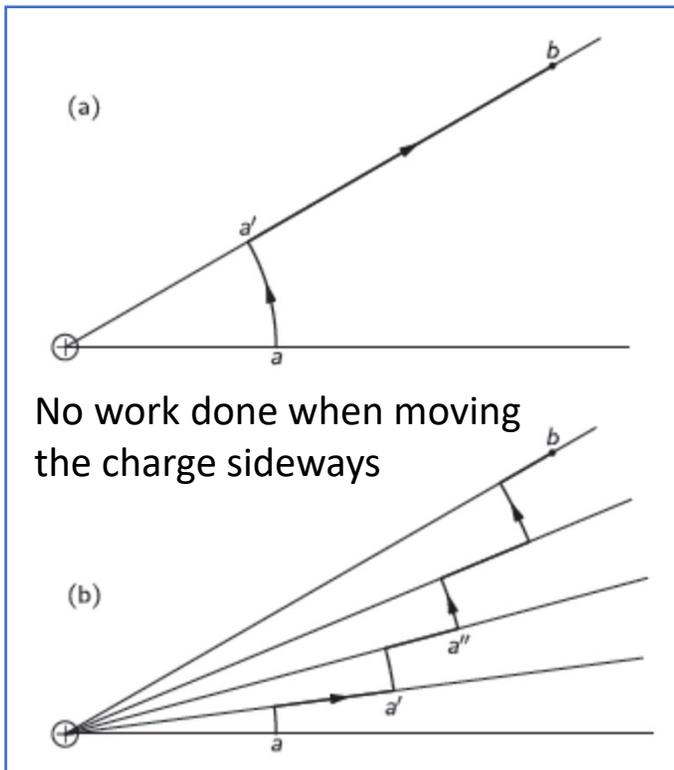
The meaning of the negative sign:  
An external force  $-\mathbf{F}$  is exerted to overcome the electrostatic force  $\mathbf{F}$ . Here  $W$  is the work done by the external force.

$W$  is **independent of the path**. Therefore,  $\oint \mathbf{E} \cdot d\mathbf{l} = 0$

But why is  $W$  independent of the path?

Let's first consider the field of a point charge.

Because the field is **radial**,  $\mathbf{E} = E(r)\hat{\mathbf{r}}$  (Coulomb's law) for a point charge.



By **superposition**, for any **electrostatic** field,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

Such a field is said to be **conservative**.

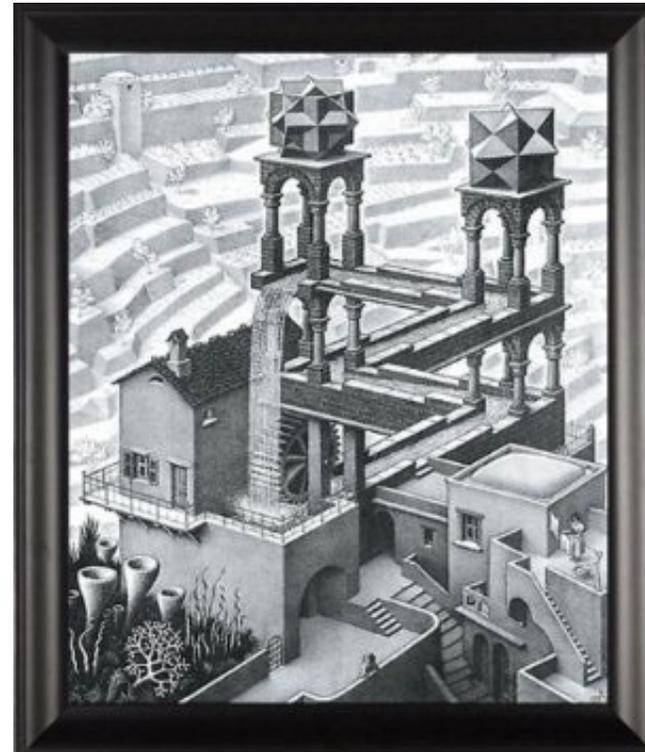
We should point out an important fact. For any **radial** force the work done is independent of the path, and there exists a **potential**. If you think about it, the entire argument we made above to show that the work integral was independent of the path depended only on the fact that the force from a single charge was **radial** and spherically symmetric. It did **not** depend on the fact that the dependence on distance was as  $1/r^2$ —there could have been any  $r$  dependence.

--Richard Feynman

Side note: A non-conservative field does not violate the conservation of energy.

Similarly, the gravitational field is also conservative, due to the similarity between Newton's law of universal gravity and Coulomb's law.

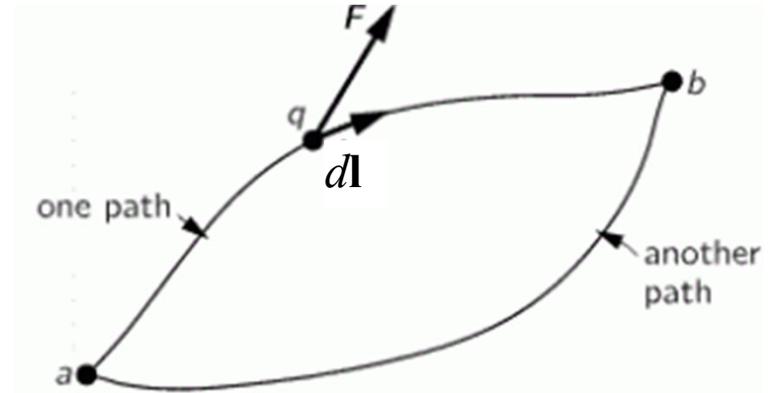
Exceptions are found only in abstract art:



Waterfall by M.C. Escher

In the physical world, you need a pump, just as you need a battery a the likes for a circuit.

Since the work to be done to move a charge  $q$  from a to b is independent of the path and proportional to  $q$ , we can define a quantity called **potential**, just like height in the gravitational field.



$$qV_a = qV_b - q \int \vec{E} \cdot d\vec{l}$$

Potential at a
Potential at b

Work to be done

$$V_a - V_b = - \int \vec{E} \cdot d\vec{l} \iff dV = - \vec{E} \cdot d\vec{l}$$

$-\mathbf{E}$  is sort of the derivative of  $V$  in 3D space. In 1D, we have  $dV = -E dl \iff E = - \frac{dV}{dl}$   
 Think about the infinitely large parallel plate capacitor.

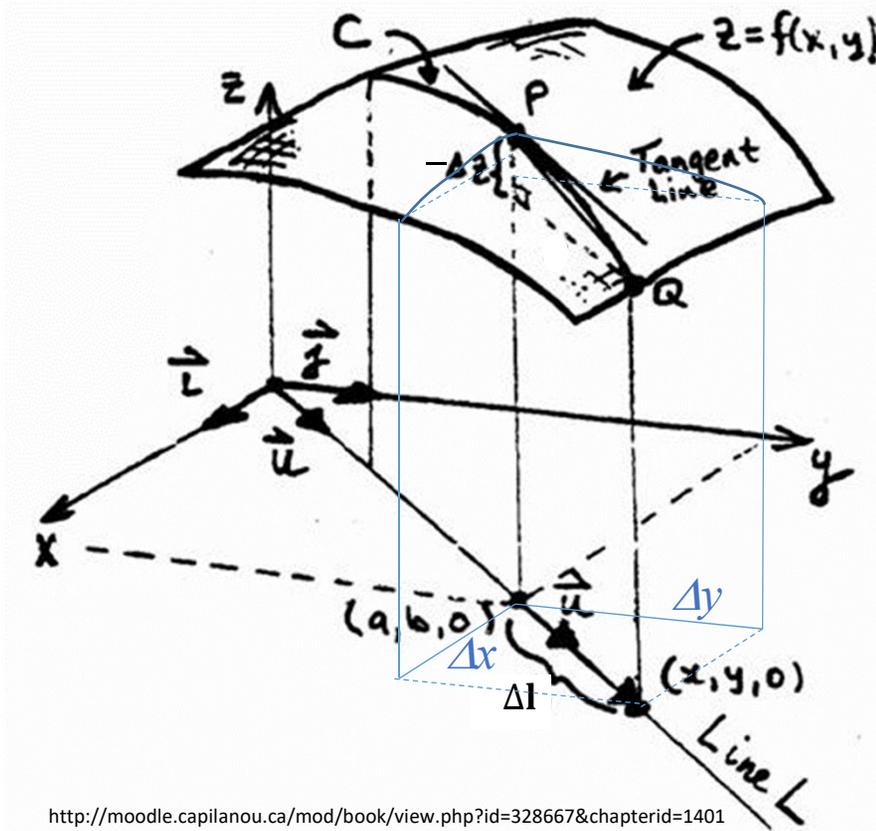
This “vector derivative” is called the **gradient**.

The **gradient** of a scalar function, or, as we call it here, a scalar field  $V(x,y,z)$  is

$$\hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}$$

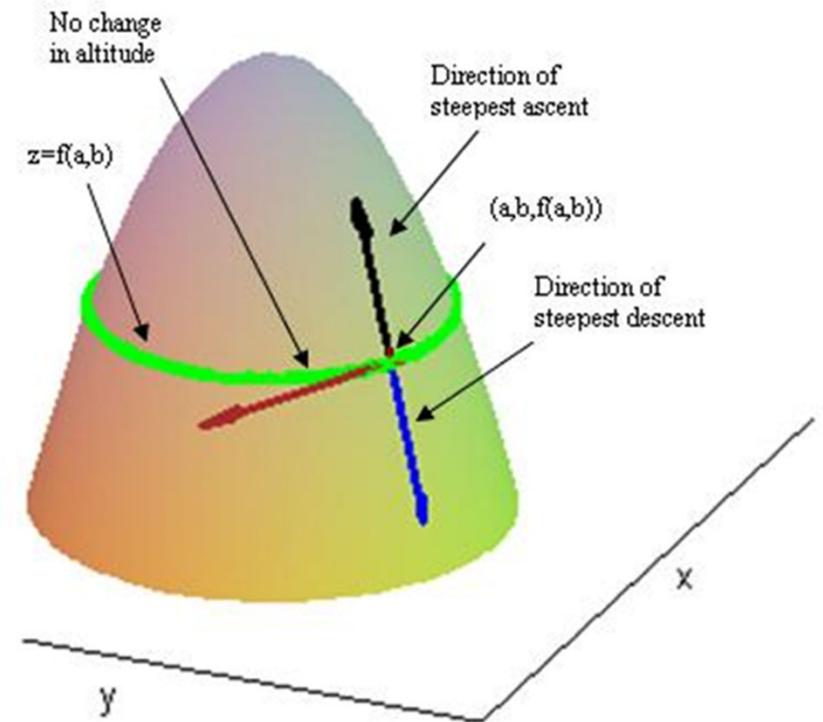
The vector sum of the three derivatives in respective directions

Visualize the gradient in 2D space



<http://moodle.capilanou.ca/mod/book/view.php?id=328667&chapterid=1401>

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The gradient is the steepest slope.  
 In the direction perpendicular to the gradient,  
 the slope is zero.

The **gradient** of a scalar function, or as we call it here, a **scalar field**  $V(x,y,z)$ , is

$$\hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}$$

The vector sum of the three derivatives in respective directions

Recall that we defined

$$\nabla = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right)$$

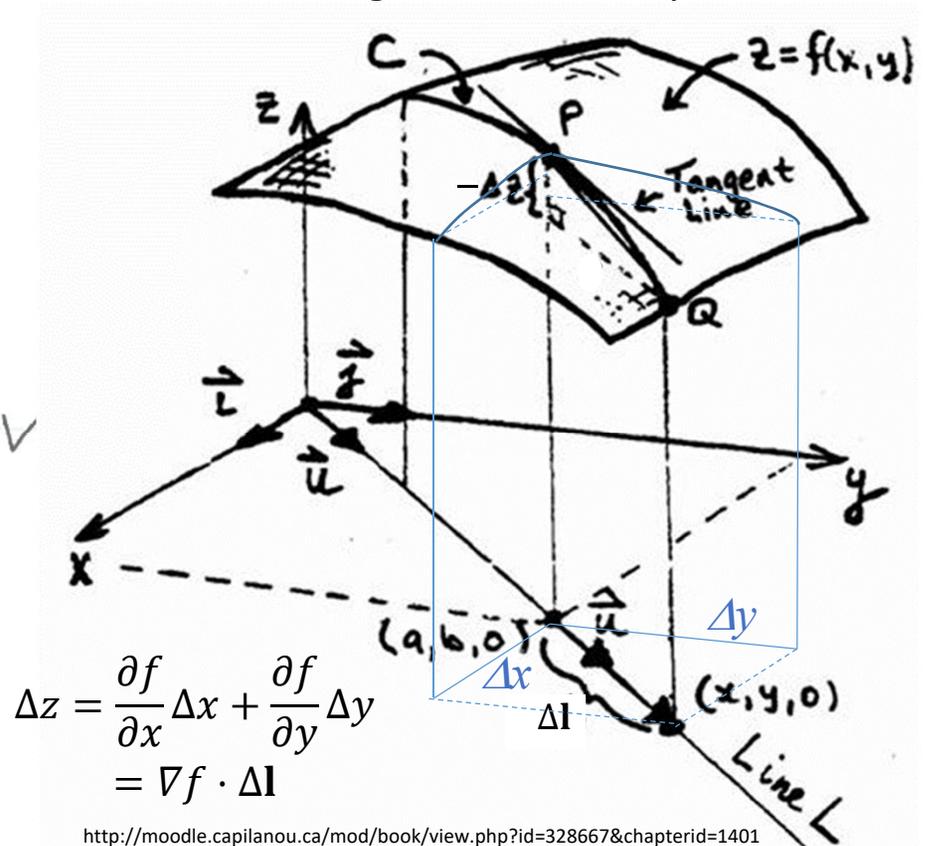
$$\therefore \nabla V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}$$

Thus we can write the **gradient** of  $V(x,y,z)$  as  $\nabla V$

$$\text{Thus } dV = -\vec{E} \cdot d\vec{l} = \nabla V \cdot d\vec{l}$$

$$\nabla V = -\vec{E}$$

Visualize the gradient in 2D space



The electric field is the negative gradient of the potential.

**Example:** potential distribution due to a point charge

$$\vec{E} = \frac{q}{4\pi\epsilon} \frac{\hat{r}}{r^2}$$

Take a **reference**  $V(\infty) = 0$ .

Recall that no work is done if we move a probe charge (not the charge  $q$ ) sideways.  
So move it right towards  $q$ .

Pay attention to the sign: a force  $-q'E$  need to push a probe charge  $q'$

$$\begin{aligned} V(R) &= V(\infty) - \int_{\infty}^R \vec{E} \cdot d\vec{r} \\ &= \frac{-q}{4\pi\epsilon} \int_{\infty}^R \frac{dr}{r^2} \\ &= \frac{q}{4\pi\epsilon} \frac{1}{r} \Big|_{\infty}^R = \frac{q}{4\pi\epsilon R} \end{aligned}$$

Why is the negative sign gone?

If in free space,  $\epsilon = \epsilon_0$

## Poisson's Equation

$$\left. \begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon} \\ \vec{E} &= -\nabla V \end{aligned} \right\} \Rightarrow \nabla \cdot (\nabla V) = -\frac{\rho}{\epsilon}$$

Recall that

$$\nabla \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$
$$\nabla V = \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z}$$

$$\begin{aligned} \nabla^2 V &\equiv \nabla \cdot (\nabla V) = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \end{aligned}$$

Notice that this is a scalar

$\nabla^2$  is called the **Laplacian** operator.

## Poisson's Equation

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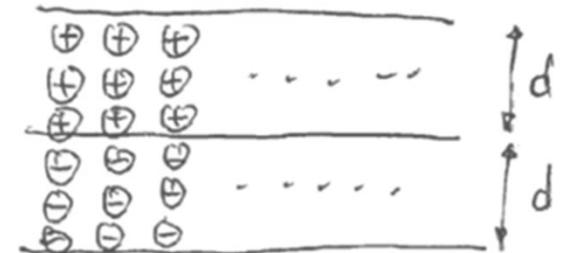
$$\begin{aligned} \nabla^2 V \equiv \nabla \cdot (\nabla V) &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left( \hat{x} \frac{\partial V}{\partial x} + \hat{y} \frac{\partial V}{\partial y} + \hat{z} \frac{\partial V}{\partial z} \right) \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \end{aligned}$$

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon}$$

Recall this problem.

One way to solve it is to use the 1D Poisson's Equation.

Here, 1D means there is no variation in the other two dimensions – the slabs are assumed to be infinitely large in lateral dimensions.



Do Problems 1, 2 of Homework 9. Read Sections 4-5 and 3-4 of textbook.  
Continue working Chapter 3.

## Current & Ohm's Law

Let's first consider the current w/o asking what drives it. (The **kinetics** of current)

Closely read these notes and Section 4-2.2 for details.

Conductor w/ **mobile** charge density  $\rho_v$ , or just  $\rho$  for short.

What are the **mobile charge carriers** in metals?

Usually the overall conductor is charge neutral.

Mobile charge carriers move at an **average net** velocity  $\mathbf{u}$ .

Special case in Fig. (a):

$$\Delta I = \frac{\Delta q'}{\Delta t} = \rho u \Delta s'$$

$$J = \frac{\Delta I}{\Delta s'} = \rho u = -neu$$

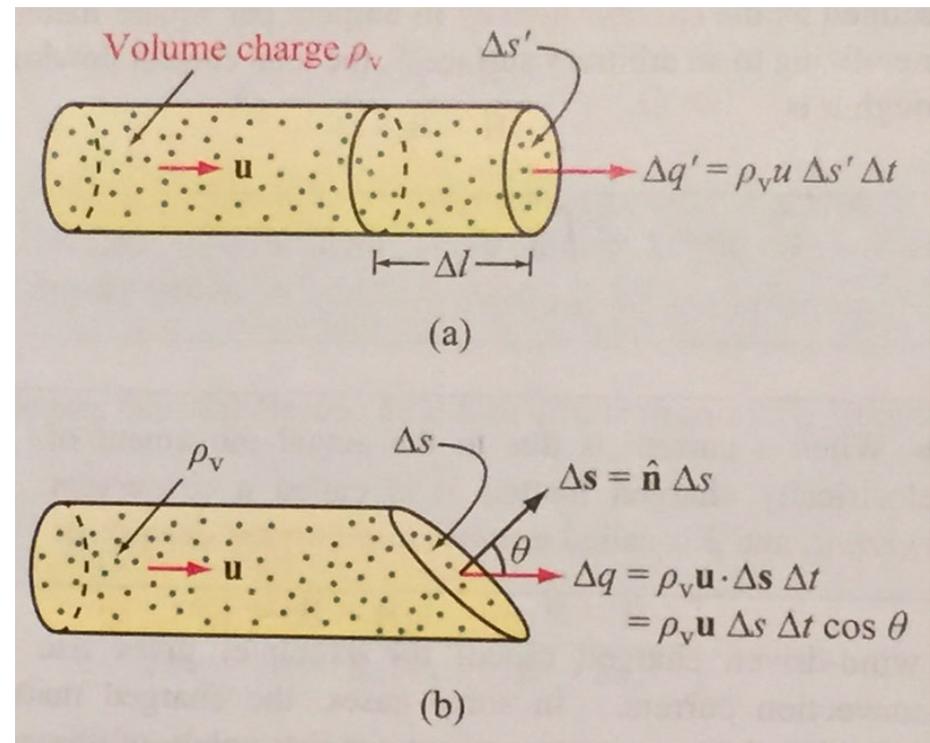
where  $n$  is the **mobile** electron density and  $e$  the electron charge.

General case in (b):  $\Delta I = \frac{\Delta q}{\Delta t} = \rho \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}$

$$\mathbf{J} = \rho \mathbf{u}$$

The unit of  $J$  is....?

$$\text{A/m}^2 = (\text{C/m}^3)(\text{m/s}) = (\text{C/s})/\text{m}^2$$



$$J = \frac{\Delta I}{\Delta s'} = \rho u = -neu$$

More generally,  $\mathbf{J} = \rho \mathbf{u}$

The unit of  $J$  is....

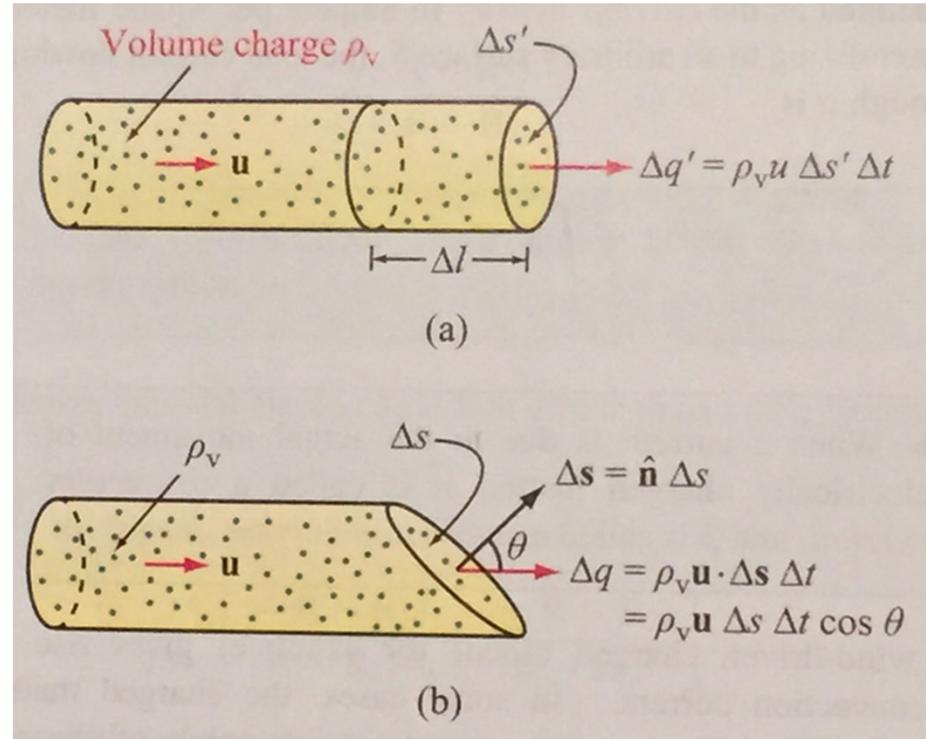
$$\text{A/m}^2 = (\text{C/m}^3)(\text{m/s}) = (\text{C/s})/\text{m}^2$$

For an arbitrary surface  $S$  (not necessarily planar),

$$I = \int_S \mathbf{J} \cdot d\mathbf{s}$$

What about a closed surface?

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = ???$$



$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$  for an arbitrary closed surface.

Kirchhoff's current law (KCL)!

Closely review these notes and textbook Section 4-2.2.

Recall that  $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$  for any arbitrary closed contour  $C$ .

What does this correspond to in circuit theory?

Kirchhoff's voltage law (KVL)

(Not exactly, if we define voltage just as potential difference between two points and include things like batteries in the circuit. We will talk about this later.)

We have NOT covered the following three slides in class.  
The pace was adjusted to balance the homework load.  
The following three slides (dynamics of current, Ohm's law)  
will be discussed after the Spring Break.

Now, back to the current – the **dynamics** of it.

The **average net** velocity  $\mathbf{u}$  is often called the **drift velocity** ( $\mathbf{v}_d$ ), as it's driven by the field  $\mathbf{E}$ :  $\mathbf{v}_d = \mu\mathbf{E}$

$\mu$ : a proportional constant (material property) called “**mobility**”

But, think about it.  $\mathbf{E} = \mathbf{F}/q$  ← Charge of the carrier.  $-e$  for the electron.

$$\mathbf{v}_d \propto \mathbf{F}$$

In semiconductor and circuit books,  $q$  stands for  $e$ .

Does this contradict Newton's second law?

Now, back to the current – the dynamics of it.

The **average net** velocity  $\mathbf{u}$  is often called the **drift velocity** ( $\mathbf{v}_d$ ), as it's driven by the field  $\mathbf{E}$ :  $\mathbf{v}_d = \mu\mathbf{E}$

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 $\mathbf{v}_d \propto \mathbf{F}$  In semiconductor and circuit books,  $q$  stands for  $e$ .

Does this contradict Newton's second law?

Here is **roughly** what happens:

On average, a charge carrier collides with something every time interval  $\tau$ . It loses/forgets its  $\mathbf{v}_d$ , or “randomizes”. Then it starts over.

$$\mathbf{J} = \rho\mathbf{u} = nq\mathbf{v}_d = nq\mu\mathbf{E} \equiv \sigma\mathbf{E}$$

charge carrier density (unit?)

charge density, not resistivity here

Conductivity  $\sigma = nq\mu$

In a semiconductor, you may have both electrons and holes, carrying  $-e$  and  $+e$  each, respectively.

$$\mathbf{J} = -nev_e + pev_h = -ne(-\mu_e \mathbf{E}) + p\mu_h \mathbf{E} = (n\mu_e + p\mu_h)e\mathbf{E} \equiv \sigma \mathbf{E}$$

electron density                      hole density

or, in the simple scalar form

$$J = nev_e + pev_h = (n\mu_e + p\mu_h)eE \equiv \sigma E$$

For a wire (or a semiconductor channel) of length  $l$  and cross section area  $A$ ,  
resistance  $R = (1/\sigma)(l/A)$

Read textbook: Section 4-6