Magnetostatics: Part 1

We present magnetostatics in comparison with electrostatics.

Sources of the fields:

Electric field **E**: Coulomb's law. Charge ➡ **E**

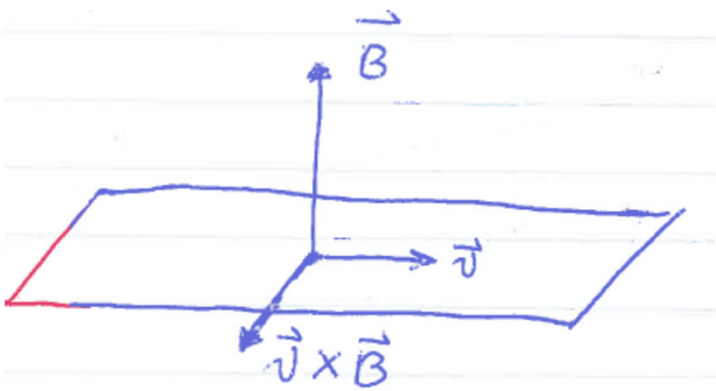
Magnetic field **B**: Biot-Savart law. Current (moving charge) ➡ **B**

Forces exerted by the fields:

Electric: $\mathbf{F} = q\mathbf{E}$ $\vec{F} = q\vec{E}$

Magnetic: $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$ $\vec{F} = q\vec{v}\times\vec{B}$

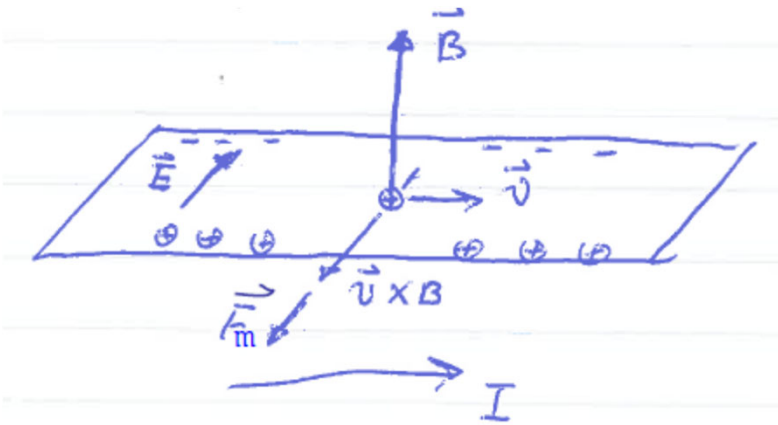
Mind the notations, both printed and hand-written



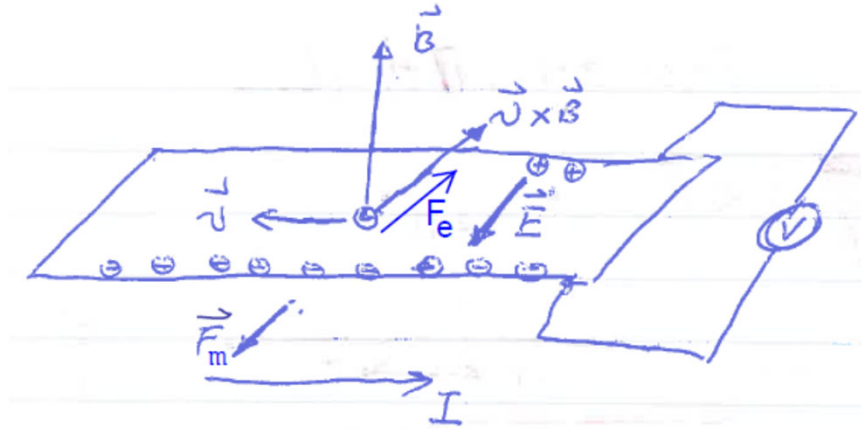
$\mathbf{F} \perp \mathbf{B}$, $\mathbf{F} \perp \mathbf{v}$

Does the magnetic force do any work to the charge?

Positive charge moving at **v** ⊥ **B**



Negative charge moving at **v** ⊥ **B**



Steady state: $E = vB$

By measuring the polarity of the induced voltage,
we can determine the sign of the moving charge.

If the moving charge carriers are in a perfect conductor, then we can have an electric field inside the perfect conductor. Does this contradict what we have learned in electrostatics?

Notice that the direction of the magnetic force is the same for both positive and negative charge carriers.

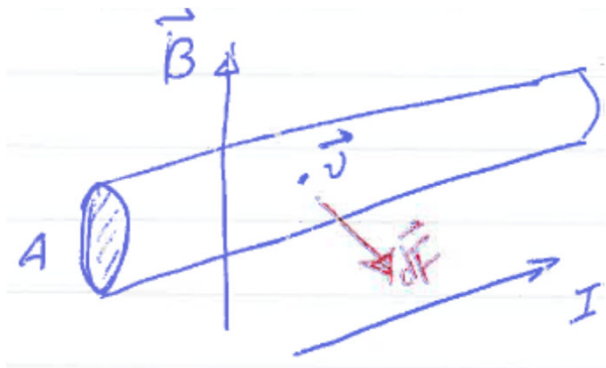
Magnetic force on a current carrying wire

The magnetic force is in the same direction regardless of the charge carrier sign.

$$I = nqvA$$

Carrier density

Charge of each carrier



If the charge carrier is negative

For a small piece of the wire **dl**

$$dQ = nqA\,dl$$

scalar

$$d\vec{F} = dQ\,\vec{v}\times\vec{B} = (nqA\,dl)\,\vec{v}\times\vec{B}$$
$$= nqvA\,d\vec{l}\times\vec{B} = I\,d\vec{l}\times\vec{B}$$

Notice that **v** // **dl**

A current-carrying wire in an external magnetic field feels the force exerted by the field. If the wire is not fixed, it will be moved by the magnetic force.

Some work must be done. Does this contradict what we just said?

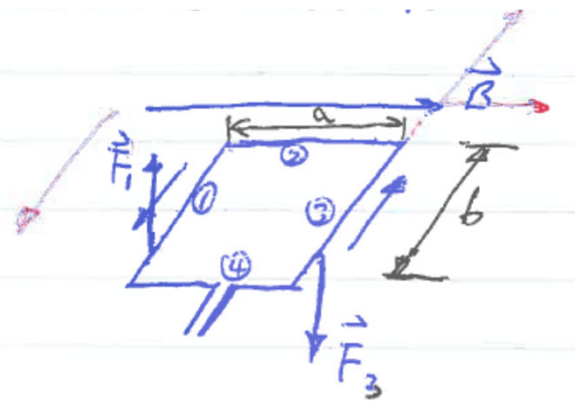
For a wire from point A to point B, $\vec{F} = I \int_A^B d\vec{\ell} \times \vec{B}$

For a wire loop, $\vec{F} = I \oint d\vec{\ell} \times \vec{B}$

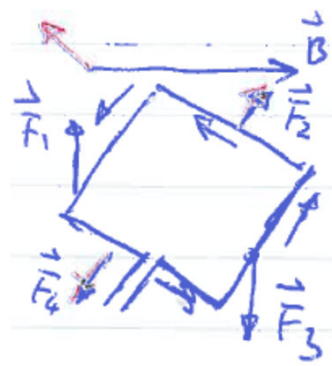
If B is a constant all along the loop, $\vec{F} = I\left(\oint d\vec{\ell}\right) \times \vec{B} = 0$

because $\oint d\vec{\ell} = 0$

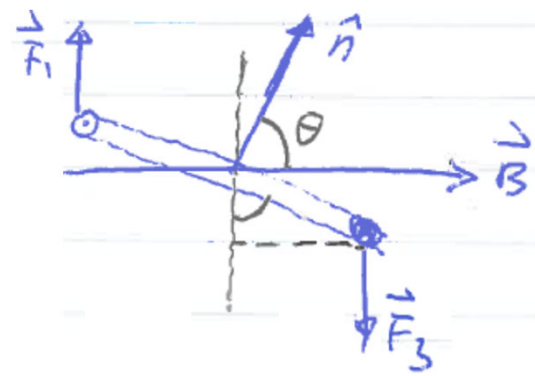
Let's look at a rectangular wire loop in a uniform magnetic field **B**.



$\mathbf{F}_1 + \mathbf{F}_3 = 0$, $F_2 = F_4 = 0$

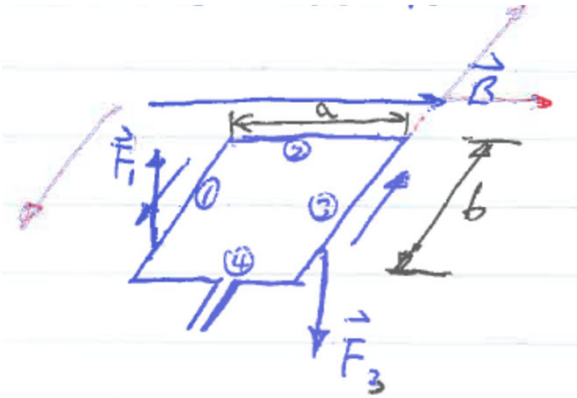


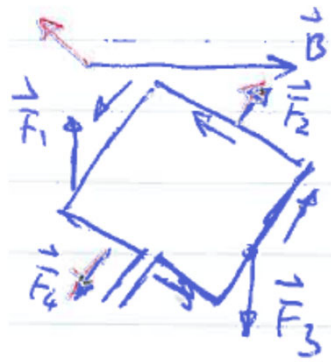
$\mathbf{F}_1 + \mathbf{F}_3 = 0$, $\mathbf{F}_2 + \mathbf{F}_4 = 0$

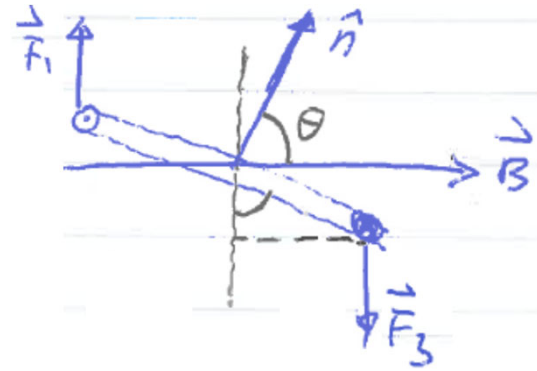


At any position, the total force is 0. Will the loop move?

A rectangular wire loop in a uniform magnetic field **B**.







$\mathbf{F}_1 + \mathbf{F}_3 = 0,\ F_2 = F_4 = 0$ $\qquad$ $\mathbf{F}_1 + \mathbf{F}_3 = 0,\ \mathbf{F}_2 + \mathbf{F}_4 = 0$

At any position, the total force is 0. Therefore the center of mass does not move.

But there is a torque **T**, driving the loop to rotate.

$$T = F_1 \frac{a}{2} \sin\theta + F_3 \frac{a}{2} \sin\theta$$
$$= F a \sin\theta = I b B a \sin\theta = I A B \sin\theta$$

Area

Torque **T** is a vector. Its direction is defined according to the right hand rule.

Define the magnetic moment $\vec{m} = I A \hat{n}$ ← Normal unit vector of the loop

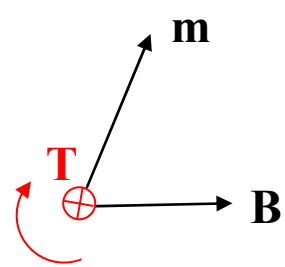
If the loop has N turns, $\vec{m} = N I A \hat{n}$

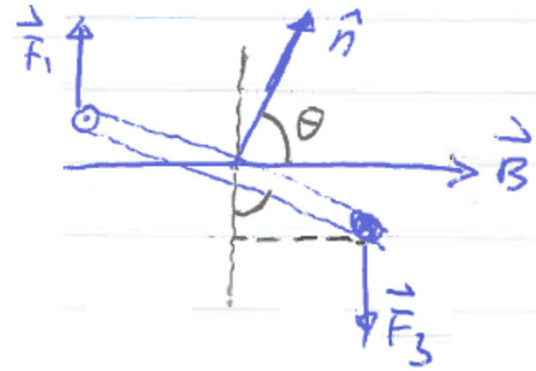
Then, $\boxed{\vec{T} = \vec{m} \times \vec{B}}$

A rectangular wire loop in a uniform magnetic field **B**.

$$\vec{m} = N I A \hat{n}$$

$$\boxed{\vec{T} = \vec{m} \times \vec{B}}$$





Although derived for a rectangular loop, this equation is general.

This is the principle of electric motors.

We know that a magnetic field does not do work.

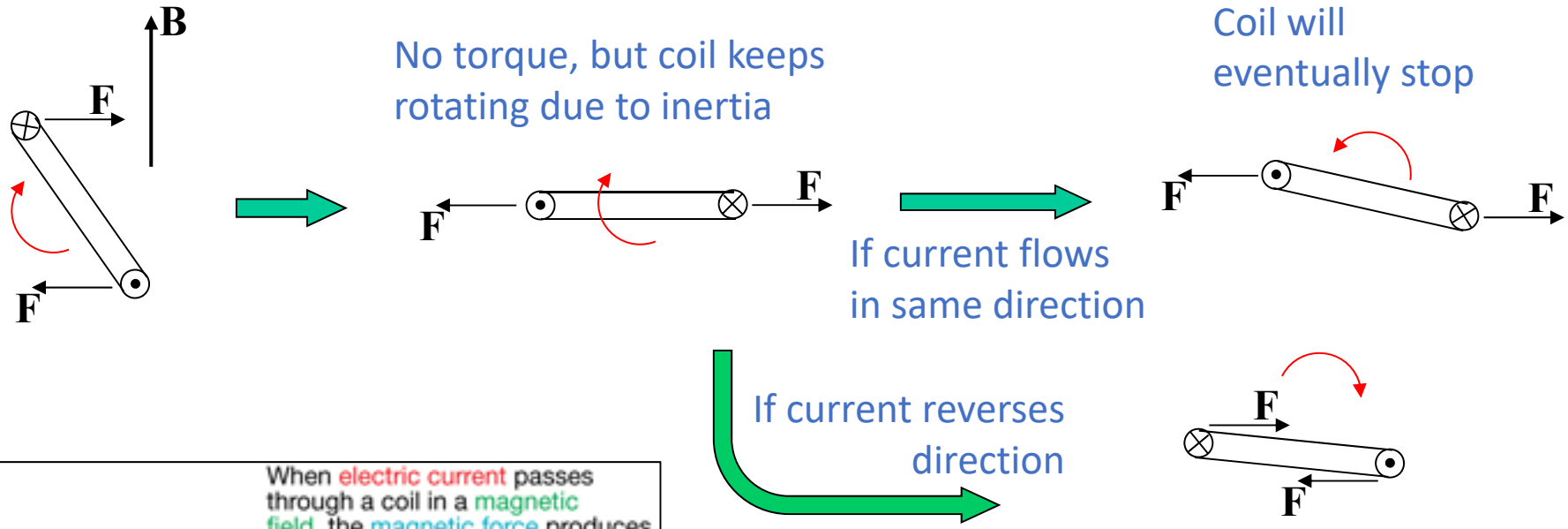
But in the motor the magnetic field provides the torque that causes the coil to rotate, and the coil can drive a load. Work is done.

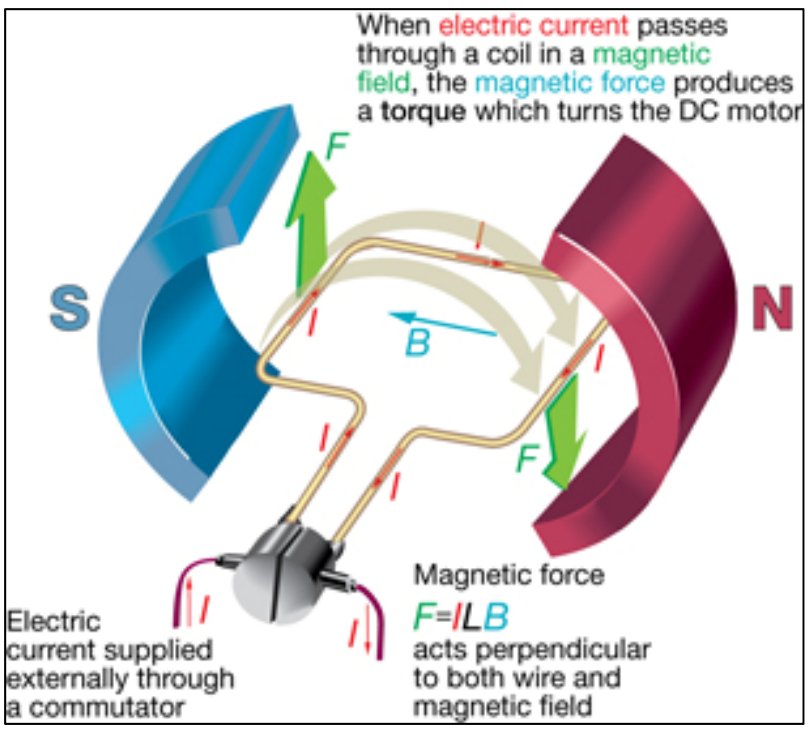
What does the work?

Hint: The magnetic field can be provided by a permanent magnet, which does not fade because the motor is running. In other words, no energy is taken from or given to the permanent magnet.

More on motors (for your possible interest)

DC motor

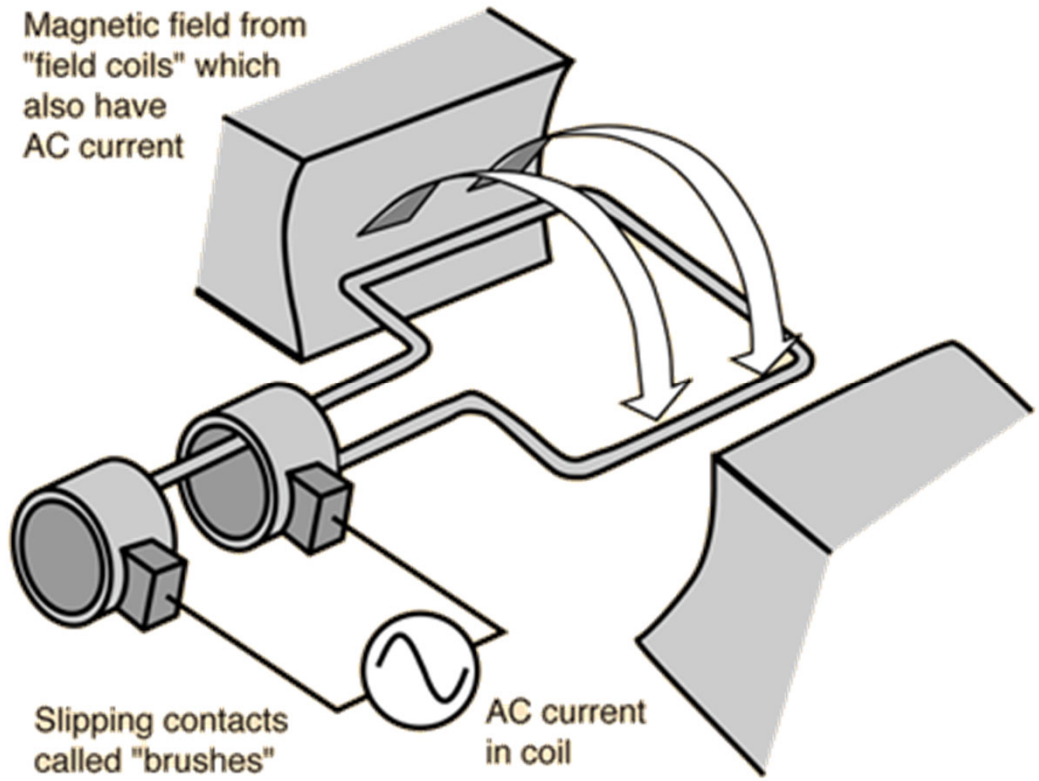




http://resource.rockyview.ab.ca/rvlc/physics30_BU/Unit_B/m4/p30_m4_l03_p4.html

See also:
https://www.youtube.com/watch?v=Y-v27GPK8M4

AC Motor



Rotates at the frequency of the sine wave: “synchronous motor”.
Random rotation direction.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html

The unit of magnetic field **B**

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

$$[\text{unit of } B] = \frac{\overset{\text{newton}}{N}}{\underset{\text{coulomb}}{C}\ \frac{m}{s}} = \frac{N}{\underset{\text{ampere}}{A}\,m} \equiv \underset{\text{tesla}}{T}$$

For charge q in electric field **E**, **F** = q**E**. Therefore vB and E are the same dimension.

$$[\text{unit of } B] = \frac{V/m}{m/s} = \frac{Vs}{m^2} \equiv T$$

We will revisit this topic later, giving you other forms of the unit tesla (T).

We have discussed the force exerted by a magnetic field on **moving** charges.

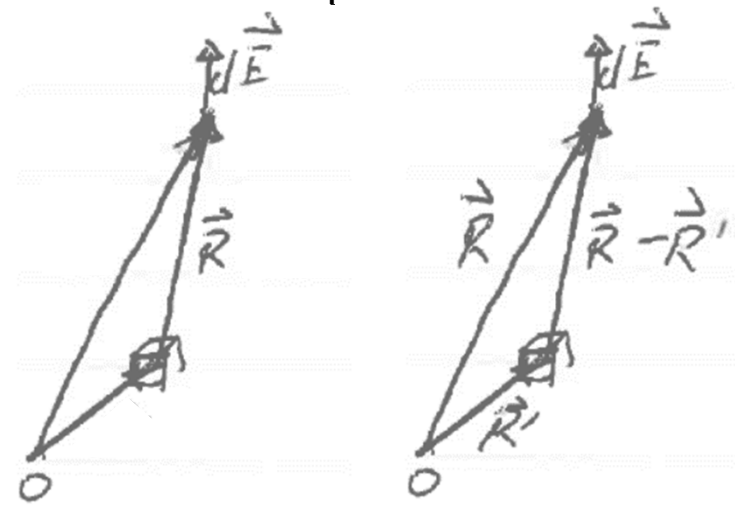
Next, let's see how the magnetic field is generated.

An electric current (i.e. moving charges) generates a magnetic field.

Again, we talk about this in comparison with the electrostatic counterpart.

$$d\vec{E} = \frac{1}{4\pi\varepsilon}\frac{\rho dV}{R^2}\hat{R}$$

Here, **R** is the vector from a volume element dV to the point where we want to find d**E**. Notice difference in notation than used in previous lecture on Coulomb's law.



Here Previously used

ρ includes only the "external" charge, not the polarization charge, which is taken care of by the material (medium) parameter ε.

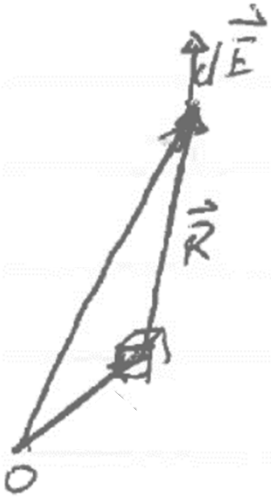
Regardless of the medium, we can write this equation in terms of **D**. $\mathbf{D} \equiv \varepsilon_0\varepsilon_r\mathbf{E} \equiv \varepsilon\mathbf{E}$

$$d\vec{D} = \frac{1}{4\pi}\frac{\rho dV}{R^2}\hat{R}$$

D is what the charge distribution ρ generates.

E is what a probe charge q feels: $\mathbf{F} = q\mathbf{E}$.

Electric field



$$d\vec{D} = \frac{1}{4\pi}\frac{\rho dV}{R^2}\hat{R}$$

$$\mathbf{D} \equiv \varepsilon_0\varepsilon_r\mathbf{E} \equiv \varepsilon\mathbf{E}$$

ρ includes only the "external" charge, not the polarization charge, which is taken care of by the material (medium) parameter ε.

D is what the charge distribution ρ generates.

E is what a probe charge q feels: $\mathbf{F} = q\mathbf{E}$.

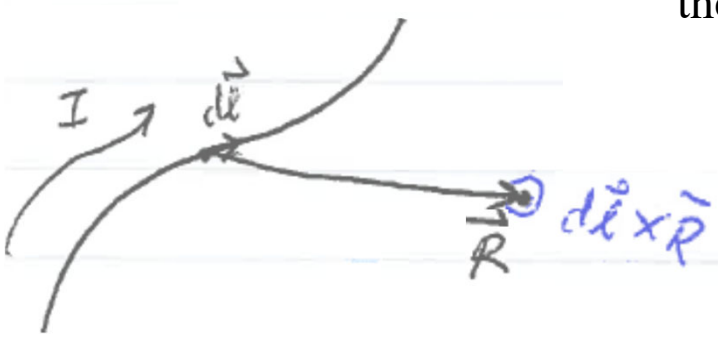
Both fields are proportional to $1/R^2$.

$d\vec{E} \parallel \hat{R}$

$d\vec{H} \perp \vec{R}$

Magnetic field

$$d\vec{H} = \frac{1}{4\pi}\frac{I\,d\vec{l}\times\hat{R}}{R^2}$$



I includes only the "external" current, not the magnetization current, which is taken care of by the material (medium) parameter μ.

$$\mathbf{B} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H}$$

H is what the current I generates.

B is what a probe charge q feels: $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$.

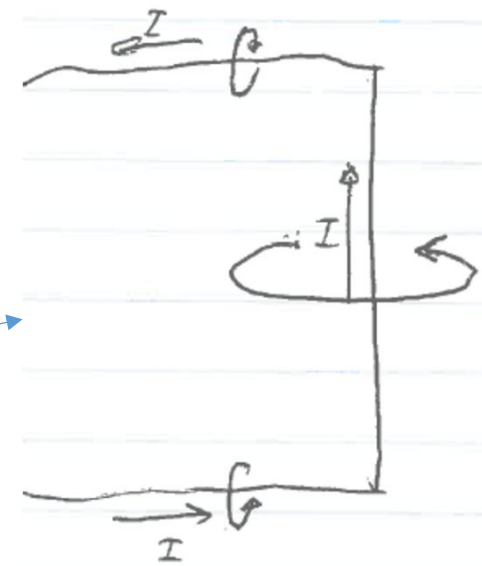
$d\mathbf{l}$ is a small segment of a wire carrying a current I

Example 1: Magnetic field of a current-carrying straight wire of finite length

Read on your own. Example 5-2 of textbook: pp. 245 in 7/E, pp. 244 in 6/E, pp. 214 in 5/E.

Just sum up piece by piece (taking the integral). This example is for you to appreciate the shape of the magnetic field in relation to the current.

But, you need to have a closed circuit. The real situation is more like this. The solution is a good approximation only far away from other parts of the circuit.



Take this good approximation to the limit of an infinitely long straight wire, we have

$$\vec{H} = \frac{I}{2\pi r}\hat{\phi}$$

Unit vector in the ϕ direction in the cylindrical coordinate

We will discuss an easier way to get to this equation later.

From this equation, you see the unit of **H** is A/m.

Using the result for the wire of a finite length, do Homework 10 Problem 1.
Using the result for the infinitely long wire, do Homework 10 Problem 2.
(Problem not well stated. Change "power cable" to "DC current-carrying wire", as we usually understand a "power cable" as two wires carrying opposite AC currents.)

Example 2: Magnetic field of a current-carrying circular wire loop

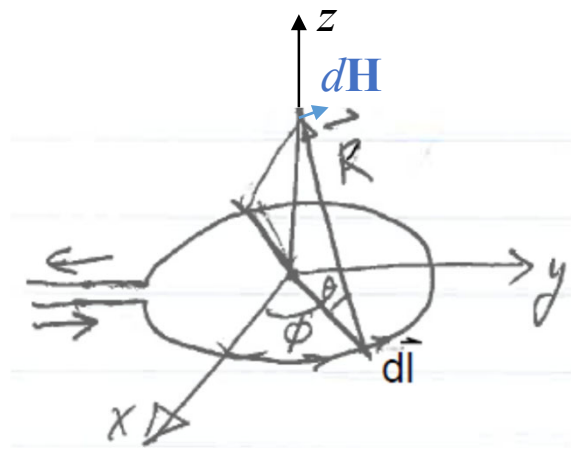
Find the magnetic filed an any location z on the axis of a current-carrying circular ring.

The gap is made small, therefore can be ignored.

So, treat is the same as Example 5-3 of textbook. But this is more realistic.

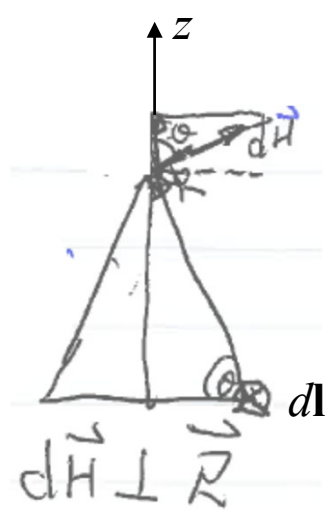
The feed wires carry opposite currents, therefore contribute no magnetic filed.

Just sum up piece by piece (taking the integral). This example is, again, for you to appreciate the shape of the magnetic field in relation to the current.



$$d\vec{H} = \frac{1}{4\pi} \frac{I\, d\vec{\ell} \times \hat{R}}{R^2}$$

$$d\vec{\ell} \perp \vec{R} \Rightarrow \left| d\vec{\ell} \times \hat{R} \right| = d\ell$$

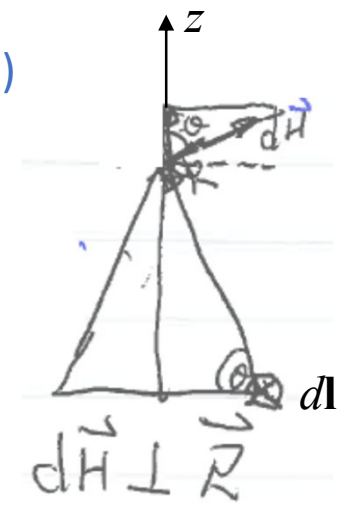


Horizontal components due to diametrical elements cancel. Therefore consider only vertical component of d**H**:

$$dH_z = \frac{I\, d\ell}{4\pi R^2} \cos\theta$$

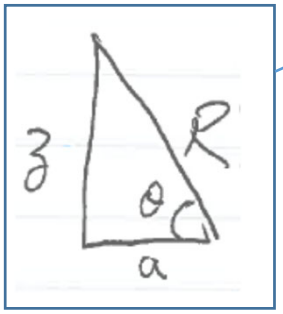
$$dH_z = \frac{I\,dl}{4\pi R^2}\cos\theta$$

Perimeter (circumference)



$$H = H_z = \oint dH_z = \oint \frac{I\,dl}{4\pi R^2}\cos\theta = \frac{I\cos\theta}{4\pi R^2}\oint dl$$

$$= \frac{I\cos\theta}{4\pi R^2}\,2\pi a = \frac{I}{2R^2}\,a\cos\theta$$



$$= \frac{I\,a}{2(a^2+z^2)}\cdot\frac{a}{\sqrt{a^2+z^2}}$$

$$= \frac{I a^2}{2(a^2+z^2)^{\frac{3}{2}}}$$

$$\boxed{H(z=0) = \frac{I a^2}{2a^3} = \frac{I}{2a}}$$

Near field

$$\boxed{H(z \gg a) = \frac{I a^2}{2|z|^3}}$$

Far field

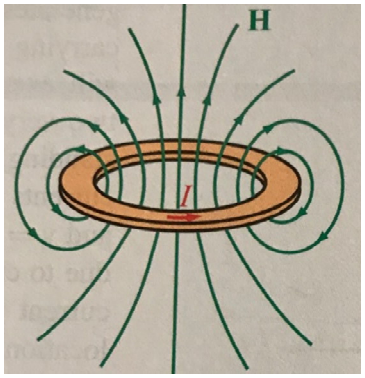
$$H(z=0) = \frac{Ia^2}{2a^3} = \frac{I}{2a}$$

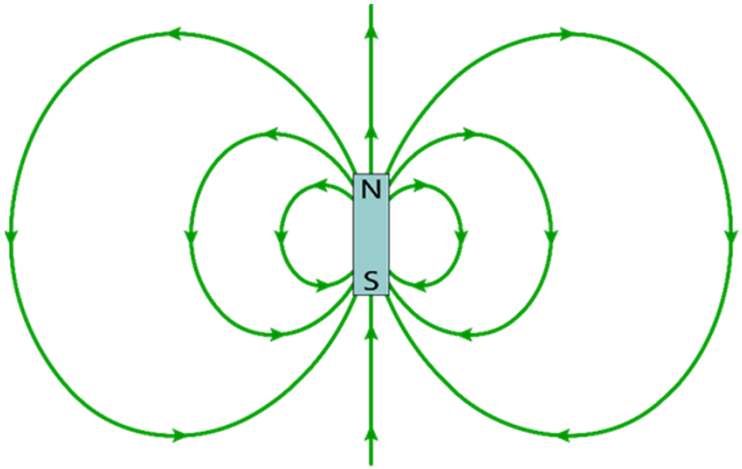
Near field

$$H(z \gg a) = \frac{Ia^2}{2|z|^3}$$

Far field

The big pictures: not limited to the *z* axis





http://www.thesciencemill.com/Research/edm/edm_paper.php

Again, far from the loop, the shape of the loop does not matter. The far field is the same as a small magnet.

For the far field, recall that we defined $m = IA = I\pi a^2$

$$\therefore H(z \gg a) = \frac{m}{2\pi|z|^3} \Rightarrow \vec{H}(z \gg a) = \frac{\vec{m}}{2\pi|z|^3}$$

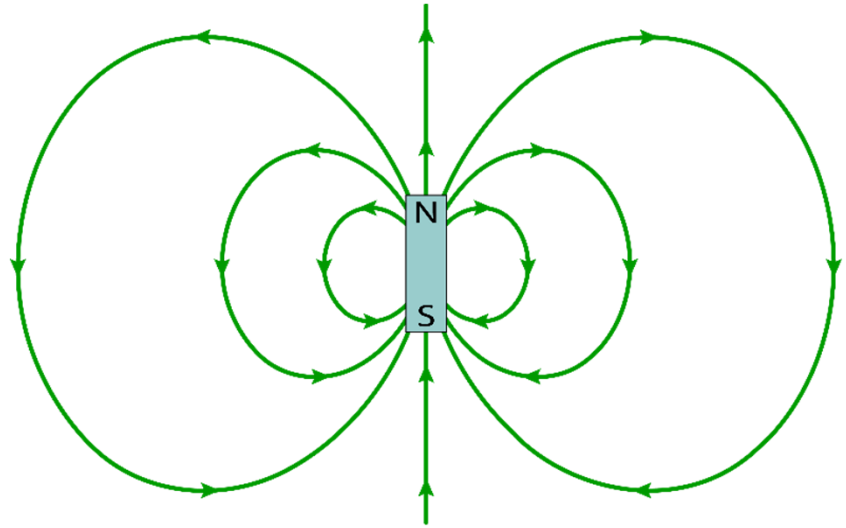
H // **m**

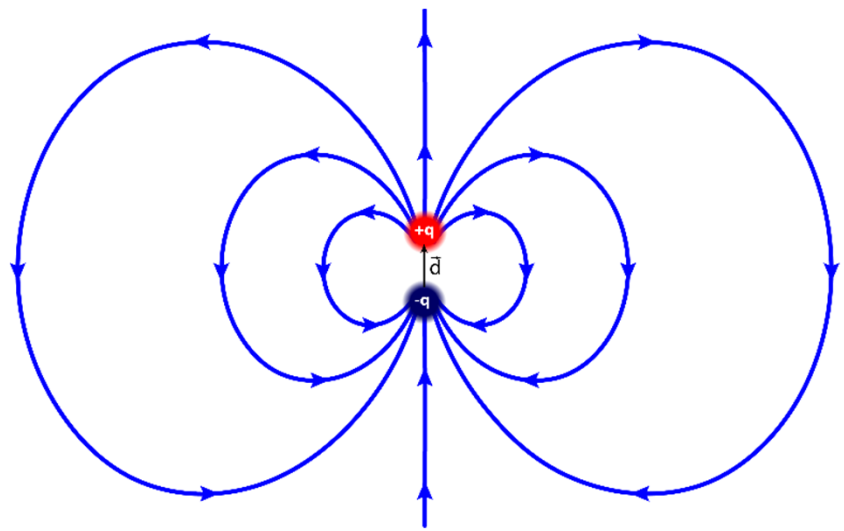
Direction of **m** follows right hand rule.

Any current carrying loop has a magnetic moment **m**, and the loop is called a magnetic dipole. It's equivalent to a small magnet. (We will talk explain why when we discuss magnetic materials.)

There is no need for a physical wire. For example, an electron orbiting a nucleus forms a magnetic dipole. (More on this when we discuss magnetic properties of materials.)

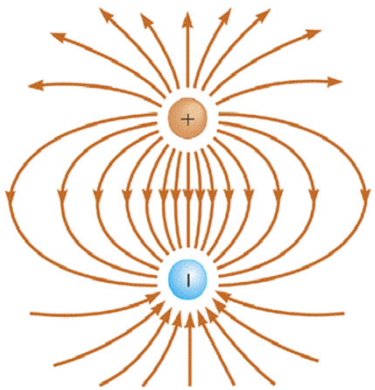
We call such a loop a magnetic dipole because its magnetic field has the same shape as the electric field of the electric dipole, when we only consider the far field.



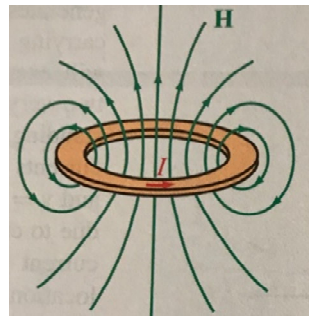


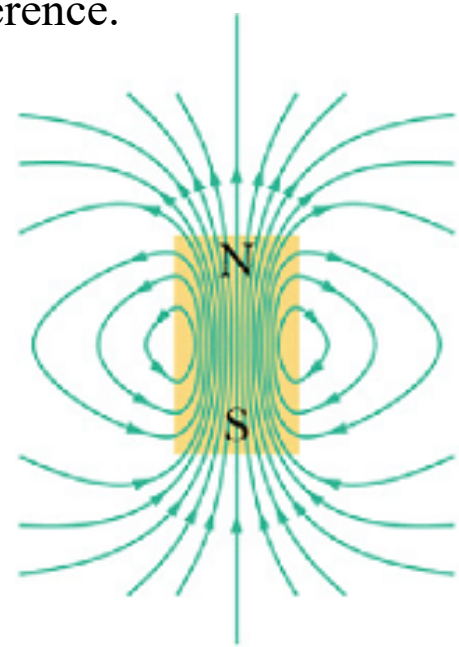
http://www.thesciencemill.com/Research/edm/edm_paper.php

If you zoom in and look at the near field, you will see a big difference.



http://physics.stackexchange.com/questions/318150/how-do-electric-field-lines-form-only-open-loops





http://www.rakeshkapoor.us/ClassNotes/classnotes.php?notes=MagneticField&index=2

Electric dipole: **E** lines come out of + charge and end at – charge. In math language,

$$\nabla \cdot \vec{D} = \rho$$

$$\oint \vec{D} \cdot d\vec{S} = \int \rho \, dV$$

Magnetic dipole: **B** lines form loops. In math language,

$$\nabla \cdot \vec{B} = 0$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

There is no such thing as a “magnetic charge”.

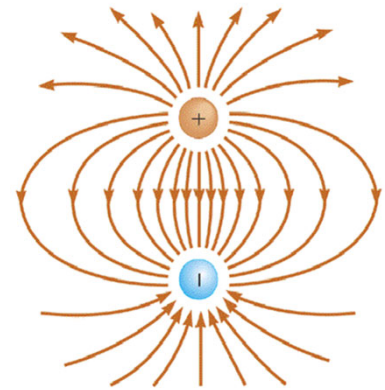
Since the electrostatic field **E** must go from the + charge into the – charge, its field lines cannot form loops. Mathematically,

$$\oint \vec{E} \cdot d\vec{\ell} = 0 \qquad \nabla \times \vec{E} = 0$$

Will explain this notation

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = \rho$$

$$\oint \vec{D} \cdot d\vec{S} = \varepsilon \oint \vec{E} \cdot d\vec{S} = \int \rho \, dV$$



Such a field is said to be conservative.

In contrast, the magnetic field must form loops.

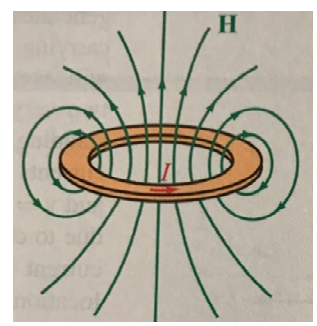
$$\nabla \cdot \vec{B} = 0 \qquad \oint \vec{B} \cdot d\vec{S} = 0$$

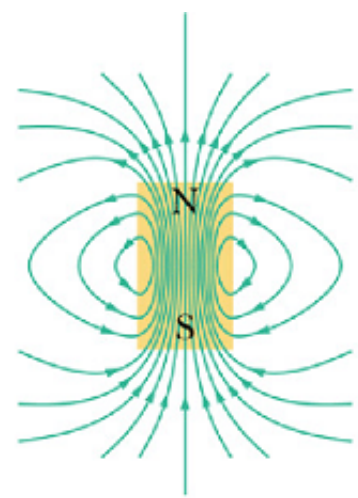
Such a field is said to be solenoidal.

The loop integral of the magnetic field is therefore finite. What is it?

$$\boxed{\oint \vec{H} \cdot d\vec{\ell} = \int \vec{J} \cdot d\vec{S} = I}$$

Ampère's law





$$\boxed{\oint \vec{H}\cdot d\vec{\ell} = \int \vec{J}\cdot d\vec{S} = I}$$ Ampère's law

Ampère's law is obeyed by all loops, regardless of size.

We shrink a loop to infinitesimally small, and define the "curl"

Unit normal vector

$$\text{curl}\,\vec{H} = \lim_{\Delta S \to 0} \frac{\vec{n}\oint \vec{H}\cdot d\vec{\ell}}{\Delta S}$$

Here, we state the way to calculate the curl in the Cartesian coordinate without proof:

$$\vec{H} = H_x\hat{x} + H_y\hat{y} + H_z\hat{z} \qquad \nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\nabla\times\vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \hat{x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \hat{y}\left(-\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z}\right) + \hat{z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

determinant

Stoke's theorem (the math)

Consider the loop integral of **H** around the contour *C*.
Imagine a dense grid.
For each small loop, by definition:

Contour *C*

$$\nabla \times \vec{H} = \text{curl}\,\vec{H} = \lim_{\Delta S \to 0} \frac{\vec{n} \oint \vec{H} \cdot d\vec{l}}{\Delta S}$$

Therefore,

$$\oint_C \vec{H} \cdot d\vec{l} = \sum \Big[\oint_{\substack{\text{small} \\ \text{loop}}} \vec{H} \cdot d\vec{l} \Big] = \sum (\nabla \times \vec{H}) \cdot \Delta \vec{S} = \int (\nabla \times \vec{H}) \cdot d\vec{S}$$

This is just math.

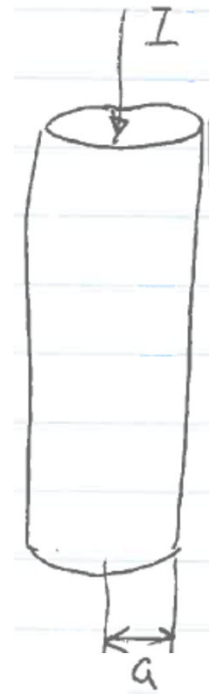
Now the physics kicks in: $\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} = I$ Integral form (big picture) of Ampère's law

Therefore, $\nabla \times \vec{H} = \vec{J}$ Differential form (small picture) of Ampère's law

Similarly, $\oint \vec{E} \cdot d\vec{l} = 0 \iff \nabla \times \vec{E} = 0$

Example 3: Magnetic field of a current-carrying infinitely-long straight wire

Wire carries current I. Current density uniformly distributed.



$$J = \frac{I}{\pi a^2}$$

For $r < a$,

$$\oint \vec{H} \cdot d\vec{\ell} = H(2\pi r) = \frac{I}{\pi a^2} \pi r^2 = I\left(\frac{r}{a}\right)^2$$

$$\therefore H = \frac{1}{2\pi r} I \left(\frac{r}{a}\right)^2 = \frac{Ir}{2\pi a^2}$$

For $r > a$,

$$\oint \vec{H} \cdot d\vec{\ell} = 2\pi r H = I \qquad \therefore H = \frac{I}{2\pi r}$$

Same result as got from Biot-Savart law in Example 1.

J can be non-uniform. For non-uniform *J*, do Homework 10 Problem 4.
Also finish Homework 10 Problem 5 (Problem 3.56 in 7/E of textbook).
For other geometries, read the following in text book:

- Example 5-5 (5-6 in 5/E of book) – Toroidal coil
- Example 5-6 (5-7 in 5/E) – Infinitely large current sheet
- Section 5-7.1 (5-8.1 in 5/E) – Solenoid

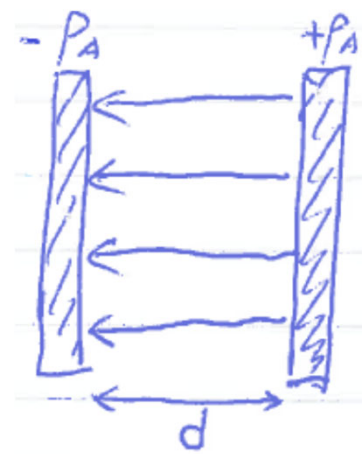
Go through the mathematical details, but more important, get a visual sense of **B** fields of currents in these geometries. Look at the figures and think.

Now, we consider the interaction between two wires, in comparison with two charged bodies in electrostatics.

What is the force between the two plates of an infinitely large parallel-plate capacitor?

$$E = \frac{\rho_A}{\varepsilon}$$

$$\therefore \frac{F}{A} = \frac{\rho_A A}{A} \frac{\rho_A}{\varepsilon} = \frac{\rho_A^2}{\varepsilon}$$

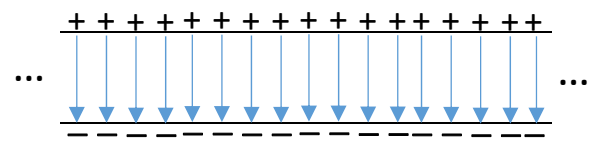


Is this right?

An old slide from a previous lecture

What we did wrong was double counting, a sort of "creative accounting".

$$E = \frac{\rho_s}{\varepsilon} = \frac{Q}{\varepsilon A}$$ is the total field due to both plates.



The force is exerted on one plate by the field due to the other. Therefore,

$$E = \frac{\rho_s}{2\varepsilon} = \frac{Q}{2\varepsilon A}$$

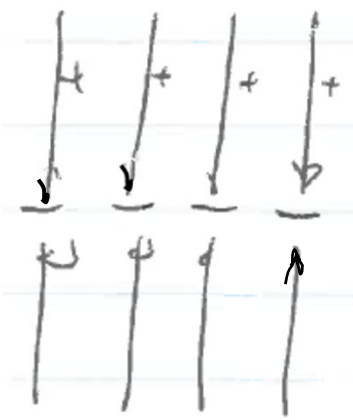
$$W = Fd = QEd = Q \cdot \frac{Q}{2\varepsilon A} d = \frac{1}{2} Q^2 \frac{d}{\varepsilon A}$$

$$C = \frac{\varepsilon A}{d}$$

$$\Rightarrow W = \frac{1}{2} \frac{Q^2}{C}$$

$$Q = CV$$

$$\Rightarrow W = \frac{1}{2} C V^2$$



The positive charge sheet in the field of the negative charge sheet

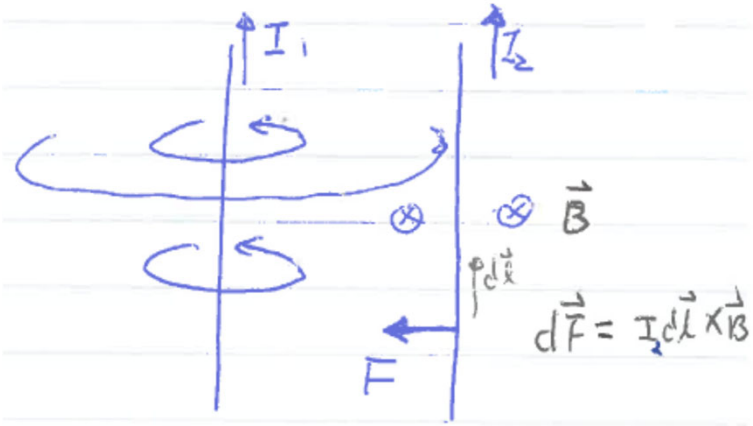
This another way to appreciate the factor ½.

Example 4: Interaction between two current-carrying infinitely-long straight wires

Similarly, we consider one wire placed in the field of the other.

The distance between the wires is *d*.

The field of wire 1 at the location of wire 2 is:



$$H = \frac{I_1}{2\pi d} \qquad B = \frac{\mu_0 I_1}{2\pi d}$$

Therefore the force on wire 2 is

$$\frac{F}{l} = \frac{1}{l} \cdot I_2 l \cdot \frac{\mu_0 I_1}{2\pi d} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Pay attention to the direction:
Two wires carrying currents in the same direction attract each other.

Do Homework 10 Problem 3.

Review textbook: Overview, Sections 5-1 through 5-3
Finish Homework 10.