

We continue to compare the electrostatic and magnetostatic fields.

The **electrostatic** field is **conservative**:  $\oint \vec{E} \cdot d\vec{l} = 0$   $\nabla \times \vec{E} = 0$

This allows us to define the potential  $V$ :

$$\nabla V = -\vec{E}$$

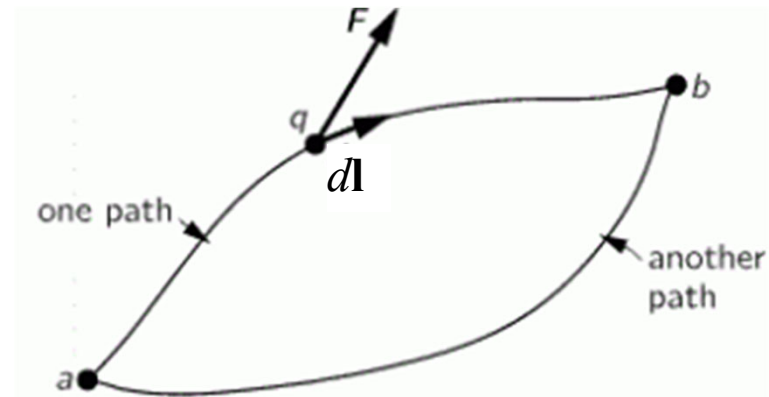
because

$$V_a - V_b = -\int \vec{E} \cdot d\vec{l}$$

is independent of the path.

If a vector field has no curl (i.e., is conservative), it must be something's gradient.

Gravity is conservative. Therefore you do see water flowing in such a loop without a pump in the physical world.



For the magnetic field,  $\nabla \cdot \vec{B} = 0$   $\oint \vec{B} \cdot d\vec{S} = 0$

If a vector field has no divergence (i.e., is solenoidal), it must be something's curl.

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

In other words, the curl of a vector field has zero divergence.

Let's use another physical context to help you understand this math:

$$\vec{J} = \nabla \times \vec{H}$$

Ampère's law

$$\nabla \cdot \vec{J} = 0 \iff \oint_S \vec{J} \cdot d\vec{s} = 0$$

Kirchhoff's current law (KCL)

Since  $\nabla \cdot \vec{B} = 0$ , we can define a vector field  $\vec{A}$  such that  $\vec{B} = \nabla \times \vec{A}$

Notice that for a given  $\vec{B}$ ,  $\vec{A}$  is not unique. For example, if  $\vec{B} = \nabla \times \vec{A}$

then  $\vec{B} = \nabla \times (\vec{A} + \vec{A}_0)$ , because  $\nabla \times \vec{A}_0 = 0$

Similarly, for the electrostatic field, the scalar potential  $V$  is not unique:

If  $\nabla V = -\vec{E}$  then  $\nabla (V + V_0) = -\vec{E}$

You have the freedom to choose the reference  $V_0$

$$\begin{aligned} \nabla \times \vec{H} &= \vec{J} && \text{(Ampère's law)} \\ \vec{B} &= \mu \vec{H} \end{aligned} \quad \left. \vphantom{\begin{aligned} \nabla \times \vec{H} &= \vec{J} \\ \vec{B} &= \mu \vec{H} \end{aligned}} \right\} \Rightarrow \nabla \times \vec{B} = \mu \vec{J} \Rightarrow$$

$$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}$$

Going through the math, you will get  $\nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu \vec{J}$

Here is what  $\nabla^2 \vec{A}$  means:  $\nabla^2 \vec{A} = \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z$

Just notation. Notice that  $\nabla^2 \vec{A}$  is a vector.

Still remember what  $\nabla^2$  means for a scalar field?

From a previous lecture:  $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

Recall that the choice for  $\mathbf{A}$  is not unique. It turns out that we can always choose  $\mathbf{A}$  such that

$$\nabla \cdot \vec{A} = 0$$

$$\begin{aligned} \nabla \times \vec{H} &= \vec{J} & (\text{Ampère's law}) \\ \vec{B} &= \mu \vec{H} \end{aligned} \quad \left. \vphantom{\begin{aligned} \nabla \times \vec{H} &= \vec{J} \\ \vec{B} &= \mu \vec{H} \end{aligned}} \right\} \Rightarrow \nabla \times \vec{B} = \mu \vec{J} \Rightarrow$$

$$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J} \Rightarrow \nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu \vec{J} \quad \left. \vphantom{\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}} \right\} \Rightarrow$$

The choice for  $\mathbf{A}$  is not unique. We choose  $\mathbf{A}$  such that  $\nabla \cdot \vec{A} = 0$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

Here is what  $\nabla^2 \vec{A}$  means:  $\nabla^2 \vec{A} = \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z$

Notice that  $\nabla^2 \vec{A}$  is a vector. Thus this is actually three equations:

$$\left\{ \begin{aligned} \nabla^2 A_x &= -\mu J_x \\ \nabla^2 A_y &= -\mu J_y \\ \nabla^2 A_z &= -\mu J_z \end{aligned} \right.$$

Recall the definition of  $\nabla^2$  for a scalar field from a previous lecture:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Poisson's equation for the magnetic field

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

is actually three equations:

$$\left\{ \begin{array}{l} \nabla^2 A_x = -\mu J_x \\ \nabla^2 A_y = -\mu J_y \\ \nabla^2 A_z = -\mu J_z \end{array} \right.$$

Compare Poisson's equation for the magnetic field with that for the electrostatic field:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon}$$

Given  $\vec{J}$ , you can solve  $\vec{A}$ , from which you get  $\vec{B}$  by

$$\vec{B} = \nabla \times \vec{A}$$

Given  $\rho$ , you can solve  $V$ , from which you get  $\vec{E}$  by

$$\vec{E} = -\nabla V$$

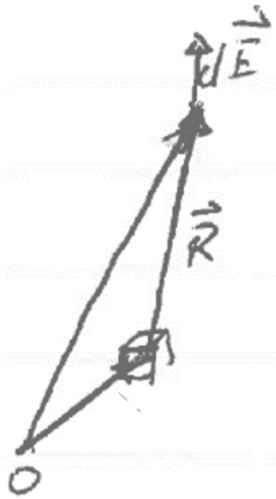
Exams (Test 2 & Final) problems will not involve the vector potential.

This is an important topic, however, if you have further interest in microwave engineering, antennas, etc. Moreover, the concepts we just discussed help you to better understand the magnetic field. For your interest, therefore, review these notes & Section 5-4 of textbook.

# Summary of methods to find magnetostatic and electrostatic fields

## Electrostatics

### Coulomb's law



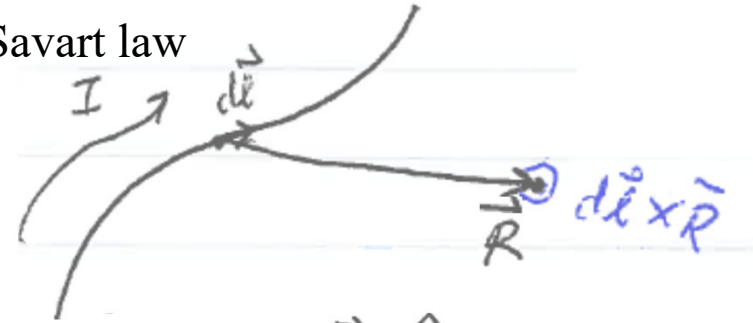
$$d\vec{E} = \frac{1}{4\pi\epsilon} \frac{\rho dV}{R^2} \hat{R}$$

$$d\vec{D} = \frac{1}{4\pi} \frac{\rho dV}{R^2} \hat{R}$$

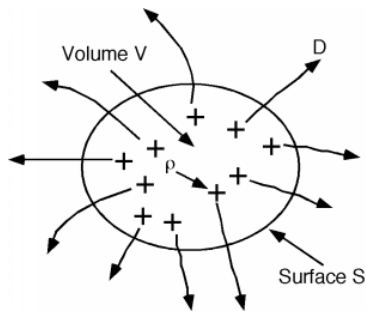
Take integral to get total field.

## Magnetostatics

### Biot-Savart law



$$d\vec{H} = \frac{1}{4\pi} \frac{I d\vec{l} \times \hat{R}}{R^2}$$



### Gauss's law

$$\oint \vec{D} \cdot d\vec{S} = \int \rho dV = Q$$

### Ampère's law



$$\oint \vec{H} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{S} = I$$

Often used to take advantage of symmetry. Simple math.

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

$$\vec{E} = -\nabla V$$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

$$\vec{B} = \nabla \times \vec{A}$$

Poisson's equations. Solve partial differential equations to get potentials, and then fields.

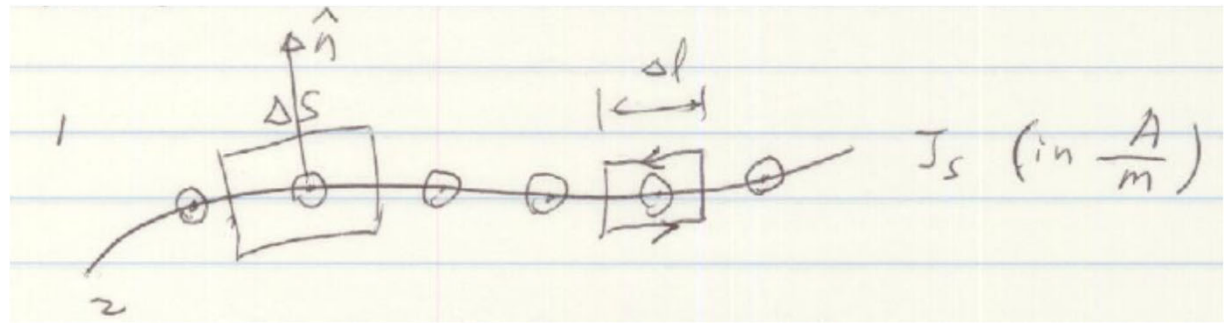
## The last topic in magnetostatics: Magnetic boundary conditions

Consider a cylinder with zero thickness, with infinitesimal top/bottom surface area  $\Delta S$

$$\nabla \cdot \vec{B} = 0 \Rightarrow$$

$$\vec{B}_1 \cdot \hat{n} \Delta S = \vec{B}_2 \cdot \hat{n} \Delta S$$

$$\Rightarrow B_{1n} = B_{2n}$$



Consider a rectangle with zero width, with infinitesimal length  $\Delta l$

$$\oint \vec{H} \cdot d\vec{l} = I$$



$$-H_{1t} \Delta l + H_{2t} \Delta l = J_s \Delta l$$



$$H_{2t} - H_{1t} = J_s$$

Review textbook Section 5-6. Do Homework 11 Problem 1.