Dynamic Fields, Maxwell's Equations (Chapter 6)

So far, we have studied static electric and magnetic fields. In the real world, however, nothing is static. Static fields are only approximations when the fields change very slowly, and "slow" is in a relative sense here.

To really understand electromagnetic fields, we need to study the dynamic fields. You will see the E & M fields are coupled to each other.

Four visual pictures to help you understand the four Maxwell's equations

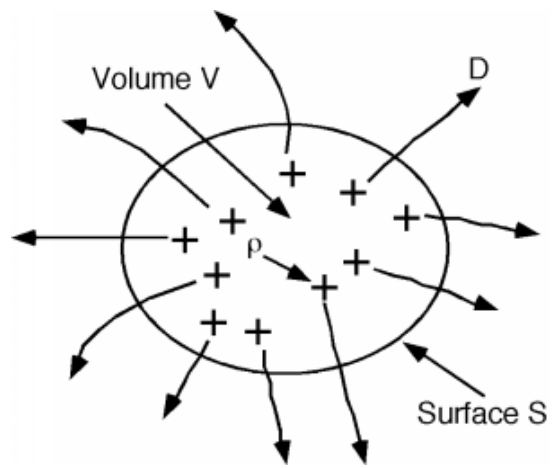
Two remain the same for dynamic and static fields. Two are different.

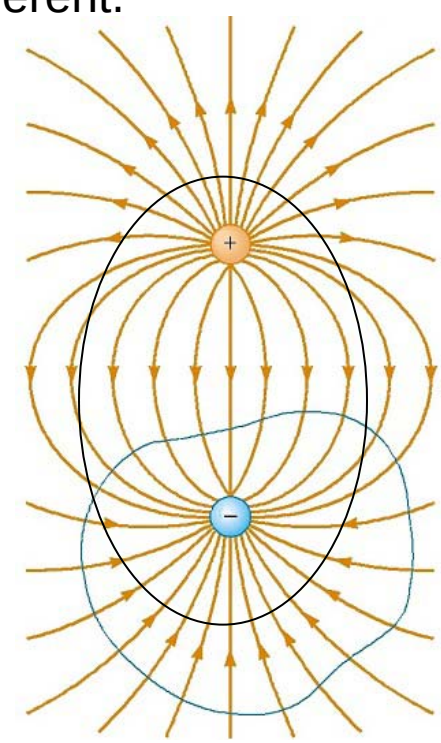
(1)

$$\oint \mathbf{E} \cdot d\mathbf{s} = \int \frac{\rho}{\varepsilon_0} dV = \frac{Q}{\varepsilon_0}$$

$$\oint \mathbf{D} \cdot d\mathbf{s} = \int \rho dV = Q$$

$$\varepsilon \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{D} = \rho$$

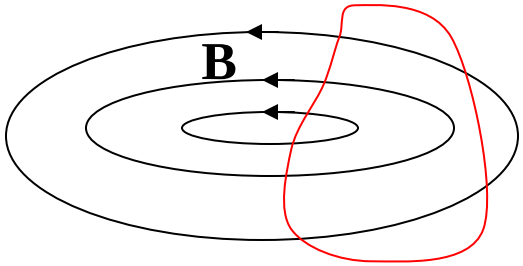




This holds for dynamic fields even when ρ changes with time.

Question: how can ρ change with time?

(2) $\oint \mathbf{B} \cdot d\mathbf{s} = 0$ $\qquad \nabla \cdot \mathbf{B} = 0$



What goes in must come out: no monopoles.
Always true, static or dynamic.

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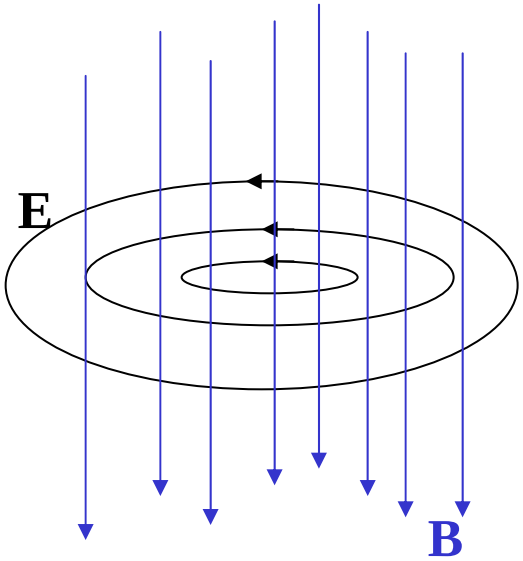
(3) The electrostatic field is conservative

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \nabla \times \mathbf{E} = 0$$

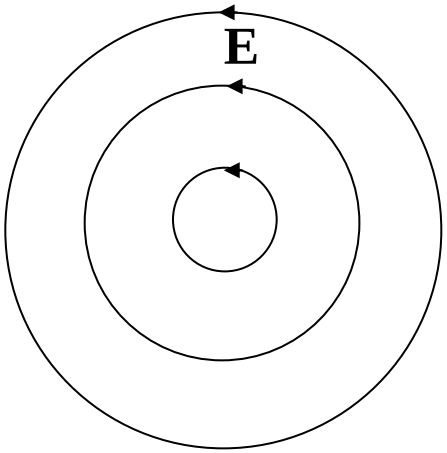
This is why we can define "potential."



Faraday's law:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Pay attention to this negative sign.



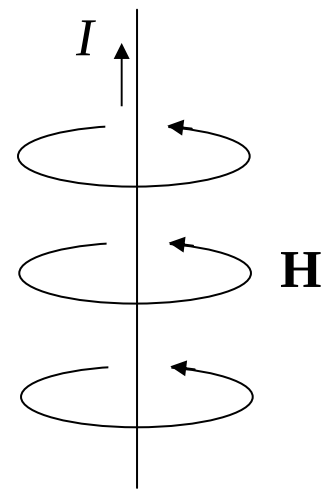
This electric field induced by a changing magnetic field is **not conservative**! It's **not** an "electrostatic field" even when $\frac{\partial \mathbf{B}}{\partial t}$ is a constant.

Cannot define a potential!

(4) Ampere's law (static)

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{S} = I$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

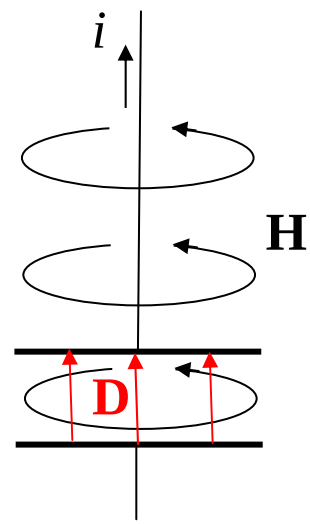


Ampere's law (dynamic)

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}) \cdot d\mathbf{S} = I + \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

Displacement current

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$



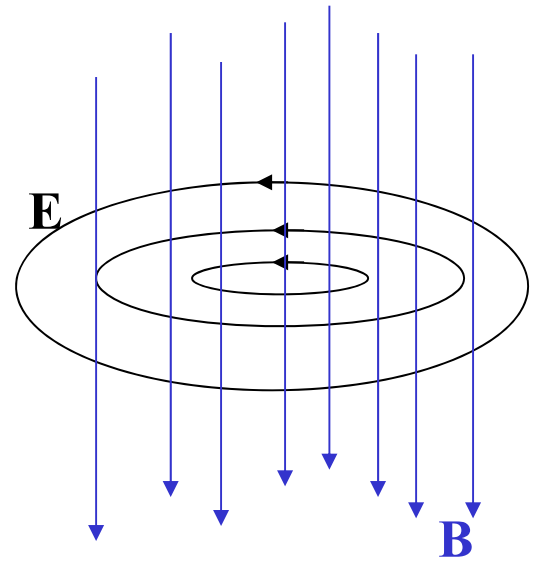
We have covered the static case in pretty much detail. Here, in the dynamic case the current could include the displacement current.

(3) and (4) are about the coupling between E & M fileds. They are the foundations of electromagnetic waves, to be discussed in Ch. 7.

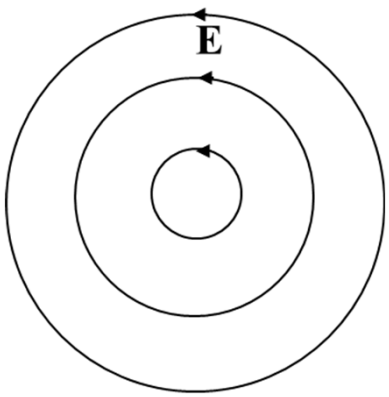
In Ch.6, we focus on Eq. (3), Faraday's law:

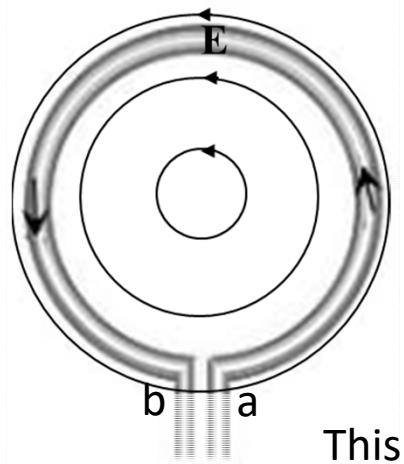
$$\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \quad \Longleftrightarrow \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Pay attention to this negative sign.



Plan view:

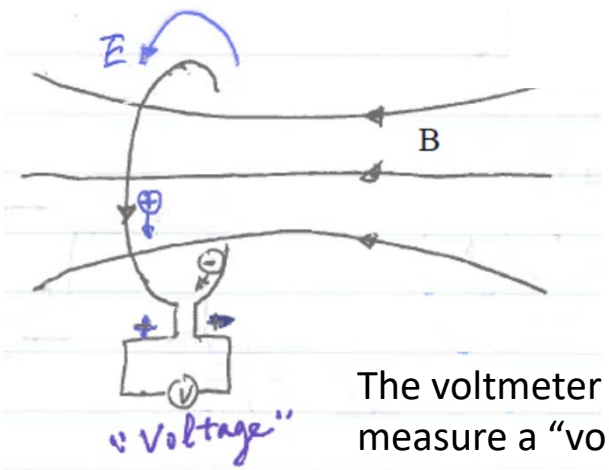




Now we placed a wire loop in this field. This electric field will drive a current.

This gap is so small that $\int_a^b \mathbf{E}\cdot d\mathbf{l} \approx \oint \mathbf{E}\cdot d\mathbf{l}$

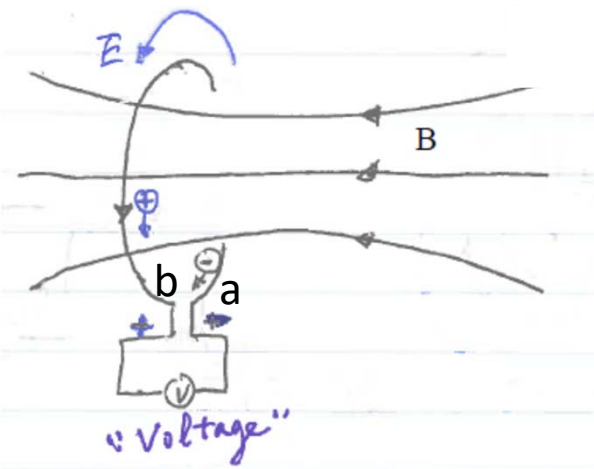
Viewed from another perspective:



The voltmeter will measure a "voltage"

This "voltage" is due to the non-electrostatic field. It is a "electromotive force." Just like that of a battery, which is due to chemistry.

$$V_{\text{emf}} = \text{emf} = \int_a^b \mathbf{E}\cdot d\mathbf{l} \approx \oint \mathbf{E}\cdot d\mathbf{l}$$



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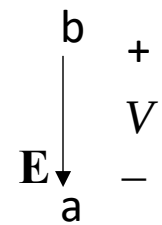
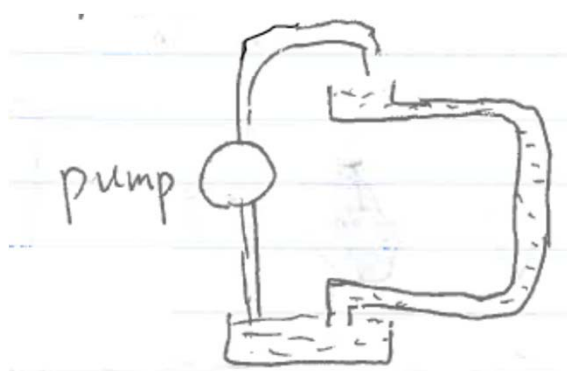
$$V_{\text{emf}} = \text{emf} = \int_a^b \mathbf{E} \cdot d\mathbf{l} \approx \oint \mathbf{E} \cdot d\mathbf{l}$$

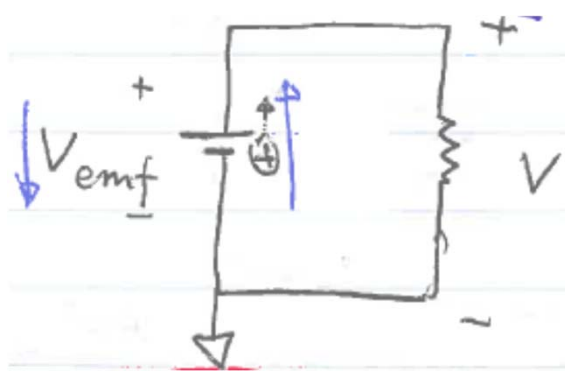
Notice that for an electrostatic field

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$$



Let's use an analogy to explain the "subtle" difference between an emf and a voltage:

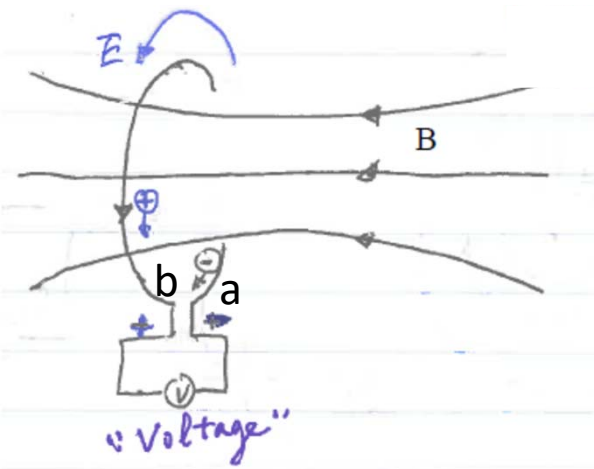




$$V = V_{\text{emf}} = \text{emf}$$

The pump works against gravity.

The battery works against the electrostatic force.



$$V = V_{\text{emf}} = \text{emf} = \int_a^b \mathbf{E}\cdot d\mathbf{l} \approx \oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

$$= -\frac{d}{dt}\left(\int \vec{B}\cdot d\vec{S}\right)$$

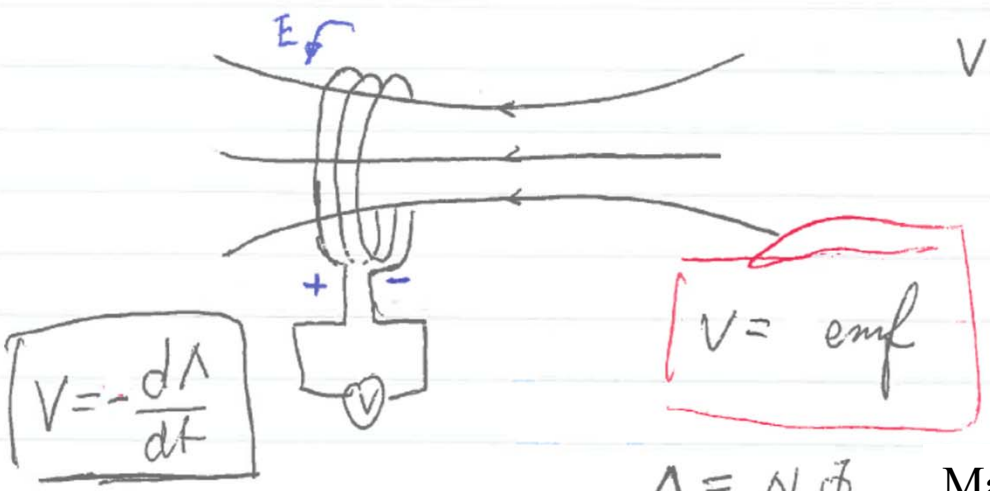
$$= -\frac{d\Phi}{dt}.$$

Define magnetic flux $\Phi = \int \vec{B}\cdot d\vec{S}$

B is therefore called the "magnetic flux density."

unit of Φ: Wb = T m^2 (weber)

We may also have a coil of *N* turns instead of a loop of just one turn:



$$V = -N\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}$$

$$= -N\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$$

$$= -N\frac{d\Phi}{dt}$$

$$= -\frac{d\Lambda}{dt}$$

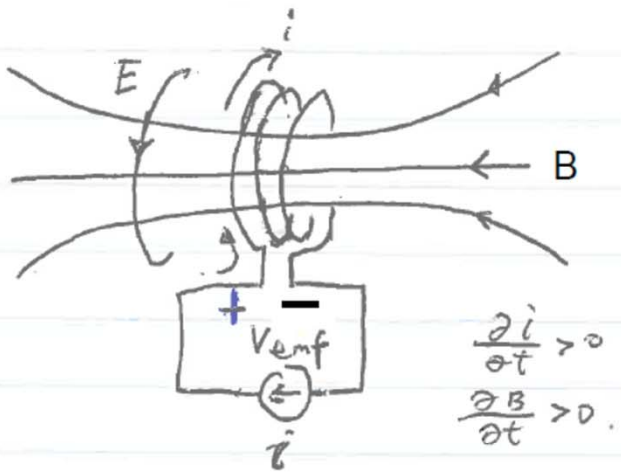
$$\boxed{V = -\frac{d\Lambda}{dt}}$$

$$\boxed{V = \text{emf}}$$

$\Lambda = N\Phi$. Magnetic flux linkage

What if you replace the voltmeter with a load resistor?

Now let's see what happens if we feed a current to the coil, when there is no external magnetic filed.



$\frac{\partial i}{\partial t} > 0$

$\frac{\partial B}{\partial t} > 0.$

i will induce a $\vec{B}$ field.

This is true, regardless of the shape or # of turns.

But for simplicity we use the expression for the B field of a very long solenoid

$B = \mu\left(\frac{N}{l}\right) I$ but keep in mind $B \propto I$ in a general case.

If I changes w/ time, so does B

$$\frac{dB}{dt} = \mu\left(\frac{N}{l}\right)\frac{di}{dt}, \quad \text{and} \quad \frac{dB}{dt} \propto \frac{di}{dt} \text{ in general}$$

Recall that $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$

$$v = -V_{emf} = -emf = -\oint \vec{E}\cdot d\vec{l} = N\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = N\frac{d\Phi}{dt} = \frac{d\Lambda}{dt}$$

negative sign due to definition of directions; induced voltage always against change of current

Define $L \equiv \frac{\Lambda}{i} \Rightarrow \Lambda = Li$

inductance

then $v = L\frac{di}{dt}$

For the coil or solenoid

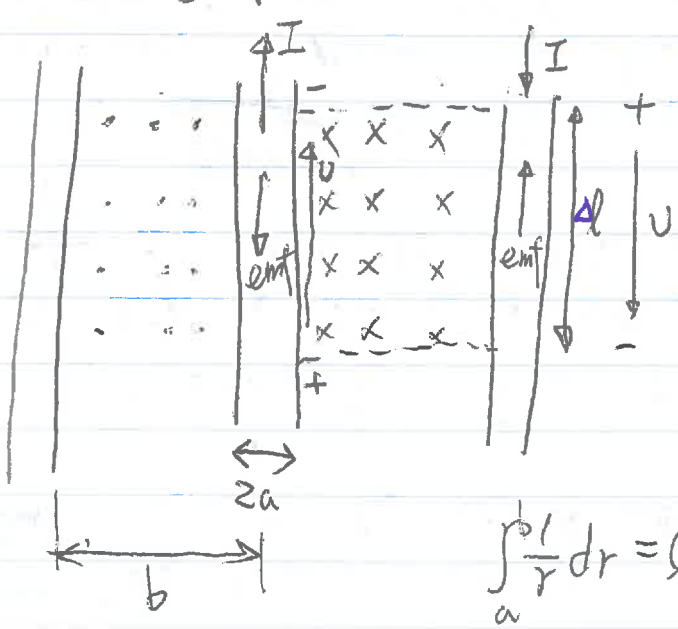
total flux linking the circuit

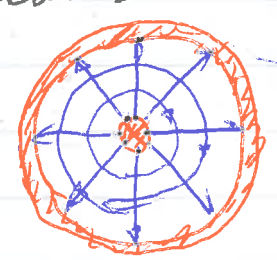
$$\Lambda = NBS = \mu\frac{N^2}{l} i S$$

$$\therefore L = \mu\frac{N^2}{l} S$$

For other geometries, read Section 5-7.2 (Fig. 5-27)

Now I'm going to give you another example of how to find the inductance of a particular geometry of conductors.





$$B = \frac{\mu I}{2\pi r}$$

$$\Lambda = \Phi = l\int_a^b B\,dr$$

$$= \frac{\mu I l}{2\pi}\int_a^b \frac{1}{r}\,dr$$

$$= \frac{\mu I l}{2\pi}\ln\frac{b}{a}$$

$$\int_a^b \frac{1}{r}\,dr = \ln r\Big|_a^b$$

(over)

$$\therefore L = \frac{\mu l}{2\pi}\ln\frac{b}{a} \quad l' = \frac{L}{\Delta l} =$$

Energy stored in an inductor

$$v = L\frac{di}{dt}$$

$$W_m = \int i v\, dt = \int i\, L\frac{di}{dt}\, dt = L\int_0^I i\, di$$

$$= \frac{1}{2} L I^2$$

Following the math in §5.8 (5.9 in 5/E)

you get $w_m = \frac{W_m}{V} = \frac{1}{2}\mu H^2$

for the infinitely long solenoid.

Analogy w/ E field

Cap	ind
$i = C\frac{dv}{dt}$	$v = L\frac{di}{dt}$
$W_E = \frac{1}{2} C V^2$	$W_m = \frac{1}{2} L I^2$
$w_E = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
derived ~~from~~ for the infinitely large cap: parallel plate	.. for the infinitely long solenoid. but ..

What's the limit here?

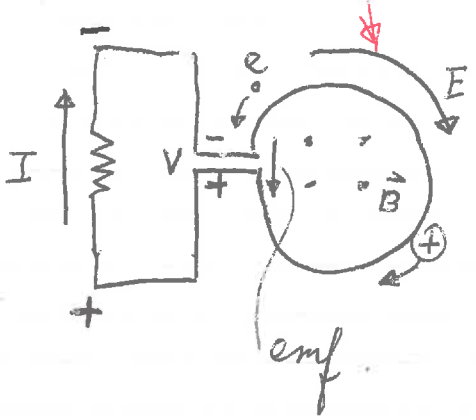
but true for general.

Lastly the unit $L = \frac{\Lambda}{I}$

$$H = \frac{Wb}{A} = \frac{T\,m^2}{A}$$

(henry)

$$\oint \vec{E}\cdot d\ell = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{S}$$



$$\frac{\partial B}{\partial t} > 0$$

Now we emphasize again the direction of the voltage induced by a changing mag. field

Compare this to Fig. 6-2 in the book.
(same # in 5/E)

The ~~voltage~~ current thru an R always flows from hi V to low V.

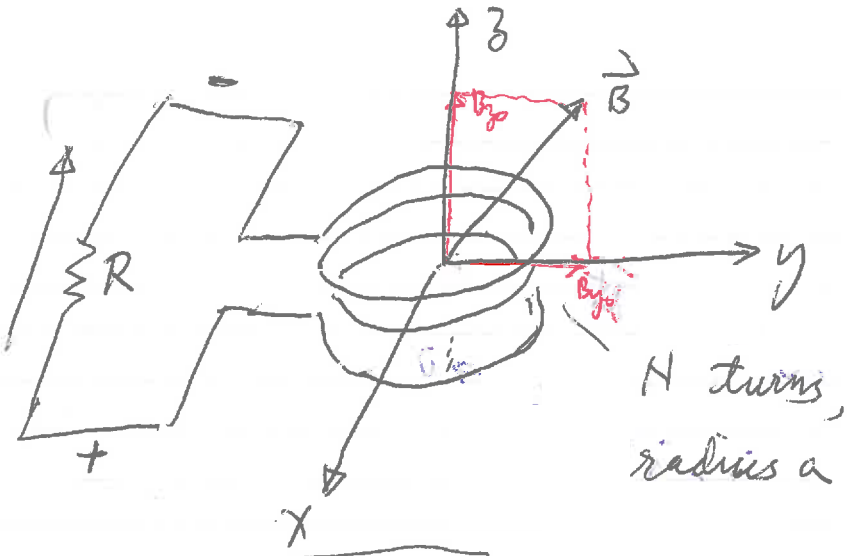
emf & V are two different concepts.

This is why I don't like the ~~[illegible]~~ discussion in the book.

The loop acts like a battery:

In the loop current flows from low V to hi V. but on the load the current flows from hi V to low V.

Thats why we say a battery. or this loop is like a pump.



$$\vec{B} = \left(\hat{y}\, B_{y0} + \hat{z}\, B_{z0}\right) \sin\omega t$$

$$\Phi = \int \vec{B} \cdot d\vec{S}$$

$$= B_{z0} \sin\omega t \, (\pi a^2)$$

$$\Lambda = N \Phi$$

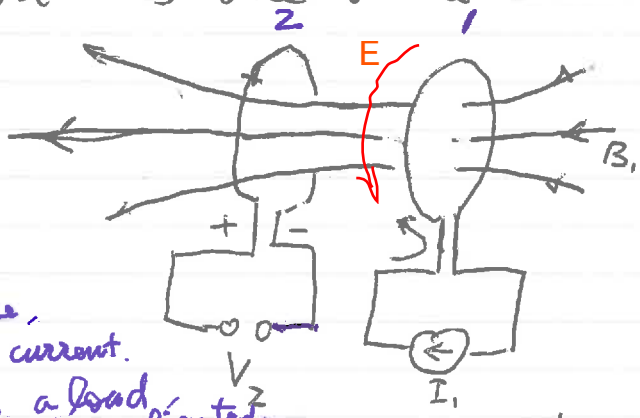
$$= N B_{z0} \pi a^2 \sin\omega t$$

$$\boxed{V = \frac{d\Lambda}{dt}} = \omega N B_{z0} \pi a^2 \cos\omega t$$

$$I = \frac{V}{R} = \frac{\pi N B_{z0} a^2 \omega}{R} \cos\omega t$$

Do Homework 11 Problems 2 through 4, and 7.

Now, let's see what happens if we put two coils very close to each other. In other words, the $\vec{B}$ through coil 2 is due to the current ~~[illegible]~~ thru coil 1.



$$\Phi_{12} = \oint_{S_2} \vec{B}_1 \cdot d\vec{S}_2 \propto I_1$$

$$\Lambda_{12} = N_2 \Phi_{12} \equiv L_{12} I_1$$

$$V_2 = \frac{d\Lambda_{12}}{dt} = L_{12} \frac{dI_1}{dt}$$

☆ Here, no current.
If there's a load, more complicated.

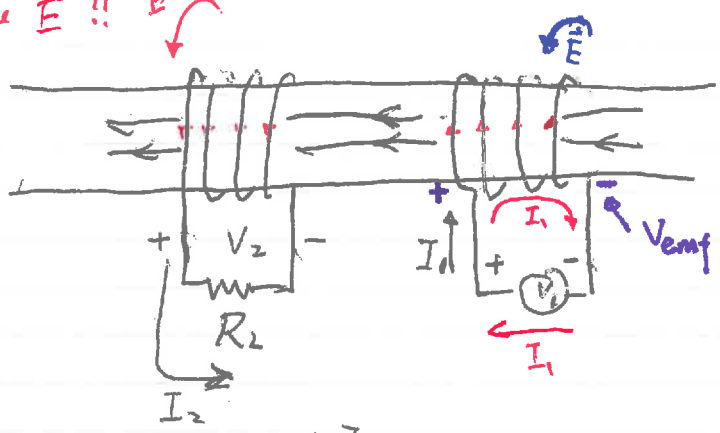
Similarly, $V_1 = \dfrac{d\Lambda_{21}}{dt} = L_{21} \dfrac{dI_2}{dt}$

If you do some "due diligence" mathematically, which we won't do here of course, you can show $L_{12} = L_{21}$. This is called the mutual inductance.

More general discussions ....

Now, let's say we have some material, with a very high μ, as the core, and the two coils are around this core. (Ideal transformer: $\mu = \infty$)

The same $\vec{E}$!! $\vec{E}$



$$V_1 = (-)N_1 \frac{d\Phi}{dt}.$$

Coil 1 must establish a voltge that equals V_1.

$$V_2 = (-)N_2 \frac{d\Phi}{dt} \qquad \therefore \quad \frac{V_1}{V_2} = \frac{N_1}{N_2}$$

How to determin the direction of the $\vec{B}$ field. For the primary coil, follow the right hand rule. For the secondary, opposite to the right hand rule.

why: energy conservation.

The primary is a load to V_1.
The secondary is a driver.

also from energy conservation.

$$V_1 I_1 = V_2 I_2 \implies \frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$R_{in} = \frac{V_1}{I_1} = \frac{\frac{N_1}{N_2} V_2}{\frac{N_2}{N_1} I_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

For sinusoidal signals $Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$

Often used to match impedances.

Do Homework 11 Problem 8.

The above are the usual hand-waving description of of the ideal transformer.

Let's go a bit deeper, just for fun.

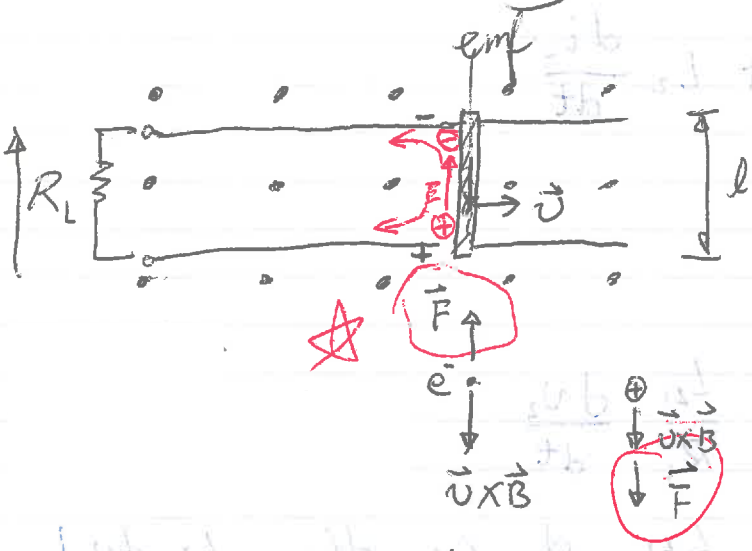
Here, V_1, V_2, I_1, and I_2 are all in phase.

$$V_1 = N_1 \frac{d\phi}{dt}$$

$$\phi \propto I_1$$

Should there be a 90-degree phase shift between the flux and the primary coil current???

To help you better understand the magnetic emf. let's see what happens if a piece of conductor is moving thru a mag. field $\vec{F} = q(\vec{v} \times \vec{B})$



$$\boxed{\text{emf} = \int (\vec{v} \times \vec{B}) \cdot d\vec{l}}$$
$$= vBl$$

$$V = vBl$$

compare the fig. to Fig 6-8 in the book
(same # in 5/E)

There's another way to look at this.

$$\Phi = B x l$$

B is const., but S is changing.

$$\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = vBl$$

$$\boxed{V = \frac{d\Phi}{dt}}$$ "Flux rule"

Read Ex. 6-3 for the non-uniform field case
(same in 5/E)

you'll see once again, the two methods are consistent w/ each other.

Flux rule emf = $\frac{d\Phi}{dt}$ describes two different phenomena:

Flux changes because circuit moves $q\vec{v} \times \vec{B}$

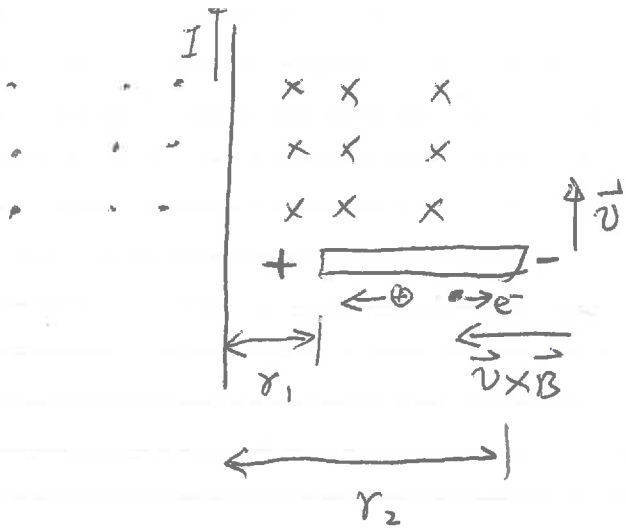
Flux changes because field changes (Faraday's law)

"We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the rule as the combined effects of two quite separate phenomena."

--Richard Feynman

Another example of induced "voltage"

(i.e. non-electrostatic driving force -- an electromotive force)



$$V = \int (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= v\int_{r_1}^{r_2} \frac{\mu_0 I}{2\pi r}\, dr$$

$$= \frac{\mu_0 I v}{2\pi}\int_{r_1}^{r_2} \frac{1}{r}\, dr$$

$$= \frac{\mu_0 I v}{2\pi}\ln\frac{r_2}{r_1}$$

Here, I give you the big picture and show you the method. If you want to plug in some #s, go to ex. 6-5 (same # in 5/E) in the book.

Do Homework 11 Problems 5, 6.

Now, you have the knowlege to understand motors & generators.

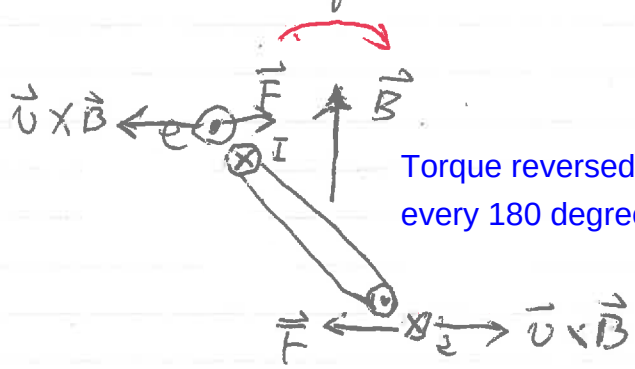
Let's first look @ motors. (dc & ac)

To save time, I'm not gonna draw the 3D picture on the board. I'll draw it in the 2D manner. you can look @ page 293 (p.p. 266 in 5/E) for the 3D picture.

Fig. 6-11

(in 6/E, or pp. 294 in 7/E)

For the motor, you feed a current to the loop. Lets see how the mag. field drives it to rotate.



Torque reversed every 180 degrees.

Recall that we actually showed this in an earlier class.

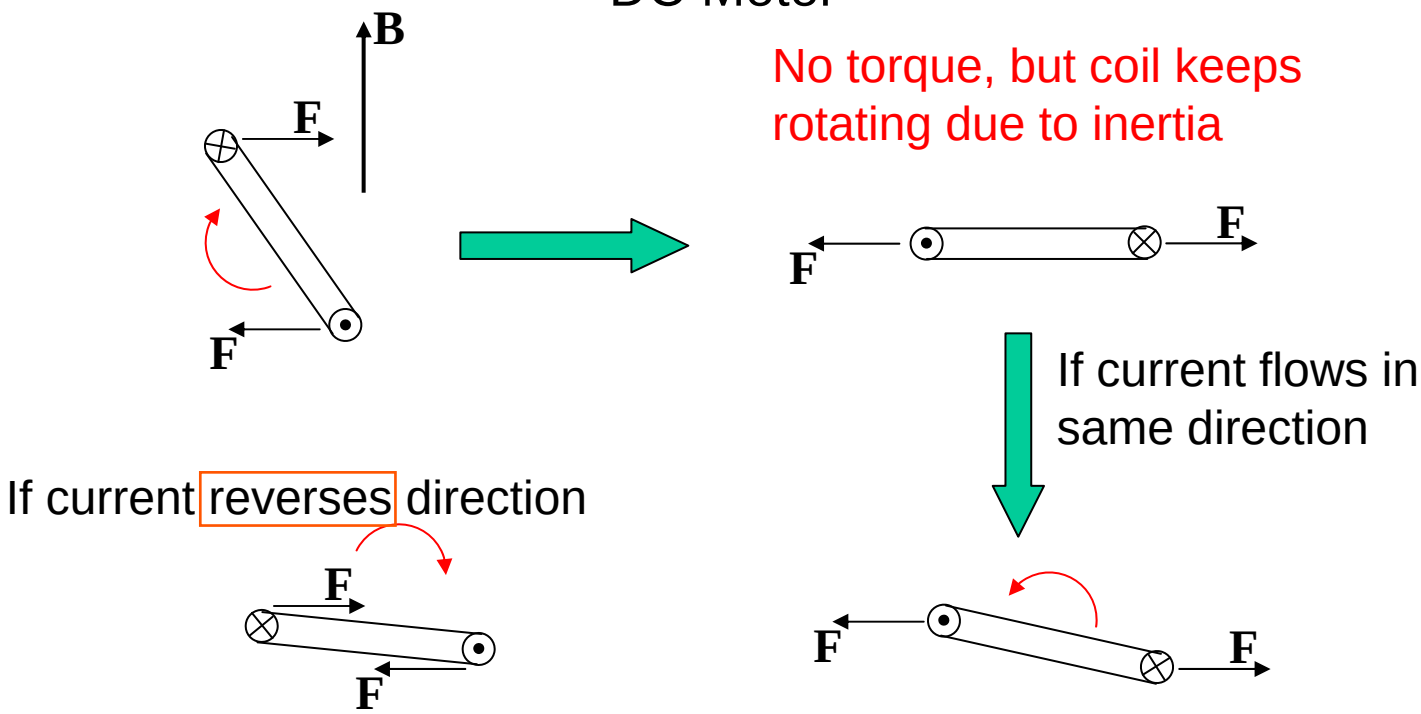
(for the dc motor)

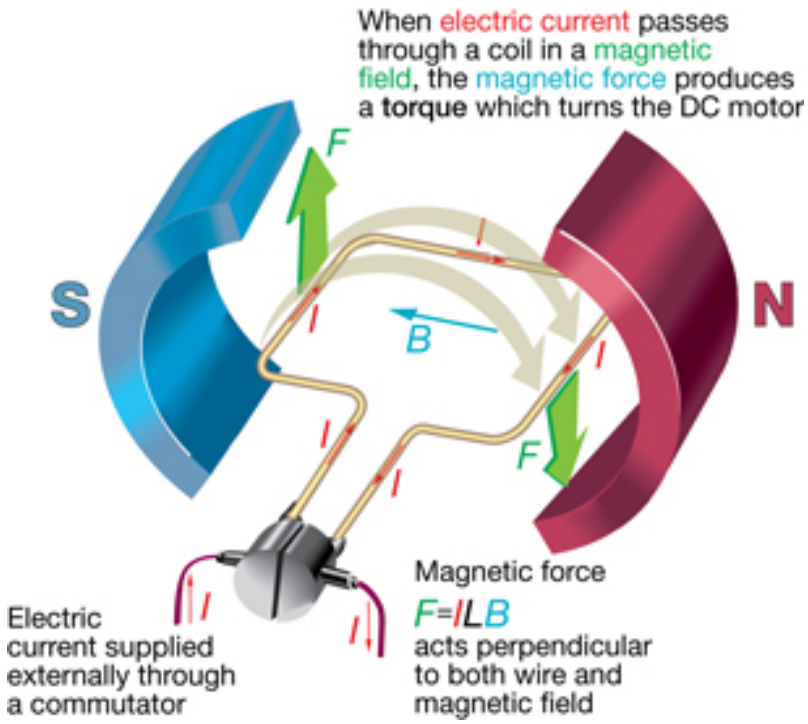
http://resource.rockyview.ab.ca/rvlc/physics30_BU/Unit_B/m4/p30_m4_l03_p4.html

https://www.youtube.com/watch?v=Y-v27GPK8M4

For positive charge $\vec{F}$ is in the same direction

DC Motor



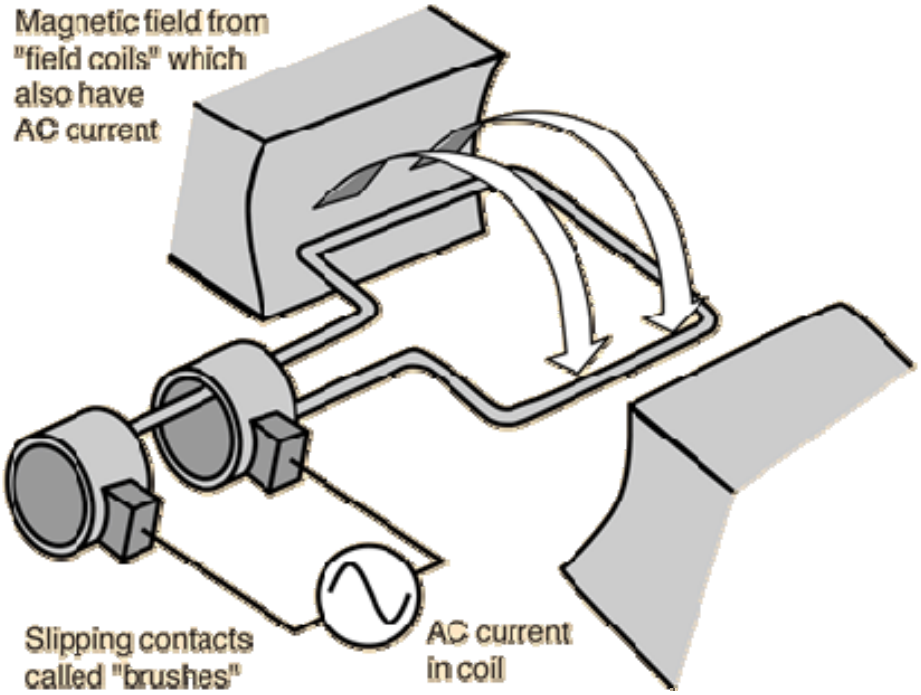


http://resource.rockyview.ab.ca/rvlc/physics30_BU/Unit_B/m4/p30_m4_l03_p4.html

See also:
https://www.youtube.com/watch?v=Y-v27GPK8M4

Fundamental: no work done by the magnetic field.

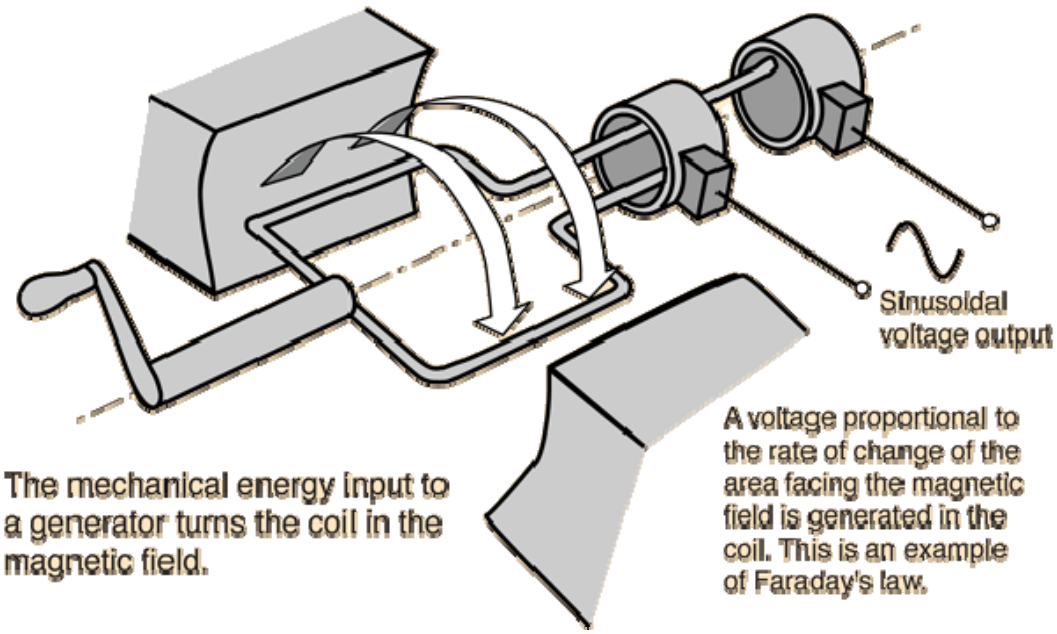
AC Motor



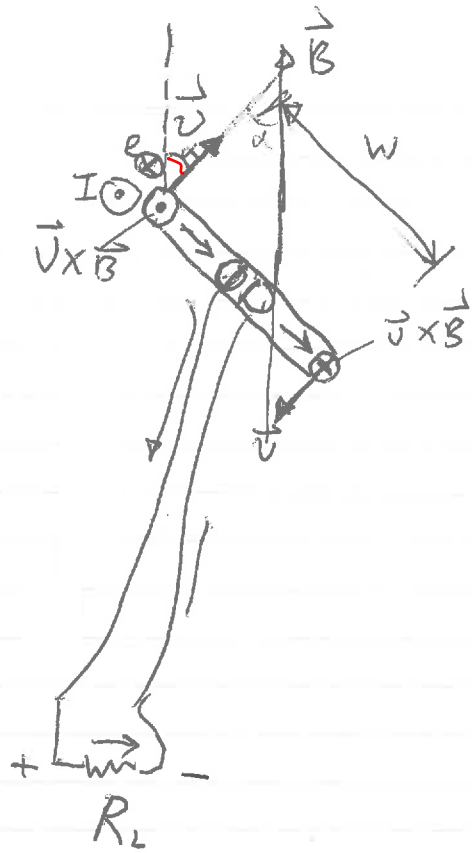
Rotates at the frequency of the sine wave: “synchronous motor”.
Random rotation direction.

http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html

Generator



http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/motorac.html



Now let's study the generator.
To generate electricity we need to turn this loop.
We spend mechanic energy to get electric energy.

Let's call this angle α

$$\alpha = \omega t + \alpha(0)$$

$$v = \omega \frac{w}{2}$$

$$\text{emf on one side} = \int (\vec{v}\times\vec{B})\cdot d\vec{l}$$

$$= \omega \frac{w}{2} B \sin\alpha \cdot l$$

$$\text{total emf} = \omega w l B \sin\alpha = \omega w l B \sin[\omega t + \alpha(0)]$$

$$= \omega A B \sin[\omega t + \alpha(0)]$$

There is a better way to get this result

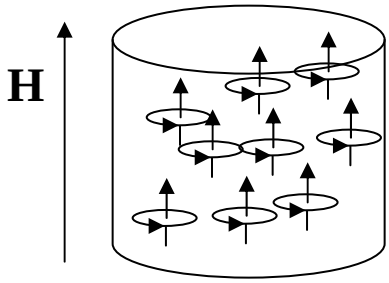
$$\Phi = \int \vec{B}\cdot d\vec{S} = \vec{B}\cdot\hat{n}\, A = AB\cos\alpha$$

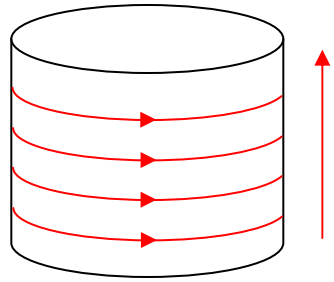
$$= AB\cos[\omega t + \alpha(0)]$$

$$\text{emf} = -\frac{d\Phi}{dt} = \omega A B \sin[\omega t + \alpha(0)]$$

Again, the fundamental concept: the magnetic field does no work.

Magnetism of Materials





$$\mathbf{M} = \lim_{\Delta V \to 0} \frac{\sum_i \mathbf{m}_i}{\Delta V}$$

$$\mathbf{M} = \chi_m \mathbf{H} \qquad J_M = \nabla \times \mathbf{M}$$

Similar: J = ∇ X H

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}} = \mu_0 (\mathbf{J} + \mathbf{J}_M) = \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{M}$$

$$= \mu_0 \nabla \times \mathbf{H} + \mu_0 \nabla \times \mathbf{M} = \mu_0 \nabla \times (\mathbf{H} + \mathbf{M})$$

$$= \mu_0 (1 + \chi_m) \nabla \times \mathbf{H}$$

$$\equiv \mu_0 \mu_r \nabla \times \mathbf{H} \equiv \mu \nabla \times \mathbf{H},$$

B = μH

where $\mu \equiv \mu_0 (1 + \chi_m) \equiv \mu_0 \mu_r$

B: effect of external & magnetization currents; felt by probe current

H: allow us to consider external current only. μ: material property

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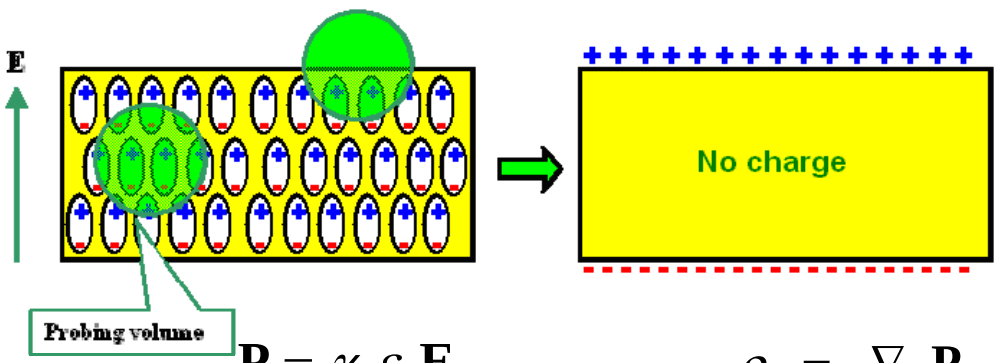


$$\mathbf{P} = \lim_{\Delta V \to 0} \frac{\sum_i \mathbf{p}_i}{\Delta V}$$

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E} \qquad \rho_P = -\nabla \cdot \mathbf{P}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho_{\text{total}} = \frac{1}{\varepsilon_0} (\rho + \rho_P) = \frac{\rho}{\varepsilon_0} - \frac{1}{\varepsilon_0} \nabla \cdot \mathbf{P}$$

Sign means dielectric polarization always against external field (charge)

$$\varepsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} = \rho$$

$$\nabla \cdot (\varepsilon_0 + \chi_e \varepsilon_0) \mathbf{E} \equiv \nabla \cdot \mathbf{D} = \rho \quad ,$$

where $\varepsilon \equiv \varepsilon_0 (1 + \chi_e) \equiv \varepsilon_0 \varepsilon_r$, $\mathbf{D} \equiv \varepsilon_0 \varepsilon_r \mathbf{E} \equiv \varepsilon \mathbf{E}$

E: Total effect of "external" and polarization charges; felt by probe charge

D: "Indirect" quantity; polarization as material property, to get E = D/ε

Dielectric polarization always works against the external electric field.

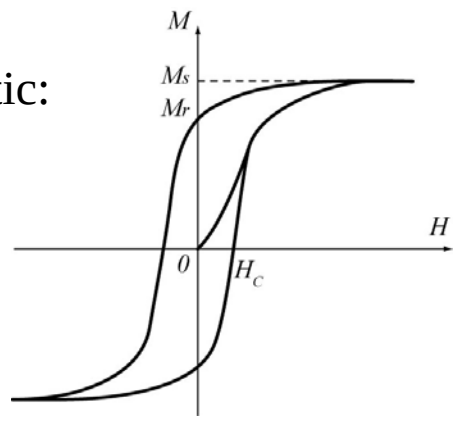
The magnetization **M**, however, may be parallel or anti-parallel to the external magnetic field **H**.

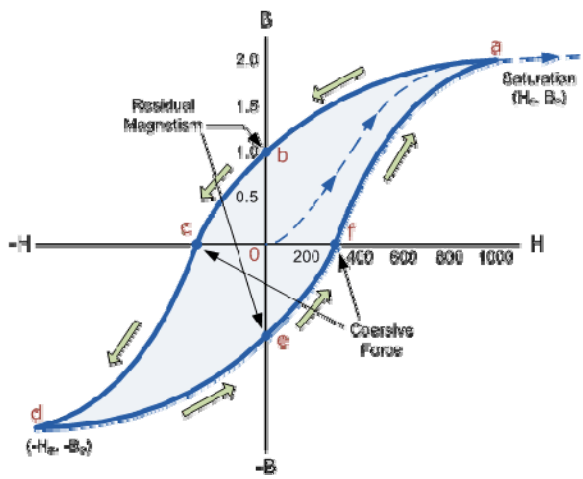
Paramagnetic: $\chi_m > 0$, $\mu_r = 1 + \chi_m > 1$

Diamagnetic: $\chi_m < 0$, $\mu_r = 1 + \chi_m < 1$

$$\boxed{\begin{array}{l}\mu_r \approx 1 \\ \mu \approx \mu_0\end{array}}$$

Ferromagnetic:
$\mu_r >> 1$,
nonlinearity,
hysteresis





The description we give here is phenomenological – no real understanding.

Now that we have tried to give you a qualitative explanation of diamagnetism and paramagnetism, we ***must*** correct ourselves and say that *it is not possible* to understand the magnetic effects of materials in any honest way from the point of view of classical physics. Such magnetic effects are a *completely quantum-mechanical phenomenon*.

Other scientists would say "heuristic"

It is, however, possible to make some ***phoney*** classical arguments and to get some idea of what is going on.

-- Richard Feynman

When we studied the electric field, we defined $\vec{D}$ and $\vec{E}$, where $\vec{D} = \varepsilon \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$.

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ $\quad$ $\nabla \cdot \vec{D} = \rho$

Here for the mag field, similarly, we have

$\vec{B} = \mu \vec{H} = \mu_r \mu_0 \vec{H}$. $\quad$ $\vec{B} = \mu_0 \vec{H} + \vec{M}$ $\quad$ $\nabla \times \vec{H} = \vec{J}$

μ_r is a property of the material, and μ_0 is a physical constant.

μ_r is dimensionless, which means its just a number without a unit.

There is an important difference between μ_r & ε_r: ε_r has a big range.

$\varepsilon_r \approx 1$ for air, $\quad$ $\varepsilon_r = 2.x$ for many polymers.

$\varepsilon_r = 3.9$ for SiO_2

and for hi-k materials it can be about 10, or 10's. or even hundreds.

There are materials w ε_r in 1000's.

But for μ_r. it's either ~ 1, or huge.

Those materials w/ huge μ_r's are called ferromagnetic. such as iron.

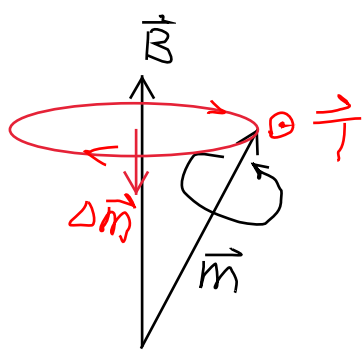
They can be permanently magnetized,

which means, when there's no current flowing around, there can be a residual B field in them.
For other materials, it's more than sufficient to use $\mu_r = 1$ for most practical purposes.

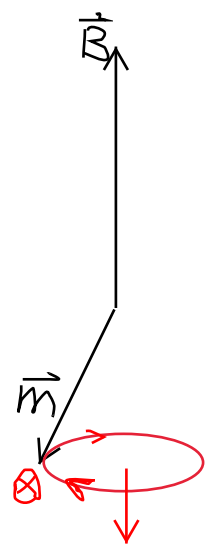
Some not-so-phoney explanations

Paramagnetic: Material contains atoms with permanent magnetic moments, which are lined up by external magnetic field.

Diamagnetic: Exhibited by atoms w/o net permanent magnetic moments, due to Larmor precession. Induced extra moment opposite to $\vec{B}$.



$$\vec{\tau} = \vec{m} \times \vec{B}$$



For an intuitive, easy-to-understand, classical analogy of Lamor precession, see precession of a gyro/spin top: https://en.wikipedia.org/wiki/Precession

All materials have diamagnetism. In paramagnetic materials, paramagnetism dominates.

Ferromagnetic: Moments line up themselves w/o external field. Should not exist had it not been for quantum mechanics. Magnetic intercation among moments too weak even at 0.1 K temperature.