Boundary Conditions

Boundaries between different media/materials

Let’s first look at the boundary between two materials in general.

In each material, we decompose the electric field into normal ($n$) and tangential ($t$) components.

Imagine a tiny rectangular loop, $\Delta l$ long and with a zero width, but with the two $\Delta l$ long sides on opposite sides of the boundary.

The boundary is not necessarily planar.

Recall that the electric field is conservative:

$$\oint \overrightarrow{E} \cdot d\overrightarrow{l} = 0$$

Therefore,

$$E_{2t}\Delta l - E_{1t}\Delta l = 0 \Rightarrow E_{2t} = E_{1t}$$

Special case: medium 1 is a perfect conductor:

$$E_1 = 0 \Rightarrow E_{1t} = 0 = E_{2t} = 0$$

The electric field at a perfect conductor surface must be perpendicular to the surface.
Imagine a tiny pie, with a zero thickness and an area $\Delta S$, and with the bottom and top on opposite sides of the boundary.

Recall Gauss’s law:

$$\oint \mathbf{D} \cdot d\mathbf{S} = Q$$

$$\Rightarrow \quad (D_{2n} - D_{1n}) \Delta S = \rho_s \Delta S$$

$$\Rightarrow \quad D_{2n} - D_{1n} = \rho_s$$

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_s$$

Special cases:

1. Interfaces between two dielectrics with

$$D_{2n} = D_{1n}, \quad \Rightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

This is why we have the discontinuity in $E$ at the surface of a charged sphere.

2. Interfaces between a perfect conductor and a dielectric or vacuum

(Medium 1 is the conductor)

$$E_1 = 0 \Rightarrow E_{1n} = 0$$

$$\varepsilon_2 E_{2n} = \rho_s$$

Does not include polarization charge
For a charged dielectric sphere with charge density $\rho$, dielectric constant $\varepsilon_r$ (thus $\varepsilon = \varepsilon_0 \varepsilon_r$), and radius $R$, find $E(r)$ and $D(r)$ for all $r$.

The system is spherically symmetric, therefore $E(r) = E(r)\hat{r}$ and $D(r) = D(r)\hat{r}$.

For $r \leq R$, 
\[ 4\pi r^2 \varepsilon E = \frac{4\pi}{3} \pi r^3 \rho \]
\[ \therefore E = \frac{1}{3\varepsilon} r \rho \]
\[ \Rightarrow E(R) = \frac{R \rho}{3\varepsilon} \]
\[ D = \frac{1}{3} r \rho \]

For $r > R$, 
\[ 4\pi r^2 \varepsilon_0 E = \frac{4\pi}{3} R^3 \rho = Q \]
\[ \therefore \left( E = \frac{Q}{4\pi \gamma^2 \varepsilon_0} \right) = \frac{R^3 \rho}{3\varepsilon_0 \gamma^2} \]

Same as point charge

Why is $E$ discontinuous and $D$ continuous?

Discontinuity unless $\varepsilon = \varepsilon_0$
Recall the infinitely large charge plane – a slide from a previous lecture

Find the electric field due to an infinitely large sheet of charge with an areal charge density \( \rho_\text{s} \). It is a 2D sheet, with a zero thickness.

By symmetry, the \( \mathbf{E} \) fields on the two sides of the sheet must be equal & opposite, and must be perpendicular to the sheet.

Imagine a cylinder (pie) with area \( A \) and zero height (thickness).

If the cylinder is at the sheet,

\[
\hat{z} \varepsilon_0 \mathbf{E} \cdot \mathbf{A} = \rho_\text{s} \mathbf{A} \quad \Rightarrow \quad \mathbf{E} = \frac{\rho_\text{s}}{\hat{z} \varepsilon_0}
\]

Recall our result for the charged disk:

\[
E (\hat{z} \to 0) = \frac{1}{\hat{z} \varepsilon_0} \left( \frac{Q}{\pi a^2} \right) = \frac{\rho_\text{s}}{\hat{z} \varepsilon_0}
\]

Actually,

\( \hat{z} \to 0 \iff \alpha \to \infty \quad \left( \frac{3}{\alpha} \right) \to 0 \)

If the cylinder is elsewhere, the net flux is 0:

What goes in comes out; no charge inside the cylinder.
Also recall the parallel-plate capacitor

What if there are two infinitely large sheets, one charged with a surface density \( +\rho_s \), and the other \( -\rho_s \). Assume \( \rho_s \) is positive for convenience.

Another look at the parallel-plate capacitor

The electric field lines starts from a positive charge and ends at a negative charge.

Gauss’s law leads to \( E = \frac{\rho_s}{\varepsilon_0} \)

You may use a negative sign to signify the “downward” direction. \( E = -\frac{\rho_s}{\varepsilon_0} \)

Sign conventions are kind of arbitrary. We just need to be self-consistent within the context.

We will discuss capacitors in greater detail later (after the Spring Break).

Review textbook Section 4-8. Do Homework 9 Problem 3.
Finish reading Chapter 3 of textbook, and finish Homework 8.