Dynamic Fields, Maxwell’s Equations (Chapter 6)

So far, we have studied static electric and magnetic fields. In the real world, however, nothing is static. Static fields are only approximations when the fields change very slowly, and “slow” is in a relative sense here.

To really understand electromagnetic fields, we need to study the dynamic fields. You will see the E & M fields are coupled to each other.

Four visual pictures to help you understand the four Maxwell’s equations

Two remain the same for dynamic and static fields. Two are different.

(1)

\[ \oint E \cdot ds = \int \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0} \]

\[ \oint D \cdot ds = \int \rho dV = Q \]

\[ \varepsilon \nabla \cdot E = \nabla \cdot D = \rho \]

This holds for dynamic fields even when \( \rho \) changes with time.

Question: how can \( \rho \) change with time?
(2) \[ \oint \mathbf{B} \cdot d\mathbf{s} = 0 \quad \text{\( \nabla \cdot \mathbf{B} = 0 \)} \]

What goes in must come out: no monopoles. Always true, static or dynamic.

(3) The electrostatic field is conservative

\[ \oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{\( \nabla \times \mathbf{E} = 0 \)} \]

This is why we can define “potential.”

Faraday’s law:

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]

Pay attention to this negative sign.

This electric field induced by a changing magnetic field is not conservative! It’s not an “electrostatic field” even when \( \frac{\partial \mathbf{B}}{\partial t} \) is a constant.

Cannot define a potential!
(4) Ampere’s law (static)
\[ \oint H \cdot dl = \oint J \cdot dS = I \]
\[ \nabla \times H = J \]

Ampere’s law (dynamic)
\[ \oint H \cdot dl = \oint \left( J + \frac{\partial D}{\partial t} \right) \cdot dS = I + \oint \frac{\partial D}{\partial t} \cdot dS \]
\[ \nabla \times H = J + \frac{\partial D}{\partial t} \]

We have covered the static case in pretty much detail. Here, in the dynamic case the current could include the displacement current.

(3) and (4) are about the coupling between E & M fields. They are the foundations of electromagnetic waves, to be discussed in Ch. 7.
In Ch. 6, we focus on Eq. (3), Faraday’s law:

$$\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad \leftrightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Pay attention to this negative sign.

Plan view:

Now we placed a wire loop in this field. This electric field will drive a current.

Viewed from another perspective:

This “voltage” is due to the non-electrostatic field. It is a “electromotive force.” Just like that of a battery, which is due to chemistry.

$$V_{\text{emf}} = \text{emf} = \oint \mathbf{E} \cdot d\mathbf{l} \approx \int \mathbf{E} \cdot d\mathbf{l}$$

The voltmeter will measure a “voltage”
This “voltage” is due to the non-electrostatic field. It is a “electromotive force.” Just like that of a battery, which is due to chemistry.

\[
V_{\text{emf}} = \text{emf} = \int_{a}^{b} \mathbf{E} \cdot d\mathbf{l} \approx \int \mathbf{E} \cdot d\mathbf{l}
\]

Notice that for an electrostatic field

\[
V = -\int_{a}^{b} \mathbf{E} \cdot d\mathbf{l}
\]

Let’s use an analogy to explain the “subtle” difference between an emf and a voltage:

The pump works against gravity.

The battery works against the electrostatic force.
Define magnetic flux

\[ \Phi = \int \mathbf{B} \cdot d\mathbf{S} \]

\( \mathbf{B} \) is therefore called the "magnetic flux density."

We may also have a coil of \( N \) turns instead of a loop of just one turn:

\[ V = -N \int \frac{\partial \mathbf{B}}{\partial t} \cdot dS \]

\[ = -N \frac{d}{dt} \int \mathbf{B} \cdot dS \]

\[ = -N \frac{d\Phi}{dt} \]

\[ = -\frac{d\Lambda}{dt} \]

Magnetic flux linkage

What if you replace the voltmeter with a load resistor?
Now let’s see what happens if we feed a current to the coil, when there is no external magnetic field.

\[ \frac{\partial B}{\partial t} > 0 \]

\[ \frac{\partial i}{\partial t} > 0 \]

This is true, regardless of the shape or \# of turns.

But for simplicity, we use the expression for the \( B \) field of a very long solenoid.

\[ B = \mu \left( \frac{N}{l} \right) I \]

but keep in mind \( B \times I \) in a general case.
If I changes w/ time, so does B.
\[
\frac{dB}{dt} = \mu \left( \frac{N}{l} \right) \frac{di}{dt}, \quad \text{and} \quad \frac{dB}{dt} \propto \frac{di}{dt} \quad \text{in general}
\]
Recall that \( \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint \vec{B} \cdot d\vec{s} \)
\[
V = \overrightarrow{\text{emf}} = -\oint \vec{E} \cdot d\vec{l} = N \frac{d}{dt} \oint \vec{B} \cdot d\vec{s} = N \frac{d}{dt} \Phi = \frac{d}{dt} \Phi
\]
Define \( L = \frac{1}{i} \Rightarrow \lambda = Li \)

then \( V = L \frac{di}{dt} \)

For the coil, or solenoid,
\( \lambda = NBS = \mu \frac{N^2 i S}{l} \)
\( L = \frac{\mu N^2}{l} S \)

Now I'm going to give you another example of how to find the inductance of a particular geometry of conductors.

\[
B = \mu \frac{I}{2\pi r} \quad \lambda = \Phi = \int_a^b B dr
\]
\[
= \mu I \frac{L}{2\pi} \int_a^b \frac{1}{r} dr
\]
\[
= \mu I \frac{L}{2\pi} \ln \frac{b}{a}
\]
\( \therefore L = \frac{\mu I}{2\pi} \ln \frac{b}{a} \quad \lambda = \frac{L}{\frac{d}{dt}} = \)

For other geometries, read Section 5-7.2 (Fig. 5-27)
Energy stored in an inductor

\[ v = L \frac{di}{dt} \]

\[ W_m = \int i v \, dt = \int i L \frac{di}{dt} \, dt = L \int_0^1 i \, di \]

\[ = \frac{1}{2} LI^2 \]

Following the math in §5.8 (5.9 in 5/E), you get

\[ w_m = \frac{W_m}{V} = \frac{1}{2} \mu H^2 \]

for infinitely long solenoid.

Analogy w/ E field

\[ \text{Cap} \]

\[ i = C \frac{dv}{dt} \]

\[ W_E = \frac{1}{2} CV^2 \]

\[ w_E = \frac{1}{2} \varepsilon E^2 \]

derived from infinitely large cap, parallel plate

but true for general

Lastly, the unit

\[ H = \frac{W_b}{A} = \frac{L}{A} \]

\[ = \frac{\mu n^2}{A} \]

(heiney)
\[ \oint E \cdot dl = -\oint \frac{\partial B}{\partial t} \cdot d\vec{s} \]

Now we emphasis again the direction of the voltage induced by a changing mag. field.

Compare this to Fig. 6-2 in the book.
(Same as in 5/E)

The **emf** current thru an R always flows from **hi V** to **low V**.

**emf** & **V** are two different concepts.

This is why I don't like the discussion in the book.

The loop acts like a battery.

In the loop, current flows from **low V** to **hi V**. But on the load, the current flows from **hi V** to **low V**.

That's why we say a battery, or this loop is like a pump.
\[ \vec{B} = (\vec{y} B_{y0} + \vec{z} B_{z0}) \sin \omega t \]
\[ \Phi = \oint \vec{B} \cdot d\vec{s} \]
\[ = B_{z0} \sin \omega t \left( \pi a^2 \right) \]
\[ \Lambda = N \Phi \]
\[ = N B_{z0} \pi a^2 \sin \omega t \]

\[ V = \frac{d\Lambda}{dt} = \omega N B_{z0} \pi a^2 \cos \omega t \]
\[ I = \frac{V}{R} = \frac{\pi N B_{z0} a^2 \omega}{R} \cos \omega t \]

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**Do Homework 11 Problems 2 through 4, and 7.**

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Now, let's see what happens if we put two coils very close to each other. In other words, the \( B \) through coil 2 is due to the current through coil 1.

\[ \Phi_{12} = \oint_{S_2} \vec{B}_1 \cdot d\vec{s} \propto I_1 \]
\[ \Lambda_{12} = N_2 \Phi_{12} \equiv L_{12} I_1 \]
\[ V_2 = \frac{d\Lambda_{12}}{dt} = L_{12} \frac{dI_2}{dt} \]

\[ V_1 = \frac{d\Lambda_{12}}{dt} = L_{21} \frac{dI_2}{dt} \]

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If you do some due diligence mathematically, which we won't do here. Of course, you can show \( L_{12} = L_{21} \). This is called the mutual inductance.

More general discussions...
Now, let's say we have some material, with a very high $\mu$, as the core, and the two coils are around this core.

(Ideal transformer: $\mu = \infty$)

$V_i = -N_1 \frac{d\Phi}{dt}$

 Coil 1 must establish a voltage that equals $V_i$.

$V_2 = uN_2 \frac{d\Phi}{dt}$

$\therefore \frac{V_i}{V_2} = \frac{N_1}{N_2}$

How to determine the direction of the $\mathbf{B}$ field.

For the primary coil, follow the right hand rule. For the secondary, opposite to the right hand rule.

Why: energy conservation.

also from energy conservation

$V_1 I_1 = V_2 I_2 \implies \frac{I_1}{I_2} = \frac{N_2}{N_1}$

$R_{in} = \frac{V_1}{I_1} = \frac{N_1}{N_2} \frac{V_2}{I_2} = \left(\frac{N_1}{N_2}\right)^2 R_2$

For sinusoidal signals

$Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_2$

Often used to match impedances.
To help you better understand the magnetic emf, let's see what happens if a piece of conductor is moving thru a mag. field $\vec{F} = \phi (\vec{v} \times \vec{B})$

$$\text{emf} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$= vB_l$$

$$V = vB_l$$

compare the fig. to Fig 6-8 in the book.

(same # in 5/6)

There's another way to look at this.

$$\Phi = B \times l$$

$$\frac{d\Phi}{dt} = B \frac{dl}{dt} = vB_l \cdot$$

"Flux rule"

Read Ex. 6-3 for the non-uniform field case.

(same in 5/5)

you'll see once again, the two methods are consistent w/ each other.
Flux rule \( \text{emf} = \frac{d\Phi}{dt} \) describes two different phenomena:

Flux changes because circuit moves \( q\vec{v} \times \vec{B} \)

Flux changes because field changes (Faraday's law)

"We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the rule as the combined effects of two quite separate phenomena."

--Richard Feynman
Another example of induced "voltage"

(i.e. non-electrostatic driving force -- an electromotive force)

\[ V = \int (\hat{u} \times \hat{B}) \cdot d\hat{l} \]

\[ = \hat{u} \int_{r_1}^{r_2} \frac{\mu_0 I}{2\pi r} \, dr \]

\[ = \frac{\mu_0 I \hat{u}}{2\pi} \ln \frac{r_2}{r_1} \]

Here, I give you the big picture and show you the method. If you want to plug in some values, go to ex. 6-5 (same as in 5/E) in the book.

Do Homework 11 Problems 5, 6.

Now, you have the knowledge to understand motors & generators.

Let's first look at motors.

To save time, I'm not gonna draw the 3D picture on the board. I'll draw it in the 2D manner. You can look on page 293 (p. 266 in 5/E) for the 3D picture.

For the motor, you feed a current into the loop. Let's show how the mag. field drives it to rotate.

Recall that we actually showed this in an earlier class.

For positive charge, \( F \) is in the same direction as \( \hat{u} \times \hat{B} \)

Torque reversed every 180 degrees.

https://www.youtube.com/watch?v=Y-v27GPK8M4
No torque, but coil keeps rotating due to inertia

If current flows in same direction

If current reverses direction

See also:
https://www.youtube.com/watch?v=Y-v27GPK8M4

Fundamental: no work done by the magnetic field.
AC Motor

Rotates at the frequency of the sine wave: “synchronous motor”. Random rotation direction.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html
Generator

The mechanical energy input to a generator turns the coil in the magnetic field.

A voltage proportional to the rate of change of the area facing the magnetic field is generated in the coil. This is an example of Faraday's law.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html
Now let's study the generator. To generate electricity, we need to turn this loop. We spend mechanical energy to get electric energy.

Let's call this angle $\alpha$

$$\alpha = \omega t + \alpha(0)$$

$$\omega = \frac{\omega}{2}$$

emf on one side = $\int (\vec{v} \times \vec{B}) \cdot d\vec{l} = \frac{\omega}{2} B \sin \alpha \cdot l$

total emf = $\omega l B \sin \alpha = \omega l B \sin [\omega t + \alpha(0)]$

$= \omega AB \sin [\omega t + \alpha(0)]$

There is a better way to get this result

$$\Phi = \int \vec{B} \cdot d\vec{S} = \vec{B} \cdot \hat{\vec{n}} \cdot \Delta = AB \cos \alpha$$

$= AB \cos [\omega t + \alpha(0)]$

$$\text{emf} = -\frac{d\Phi}{dt} = \omega AB \sin [\omega t + \alpha(0)]$$

Again, the fundamental concept: the magnetic field does no work.
\[ \nabla \times B = \mu_0 J_{\text{total}} = \mu_0 (J + J_M) = \mu_0 J + \mu_0 \nabla \times M = \mu_0 \nabla \times H + \mu_0 \nabla \times M = \mu_0 \nabla \times (H + M) \]

\[ = \mu_0 (1 + \chi_m) \nabla \times H \]

\[ \equiv \mu_0 \mu_r \nabla \times H \equiv \mu \nabla \times H, \]

where \( \mu \equiv \mu_0 (1 + \chi_m) \equiv \mu_0 \mu_r \)

\[ B = \mu H \]

\[ \text{Magnetism of Materials} \]

\[ \text{Similar: } J = \nabla \times H \]

\[ \nabla \cdot E = \frac{1}{\varepsilon_0} \rho_{\text{total}} = \frac{1}{\varepsilon_0} (\rho + \rho_P) = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0} \nabla \cdot P \]

\[ \rho_P = -\nabla \cdot P \]

\[ \varepsilon_0 \nabla \cdot E + \nabla \cdot P = \rho \]

\[ \nabla \cdot (\varepsilon_0 + \chi_e \varepsilon_0)E \equiv \nabla \cdot D = \rho, \]

where \( \varepsilon \equiv \varepsilon_0 (1 + \chi_e) \equiv \varepsilon_0 \varepsilon_r, \quad D \equiv \varepsilon_0 \varepsilon_r E \equiv \varepsilon E \)

\[ P = \chi_e \varepsilon_0 E \]

\[ P = \text{No charge} \]

\[ P = \lim_{\Delta V \to 0} \frac{\sum p_i}{\Delta V} \]

\[ \text{E: Total effect of "external" and polarization charges; felt by probe charge} \]

\[ \text{D: "Indirect" quantity; polarization as material property, to get } E = D/\varepsilon \]
Dielectric polarization always works against the external electric field.

The magnetization $\mathbf{M}$, however, may be parallel or anti-parallel to the external magnetic field $\mathbf{H}$.

**Paramagnetic:** $\chi_m > 0$, $\mu_r = 1 + \chi_m > 1$

**Diamagnetic:** $\chi_m < 0$, $\mu_r = 1 + \chi_m < 1$ \[ \frac{\mu_r}{\mu} \approx \frac{1}{\mu_0} \]

**Ferromagnetic:** $\mu_r >> 1$,
nonlinearity, hysteresis

The description we give here is phenomenological – no real understanding.

Now that we have tried to give you a qualitative explanation of diamagnetism and paramagnetism, we must correct ourselves and say that it is not possible to understand the magnetic effects of materials in any honest way from the point of view of classical physics. Such magnetic effects are a completely quantum-mechanical phenomenon.

Other scientists would say "heuristic".

It is, however, possible to make some phoney classical arguments and to get some idea of what is going on.

-- Richard Feynman
When we studied the electric field, we defined $\mathbf{D}$ and $\mathbf{E}$, where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad \nabla \cdot \mathbf{D} = \rho$$

Here for the mag field, similarly, we have

$$\mathbf{B} = \mu_0 \mathbf{H} = \mu_r \mu_0 \mathbf{H} \quad \mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M} \quad \nabla \times \mathbf{H} = \mathbf{J}$$

$\mu_r$ is a property of the material, and $\mu_0$ is a physical constant.

$\mu_r$ is dimensionless, which means it's just a number without a unit.

There is an important difference between $\mu_r$ and $\varepsilon_r$. $\varepsilon_r$ has a big range.

$\varepsilon_r \approx 1$ for air, $\varepsilon_r = 2.4$ for many polymers.

$\varepsilon_r = 3.9$ for SiO$_2$ and for dielectric materials it can be about 10, or 10's, or even hundreds.

There are materials with $\varepsilon_r$ in thousands.

But for $\mu_r$, it's either ~1, or huge.

Those materials with huge $\mu_r$'s are called ferromagnetic, such as iron.

They can be permanently magnetized.
Which means, when there’s no current flowing around, there can be a residual $B$ field in them. For other materials, it’s more than sufficient to use $\mu_r = 1$ for most practical purposes.