Boundary Conditions

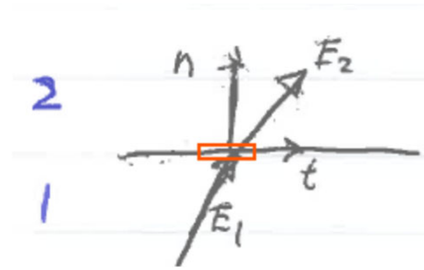
Boundaries between different media/materials

Let's first look at the boundary between two materials in general.

In each material, we decompose the electric field into normal (*n*) and tangential (*t*) components.

Imagine a tiny rectangular loop, *Δl* long and with a zero width, but with the two *Δl* long sides on opposite sides of the boundary.

The boundary is not necessarily planar.



Recall that the electric field is conservative: $\oint \vec{E} \cdot d\vec{l} = 0$

Therefore, $E_{2t}\,\Delta l - E_{1t}\,\Delta l = 0 \Rightarrow \boxed{E_{2t} = E_{1t}}$

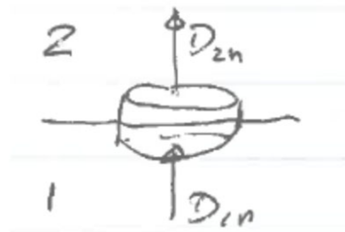
Special case: medium 1 is a perfect conductor:

$$E_1 = 0 \Rightarrow E_{1t} = 0 = E_{2t} = 0$$

The electric field at a perfect conductor surface must be perpendicular to the surface.

Imagine a tiny pie, with a zero thickness and an area ΔS , and with the bottom and top on opposite sides of the boundary.

Recall Gauss's law: $\oint_s \vec{D} \cdot d\vec{S} = Q$



$$\Rightarrow \quad (D_{2n} - D_{1n})\Delta S = \rho_s \Delta S$$

$$\Rightarrow \quad D_{2n} - D_{1n} = \rho_s$$

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \rho_s$$

Special cases:

Does not include polarization charge

1. Interfaces between two dielectrics with $\rho_s = 0$

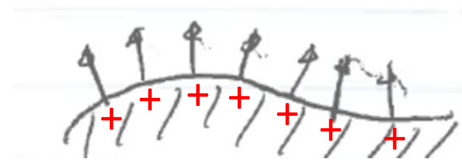
$$\boxed{D_{2n} = D_{1n} \,, \quad \Leftrightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}}$$

This is why we have the discontinuity in E at the surface of a charged sphere.

2. Interfaces between a perfect conductor and a dielectric or vacuum

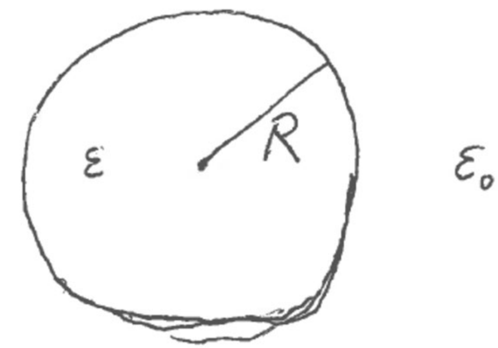
(Medium 1 is the conductor)

$$\boxed{\begin{array}{l} E_1 = 0 \quad \Rightarrow E_{1n} = 0 \\ \varepsilon_2 E_{2n} = \rho_s \end{array}}$$



Slide from a previous lecture

For a charged dielectric sphere with charge density ρ, dielectric constant ε_r (thus $\varepsilon = \varepsilon_0\varepsilon_r$), and radius R, find $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ for all $\mathbf{r}$.



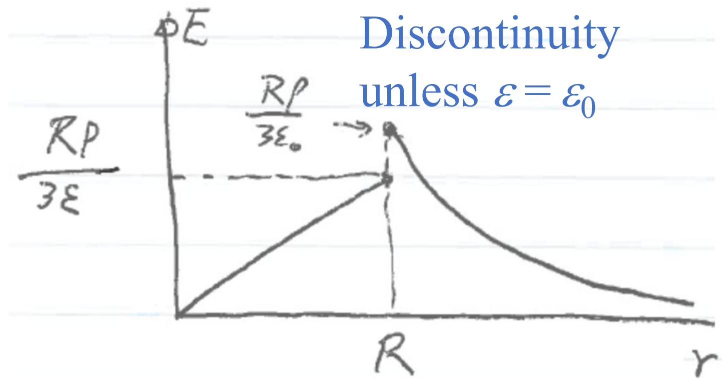
The system is spherically symmetric, therefore $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$ and $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$.

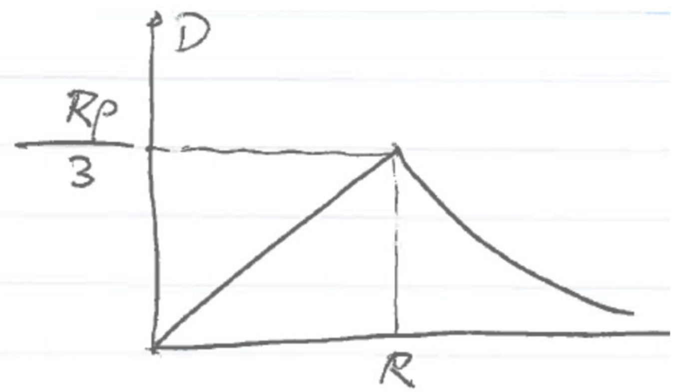
For $r \le R$, $\cancel{4\pi} r^2 \varepsilon E = \frac{\cancel{4}}{3} \cancel{\pi} r^3 \rho$

$\therefore E = \frac{1}{3\varepsilon} r\rho \quad \Rightarrow \quad E(R) = \frac{R\rho}{3\varepsilon}$

$D = \frac{1}{3} r\rho \qquad\qquad D(R) = \frac{R\rho}{3}$

Discontinuity unless $\varepsilon = \varepsilon_0$





For $r > R$, $4\pi r^2 \varepsilon_0 E = \frac{4}{3}\pi R^3 \rho = Q$

$\therefore \boxed{E = \frac{Q}{4\pi r^2 \varepsilon_0}} = \frac{R^3\rho}{3\varepsilon_0 r^2}$

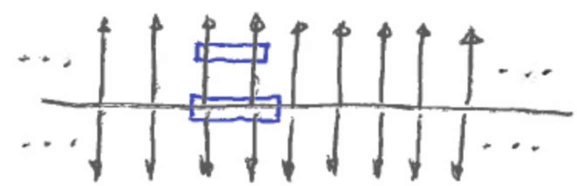
Same as point charge

Why is E discontinuous and D continuous?

Recall the infinitely large charge plane – a slide from a previous lecture

Find the electric field due to an infinitely large sheet of charge with an areal charge density ρ_S. It is a 2D sheet, with a zero thickness.

By symmetry, the **E** fields on the two sides of the sheet must be equal & opposite, and must be perpendicular to the sheet.



Imagine a cylinder (pie) with area A and zero height (thickness).

If the cylinder is at the sheet,

$$2\varepsilon_0 E A = \rho_s A \quad \Longrightarrow \quad E = \frac{\rho_s}{2\varepsilon_0}$$

Treat **E** as a scalar, since we already know the direction.

Recall our result for the charged disk: $E(z \to 0) = \dfrac{1}{2\varepsilon_0} \dfrac{Q}{\pi a^2} = \dfrac{\rho_s}{2\varepsilon_0}$

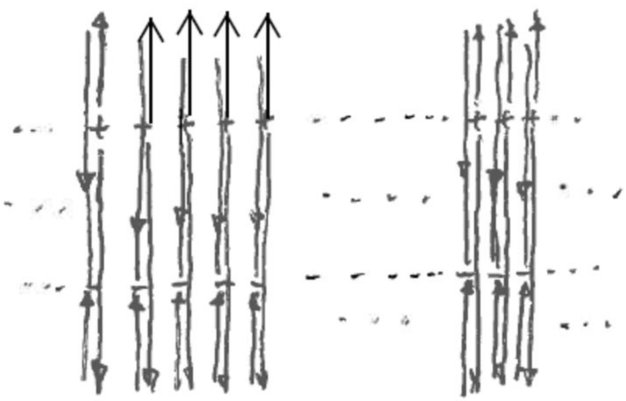
Actually, $z \to 0 \quad \Longleftrightarrow \quad a \to \infty \quad \left(\frac{z}{a}\right) \to 0$

If the cylinder is elsewhere, the net flux is 0:

What goes in comes out; no charge inside the cylinder.

Also recall the parallel-plate capacitor

What if there are two infinitely large sheets, one charged with a surface density $+\rho_S$, and the other $-\rho_S$. Assume ρ_S. is positive for convenience.

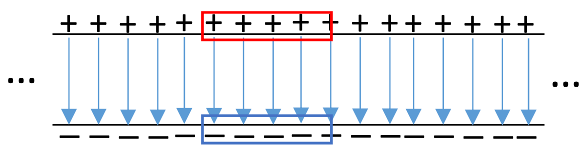


$$\leftarrow E = \frac{\rho_s}{2\varepsilon_0} - \frac{\rho_s}{2\varepsilon_0} = 0$$

$$\leftarrow E = -\frac{\rho_s}{2\varepsilon_0} + \frac{-\rho_s}{2\varepsilon_0} = -\frac{\rho_s}{\varepsilon_0}$$

$$\leftarrow E = -\frac{\rho_s}{2\varepsilon_0} + \frac{\rho_s}{2\varepsilon_0} = 0$$

Another look at the parallel-plate capacitor



The electric field lines starts from a positive charge and ends at a negative charge.

Gauss's law leads to $E = \frac{\rho_s}{\varepsilon_0}$

You may use a negative sign to signify the "downward" direction. $E = -\frac{\rho_s}{\varepsilon_0}$

Sign conventions are kind of arbitrary. We just need to be self-consistent within the context.

We will discuss capacitors in greater detail later (after the Spring Break).

Review textbook Section 4-8. Do Homework 9 Problem 3.
Finish reading Chapter 3 of textbook, and finish Homework 8.