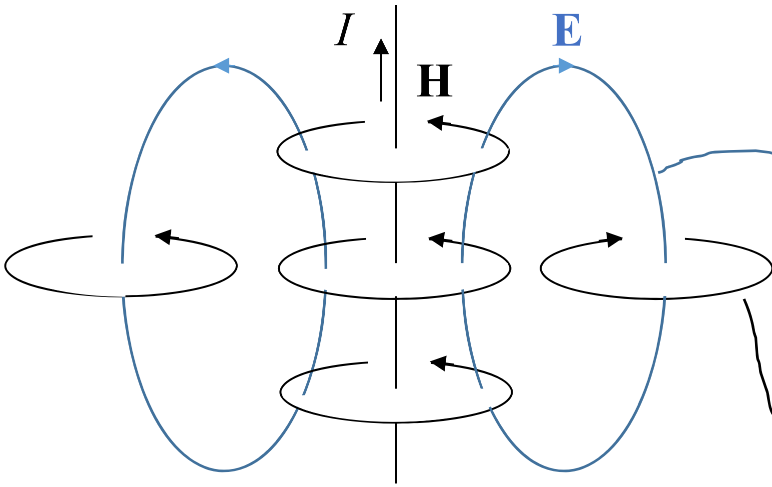
Plane Wave

Our discussion on dynamic electromagnetic field is incomplete.



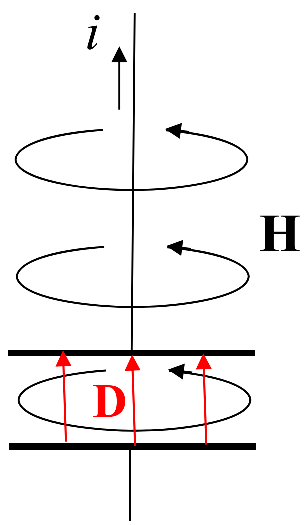
An AC current induces a magnetic field, which is also AC and thus induces an AC electric field.

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int \mathbf{J}\cdot d\mathbf{S} = I \qquad \nabla\times\mathbf{H} = \mathbf{J}$$

$$\boxed{\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

This AC electric field induces an AC magnetic field.

This goes on and on…

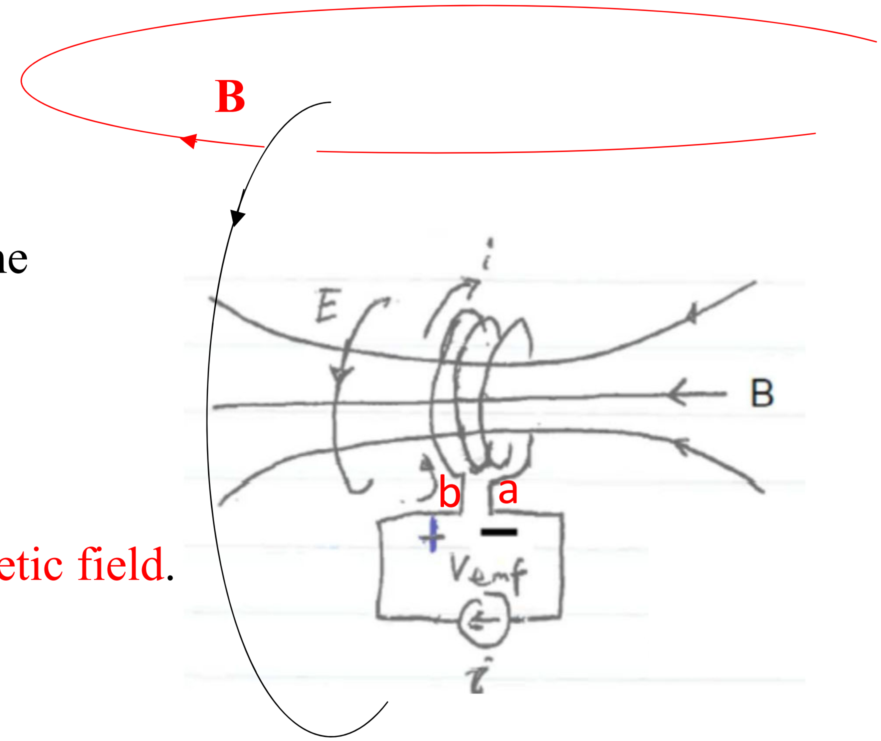


Generalization: include a capacitor and consider the displacement current. The first step is then

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int (\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t})\cdot d\mathbf{S} = I + \int \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

For the inductor, the AC magnetic field inside the coil induces an AC electric field, responsible for the emf of the coil; this is how the inductor works.



$$\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad\qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This AC electric field in turn induces an AC magnetic field.

$$\boxed{\oint \mathbf{H}\cdot d\mathbf{l} = \int \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S} \qquad\qquad \nabla\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}}$$

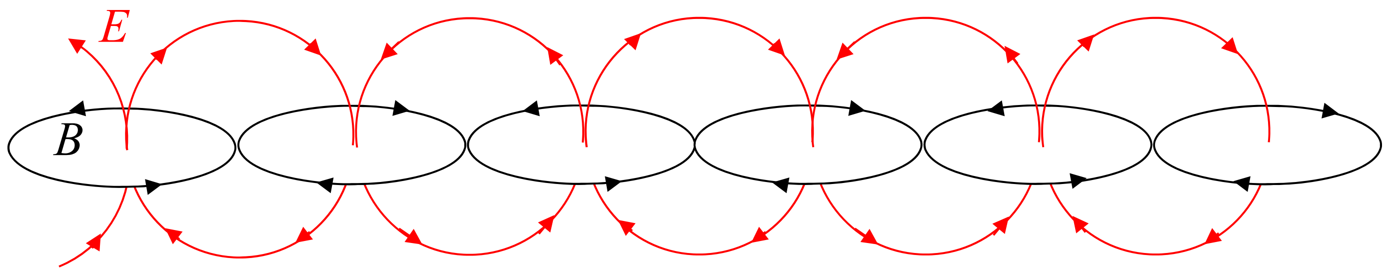
This AC magnetic field induces an AC electric field. This goes on and on…

So, in principle everything is an antenna. Not necessarily a good one.

In many situations we do not consider the “on and on” process, especially for low frequencies.

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t \qquad\qquad \frac{d}{dt}\cos\omega t = -\omega\sin\omega t$$

The “on and on” process is wave propagation.



Somehow start with a changing electric field E, say $E \propto \sin\omega t$

The changing electric field induces a magnetic field, $B \propto \dfrac{\partial E}{\partial t} \propto \cos\omega t$

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \qquad \mathbf{D} \equiv \varepsilon_0\varepsilon_r\mathbf{E} \equiv \varepsilon\mathbf{E} \qquad \mathbf{B} = \mu\mathbf{H} = \mu_r\mu_0\mathbf{H}$$

As the induced magnetic field is changing with time, it will in turn induce an electric field

$E \propto -\dfrac{\partial B}{\partial t} \propto \sin\omega t$

$\boxed{\text{Notice that } \dfrac{d}{dt}\cos\omega t = -\omega\sin\omega t}$

Negative signs cancel

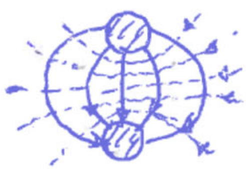
$$\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

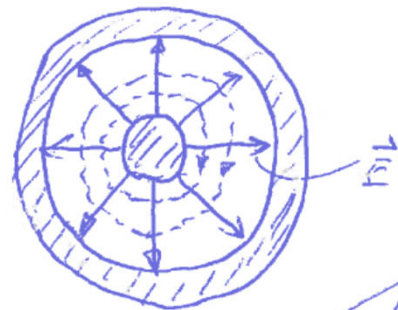
And on and on....

Just as the mechanical wave on a string.

Electromagnetic wave propagation is a consequence of dynamic electromagnetic fields, and is therefore ubiquitous.

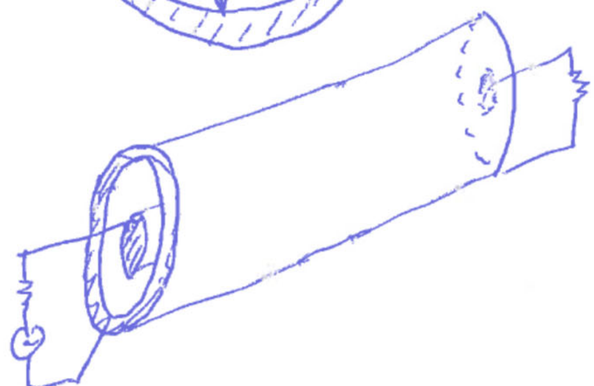
Recall the transmission lines.



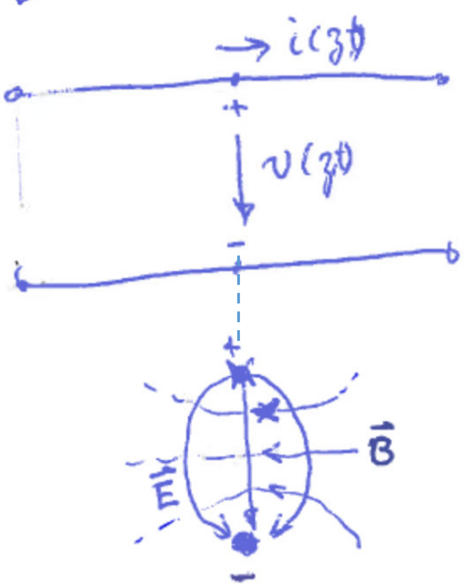


A transmission line is made of two "wires", or two conductors.

Recall boundary conditions. Electric field lines start/end at conductor surfaces, where there is charge.

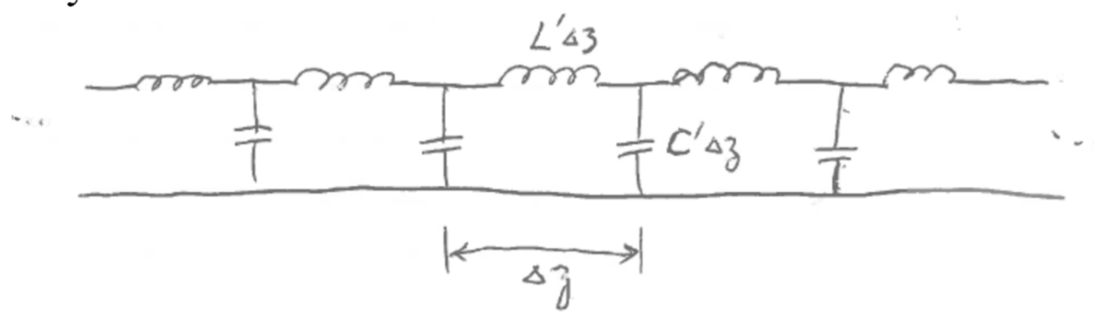


Local voltage $v(z)$ can be defined at location z. $v(z)$ is simply the integral of the **E** field from one conductor to the other.

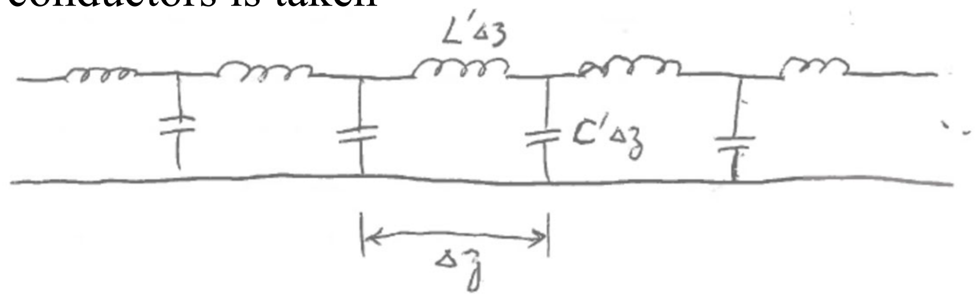


Local current $i(z)$ can also be defined at location z. $i(z)$ is simply the loop integral of the **H** field around a wire.

The electromagnetic field between the two conductors is taken care of by a distributed circuit model:



The electromagnetic field between the two conductors is taken care of by a distributed circuit model:



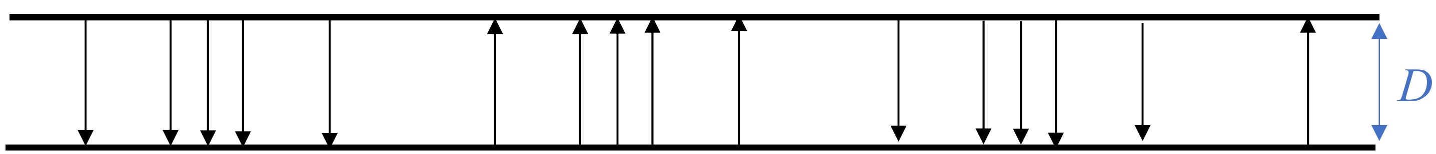
From this circuit model, we derived two formally identical partial differential equations of voltage $v(z)$ and current $i(z)$ – the telegrapher's equations:

$$\left.\begin{aligned}\frac{\partial^2 v}{\partial z^2} &= L'C'\frac{\partial^2 v}{\partial t^2}\\ \frac{\partial^2 i}{\partial z^2} &= L'C'\frac{\partial^2 i}{\partial t^2}\end{aligned}\right\}$$

Let $v_p = \dfrac{1}{\sqrt{L'C'}}$, we have $\dfrac{\partial^2 v}{\partial z^2} = \dfrac{1}{v_p^2}\dfrac{\partial^2 v}{\partial t^2}$

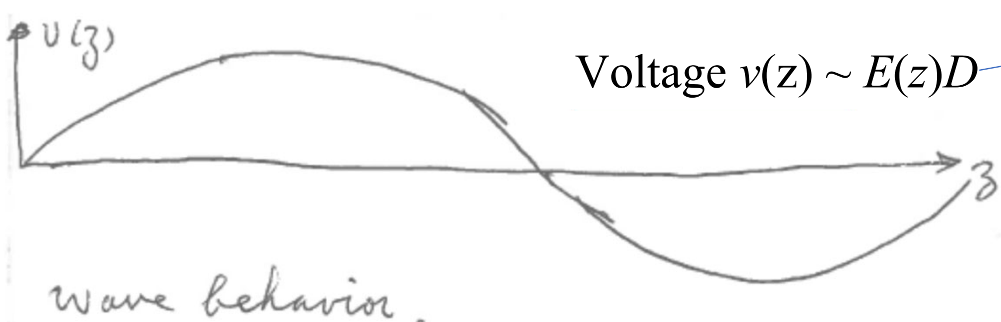
and formally identical equation of $i(z)$: $\dfrac{\partial^2 i}{\partial z^2} = \dfrac{1}{v_p^2}\dfrac{\partial^2 i}{\partial t^2}$

The solutions to these wave equations are voltage and current waves.



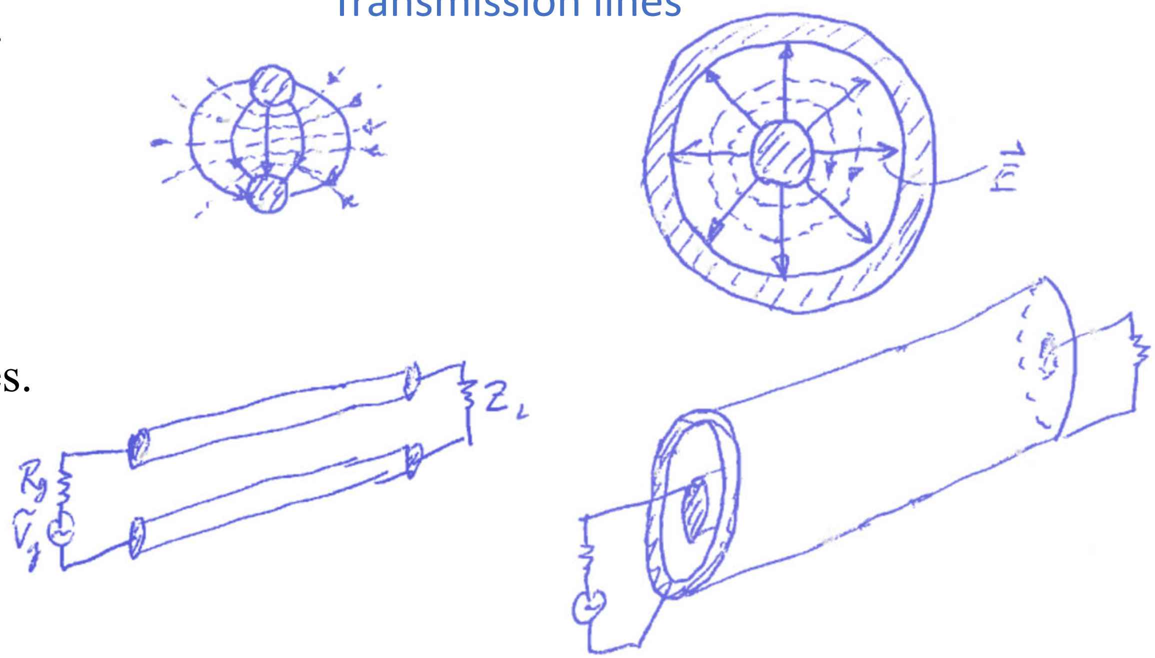
distance

Voltage $v(z) \sim E(z)D$



Transmission lines are waveguides. The two conductors confine the electromagnetic field, and therefore the wave propagate along the longitudinal direction.

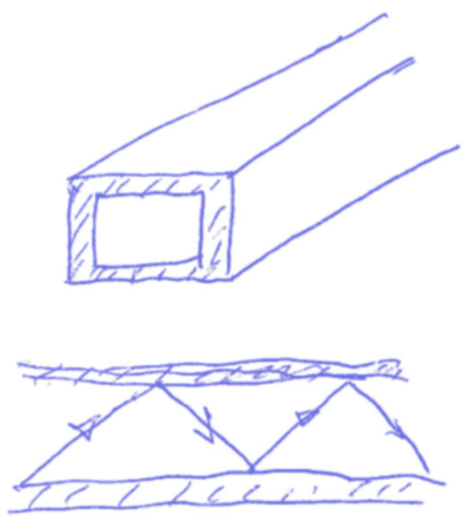
Transmission lines



There are other types of waveguides.

In general, you do not need two conductors to guide an EM wave.

A metal tube is a waveguide:

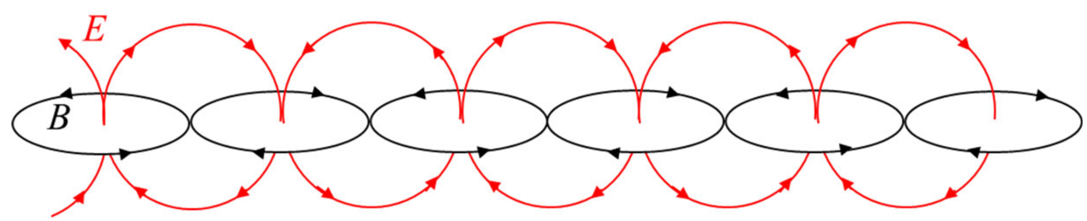


Rectangular waveguide

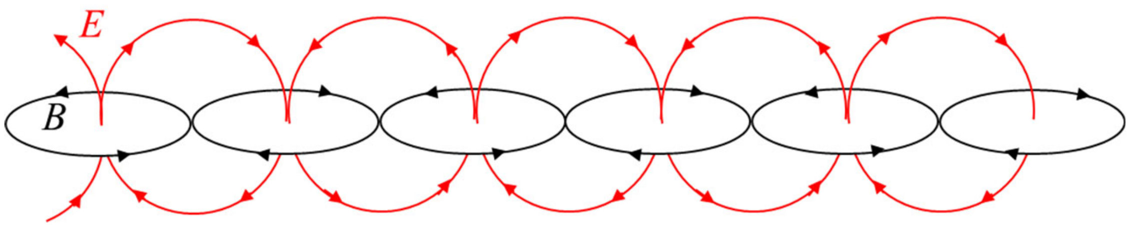
Here, you cannot define local voltages and currents.

You may imagine a very coarse ray optics picture: metal walls are like mirrors. But this is not accurate. Ray optics breaks down when waveguide dimensions are comparable to the wavelength.

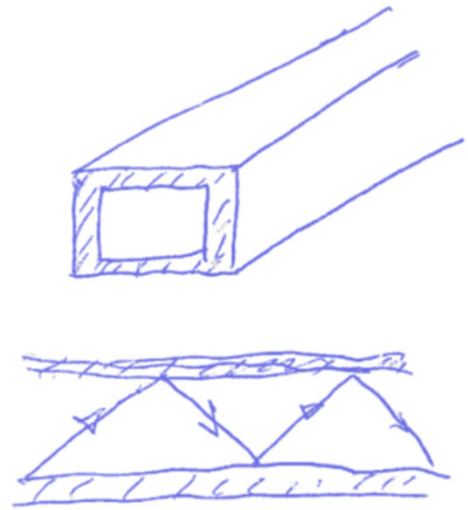
The electromagnetic wave also propagates in free space:



For waveguides that are not transmission lines and for free space, we cannot define even local voltages $v(z,t)$ and local currents $i(z,t)$.



EM wave in free space



Rectangular waveguide

We must resort to the "real" electromagnetic field theory.

$$\boxed{i = C\frac{\mathrm{d}v}{\mathrm{d}t}}$$

$$\boxed{v = L\frac{\mathrm{d}i}{\mathrm{d}t}}$$

Changing voltage → current

Changing current → voltage

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

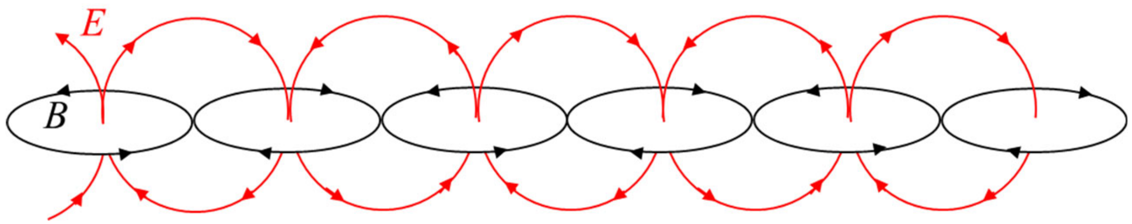
The changing electric field induces a magnetic field.

The changing magnetic field induces an electricfield.

By doing similar math (describing the coupling), we can work out similar wave equations of the fields.

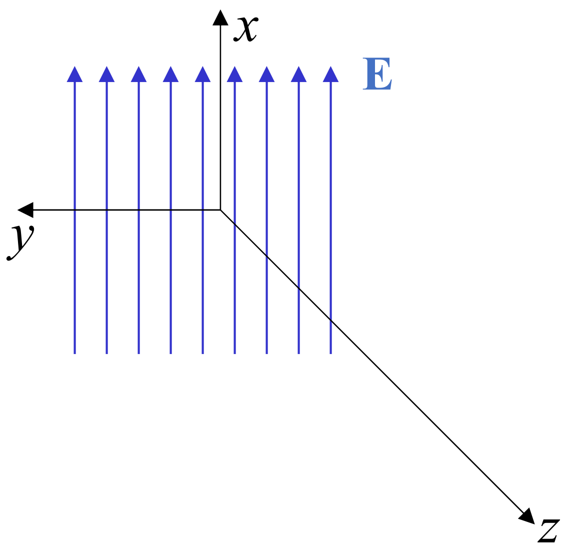
Just a bit more complicated, since fields are vectors.

Now, we use the simplest case to illustrate the electromagnetic field theory of waves.



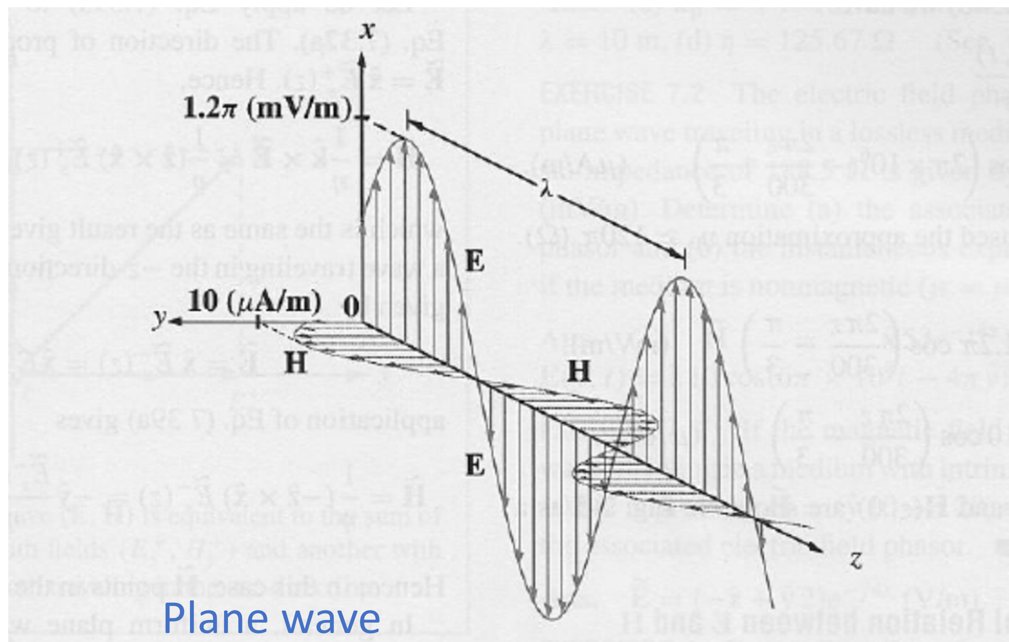
General case,
more complicated.
(Not plane wave)

To make the math simple, we assume infinitely large wave fronts (and source)



The wave fronts are parallel to the *x*-*y* plane.

No variation with regard to *x* or *y*, thus one-dimensional (1D) problem – simple math.



Plane wave

Figure 7-5: Spatial variations of **E** and **H** at $t = 0$ for the plane wave of Example 7-1.

A familiar picture you have seen before

Actually it's more like this:

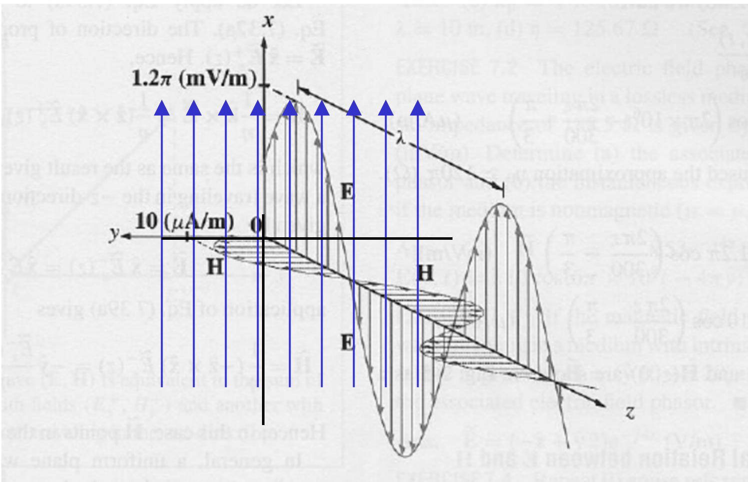
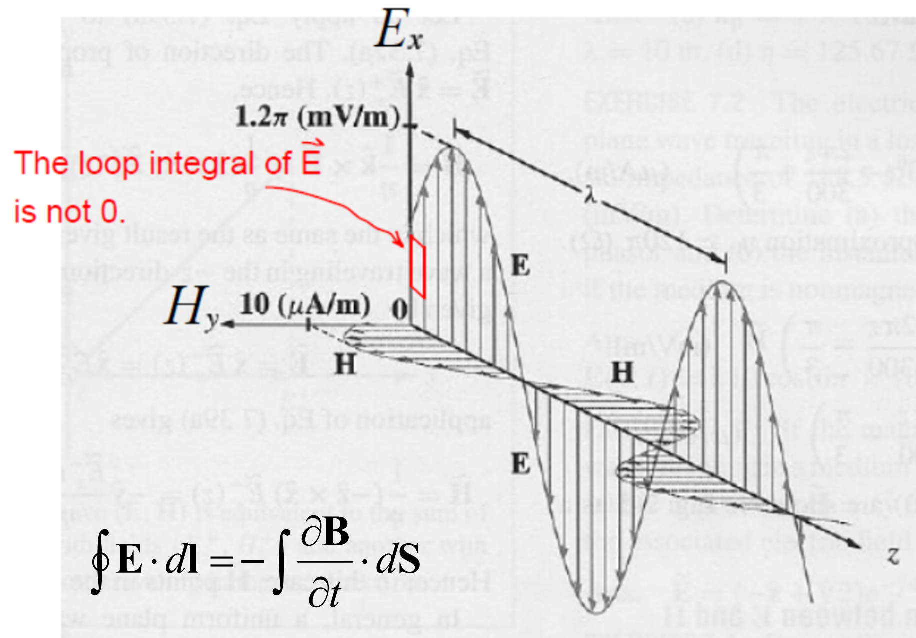


Figure 7-5: Spatial variations of E and H at $t = 0$ for the plane wave of Example 7-1.

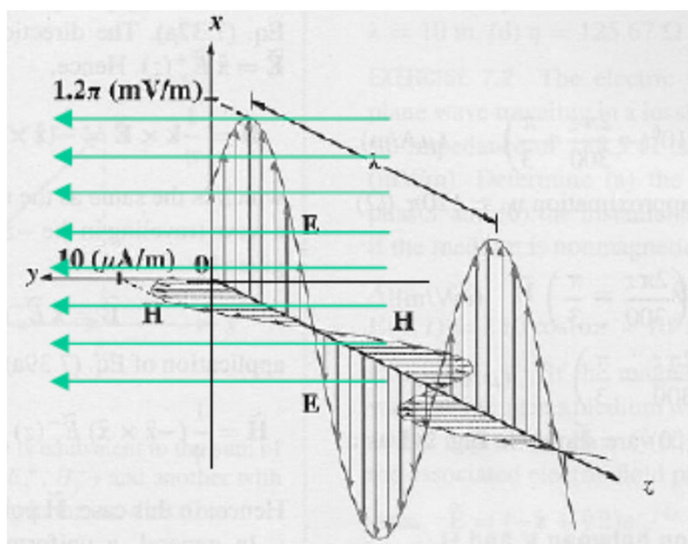


Thus there must be **B** field in *y* direction:

Take-home messages:

The EM plane wave is a transverse wave.

E ⊥ **H**.



This is the big picture of plane waves.

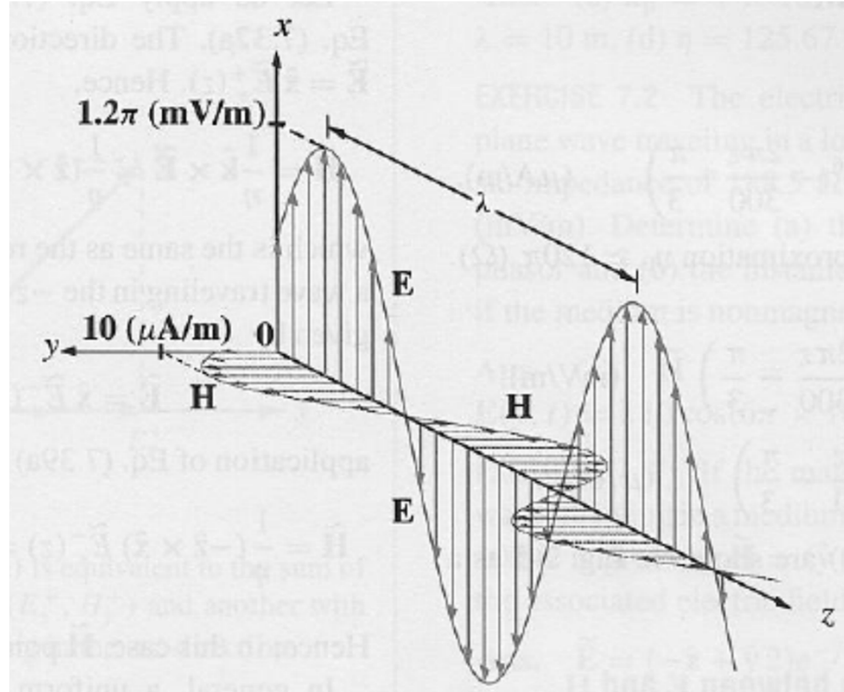
We will go through the formal math in the following.

We have the freedom to call the propagation direction z.

We assume no variation with regard to x or y.

$$\frac{\partial}{\partial x} \to 0, \quad \frac{\partial}{\partial y} \to 0 .$$

$$\nabla \to \hat{z}\frac{\partial}{\partial z} , \quad \nabla^2 \to \frac{\partial^2}{\partial z^2}$$



Now we prove that the EM plane wave is a transverse wave, i.e. **E** and **H** are parallel to the x-y plane.

$$\boxed{\oint \mathbf{H}\cdot d\mathbf{l} = \int \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}}$$

The changing electric field induces a magnetic field.

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{H} = \hat{z}\frac{\partial}{\partial z} \times (\hat{x} H_x + \hat{y} H_y + \hat{z} H_z)$$

$$= \hat{z}\times\hat{x}\frac{\partial H_x}{\partial z} + \hat{z}\times\hat{y}\frac{\partial H_y}{\partial z}$$

$$= \hat{y}\frac{\partial H_x}{\partial z} - \hat{x}\frac{\partial H_y}{\partial z}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{H} = \hat{z}\frac{\partial}{\partial z} \times (\hat{x} H_x + \hat{y} H_y + \hat{z} H_z)$$

$$= \hat{z} \times \hat{x} \frac{\partial H_x}{\partial z} + \hat{z} \times \hat{y} \frac{\partial H_y}{\partial z}$$

$$= \hat{y} \frac{\partial H_x}{\partial z} - \hat{x} \frac{\partial H_y}{\partial z}$$

Alternatively, $\nabla \times \vec{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \hat{x}\left(-\frac{\partial H_y}{\partial z}\right) - \hat{y}\left(-\frac{\partial H_x}{\partial z}\right)$

Anyway, you see $\nabla \times \vec{H}$ is in the *x*-*y* plane, i.e., $(\nabla \times \vec{H}) \cdot \hat{z} = 0$. i.e. $\nabla \times \vec{H} \perp \hat{z}$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \Rightarrow \quad \frac{\partial \vec{E}}{\partial t} \cdot \hat{z} = 0 \quad , \quad \text{i.e.} \quad \frac{\partial \vec{E}}{\partial t} \perp \hat{z} \quad \Rightarrow$$

$$\frac{\partial E_z}{\partial t} = 0 \quad \Rightarrow \quad E_z = \text{Constant}$$

If there is E_z, it does not change, therefore is not part of the wave, but just a DC background.

Therefore, plane wave $\mathbf{E}(z,t)$ is a transverse wave.

The above is better visualized considering the integral form of the equation:

$$\oint \mathbf{H}\cdot d\mathbf{l} = \int \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{S} \qquad \frac{\partial D_z}{\partial t}A = \oint \mathbf{H}_{//}\cdot \mathrm{d}\mathbf{l} = 0$$

Loop in the *x*-*y* plane

Similarly, we can show $\mathbf{H}(z,t)$ is a transverse:

$$\boxed{\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

The changing magnetic field induces an electricfield.

$$\nabla\times\vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \Rightarrow \frac{\partial H_z}{\partial t} = 0 \Rightarrow$$ H_z not part of wave.

Better visualization using the integral form of the equation:

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \qquad \frac{\partial B_z}{\partial t}A = \oint \mathbf{E}_{//}\cdot \mathrm{d}\mathbf{l} = 0$$

Now we have shown $\mathbf{E}\perp\hat{\mathbf{z}}$ and $\mathbf{H}\perp\hat{\mathbf{z}}$, i.e. the EM plane wave is a transverse wave.

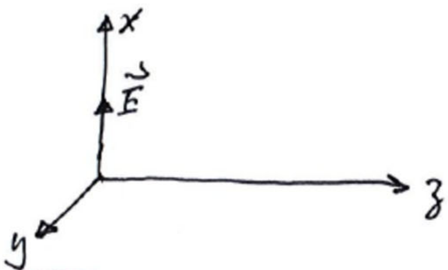
Note: Not all EM waves are plane waves.

We have yet to prove **E** ⊥ **H**.

Now that **E** ⊥ $\hat{\mathbf{z}}$ and **H** ⊥ $\hat{\mathbf{z}}$, We can call the direction of **E** the *x* direction.

$\vec{E} = \hat{x} E_x \equiv \hat{x} E.$ We now have the freedom to drop the subscript *x* in E_x.

Then, $\frac{\partial \vec{E}}{\partial t} = \hat{x} \frac{\partial E}{\partial t}$



We know that

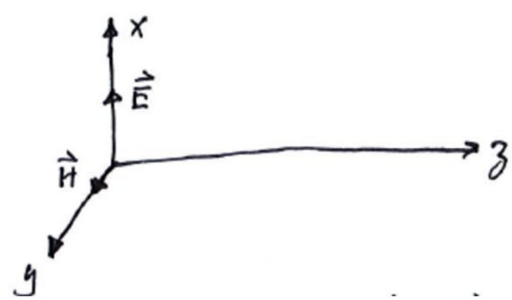
$$\varepsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} = \boxed{-\hat{x} \frac{\partial H_y}{\partial z}} + \hat{y} \frac{\partial H_x}{\partial z}.$$

And, $$\varepsilon \frac{\partial \vec{E}}{\partial t} = \boxed{\varepsilon \hat{x} \frac{\partial E}{\partial t}} \longrightarrow \boxed{\varepsilon \frac{\partial E}{\partial t} = -\frac{\partial H_y}{\partial z}} \quad (1)$$

$$\therefore \frac{\partial H_x}{\partial z} = 0 \Rightarrow \boxed{H_x = 0}$$

$$\vec{H} = \hat{y} H_y \equiv \hat{y} H$$

$$\boxed{\vec{H} \perp \vec{E}}$$



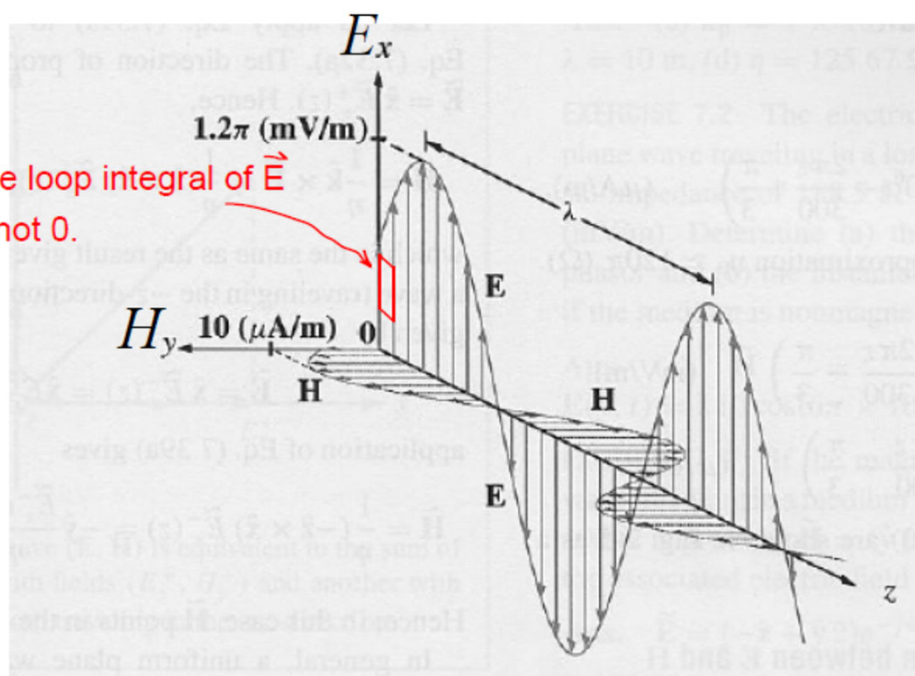
Similarly, we can use $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$ to get $H_x = 0$ How? Try to do it.

and the by-product $\boxed{\frac{\partial E}{\partial z} = -\mu \frac{\partial H}{\partial t}}$ (2)

We now have the freedom to drop the subscript *y* in H_y.

Recall this picture:

The loop integral of $\vec{E}$ is not 0.



$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\frac{\partial B_y}{\partial t} A = \oint \mathbf{E} \cdot \mathrm{d}\mathbf{l}$$

Loop in *x-z* plane as shown

Thus there must be **B** field in *y* direction:

Once again, better visualization using the integral form of the equation.

Now, we make use of the two "by-product" equations to derive the wave equation.

$$(1):\quad \varepsilon\frac{\partial E}{\partial t} = -\frac{\partial H}{\partial z} \Rightarrow \begin{cases} \varepsilon\dfrac{\partial^2 E}{\partial t^2} = -\dfrac{\partial^2 H}{\partial z\,\partial t} & (3) \\ \varepsilon\dfrac{\partial^2 E}{\partial z\,\partial t} = -\dfrac{\partial^2 H}{\partial z^2} & (4) \end{cases}$$

$$(2):\quad \mu\frac{\partial H}{\partial t} = -\frac{\partial E}{\partial z} \Rightarrow \begin{cases} \mu\dfrac{\partial^2 H}{\partial t^2} = -\dfrac{\partial^2 E}{\partial z\,\partial t} & (5) \\ \mu\dfrac{\partial^2 H}{\partial z\,\partial t} = -\dfrac{\partial^2 E}{\partial z^2} & (6) \end{cases}$$

$$\left.\begin{matrix}(3)\\(6)\end{matrix}\right\} \Rightarrow \frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon\mu}\frac{\partial^2 E}{\partial z^2} \equiv v_p^2\frac{\partial^2 E}{\partial z^2}$$

$$\left.\begin{matrix}(4)\\(5)\end{matrix}\right\} \Rightarrow \frac{\partial^2 H}{\partial t^2} = \frac{1}{\varepsilon\mu}\frac{\partial^2 H}{\partial z^2} \equiv v_p^2\frac{\partial^2 H}{\partial z^2}$$

We have the freedom to drop the subscripts.

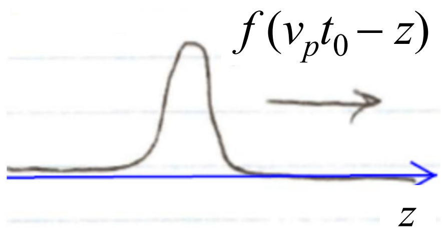
$$\boxed{\begin{gathered} \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 E_x}{\partial t^2}, \qquad \frac{\partial^2 H_y}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 H_y}{\partial t^2} \\ v_p = 1/\sqrt{\varepsilon\mu} = 1/\left(\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_r\mu_r}\right) = \frac{c}{\sqrt{\varepsilon_r\mu_r}} = \frac{c}{n} \end{gathered}}$$

Two formally identical wave equations.

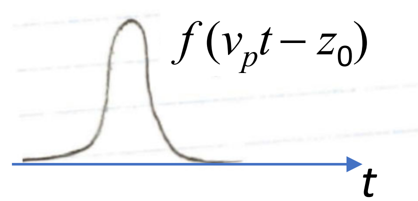
Review along with and compare with the telegrapher's equation.

Recall that for the telegrapher's equation $\dfrac{\partial^2 v}{\partial z^2} = \dfrac{1}{v_p^2}\dfrac{\partial^2 v}{\partial t^2}$

$v = f\,(v_p t - z)$ is a general solution to this equation. Here, $f(\ )$ is an arbitrary function.



Snapshot at $t = t_0$



Waveform at $z = z_0$

What is the other general solution?

Similarly, for the wave equations of E and H, $\dfrac{\partial^2 E_x}{\partial z^2} = \dfrac{1}{v_p^2}\dfrac{\partial^2 E_x}{\partial t^2}$, $\dfrac{\partial^2 H_y}{\partial z^2} = \dfrac{1}{v_p^2}\dfrac{\partial^2 H_y}{\partial t^2}$

E and H each has a general solution in the form $f\,(v_p t - z)$.

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 E_x}{\partial t^2}, \quad \frac{\partial^2 H_y}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 H_y}{\partial t^2}$$

E and H each has a general solution in the form $f\,(v_p t - z)$: waves propagating towards the $+z$.

There are, of course, waves propagating towards the $-z$: another general solution $f\,(v_p t + z)$.

Next, we seek time-harmonic special solutions in the form of

$$E_x(z, t) = \mathrm{Re}\left[\tilde{E}_x(z)\, e^{j\omega t}\right]$$

as we did with transmission lines.

What is the function f for this kind of special solution?

$$E_x(z,t) = \mathrm{Re}\left[\tilde{E}_x(z)\, e^{j\omega t}\right]$$

$$\frac{\partial}{\partial t} \to j\omega\,,$$

$$\frac{\partial^2}{\partial t^2} = -\omega^2$$

Convert ordinary differential equations to algebraic equations, and partial differential equations to ordinary differential equations.

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 E_x}{\partial t^2} \Rightarrow \frac{d^2 \tilde{E}_x}{dz^2} = -\frac{1}{v_p^2}\,\omega^2 \tilde{E}_x$$

$$\text{Let } k = \frac{\omega}{v_p}$$

Wave vector.
Will explain why it's called the wave **vector**.

$$k = \frac{\omega}{v_p} = \omega\sqrt{\varepsilon\mu}$$

$$\Rightarrow \frac{d^2 \tilde{E}_x}{dz^2} = -k^2 \tilde{E}_x$$

This is the same equation as we solved for the voltage or current of the transmission line. Therefore the same solution:

$$\tilde{E}_x = E_{x_0}^+ e^{-jkz} + E_{x_0}^- e^{jkz}$$

What does this mean?

$$\tilde{E}_x = E_{x_0}^{+} e^{-jkz} + E_{x_0}^{-} e^{jkz}$$

Complex amplitudes

What is the corresponding $\mathbf{E}(z,t)$?

This is two waves, characterized by k and $-k$ along the z axis.

More generally, a wave can propagate in any direction, and can be characterized by a vector **k** in the propagation direction. Thus the name wave **vector**.

The directions of **E**, **H**, and **k** follow this right hand rule.

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{k}) \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \phi(\mathbf{k}))$$

$$\tilde{\mathbf{E}}(\mathbf{k}) = \tilde{\mathbf{E}}_0(\mathbf{k}) e^{-\mathbf{k} \cdot \mathbf{r} + \phi(\mathbf{k})}$$

Let's consider a wave propagating in one direction, as we did with transmission lines.

$$\tilde{E}_x = E_{x_0}^+ e^{-jkz}$$

$$\vec{E}(z,t) = \mathrm{Re}\left[\hat{x} E_{x_0}^+ e^{j(\omega t - kz)}\right]$$

$$\vec{E}(z,t) = \hat{x}\left|E_{x_0}^+\right| \cos(\omega t - kz + \phi_0)$$

$$E_{x_0}^+ = \left|E_{x_0}^+\right| e^{j\phi_0}$$

Before we go further, let's get the notations right:
physical quantities vs. phasors, scalars vs. vectors

$$\tilde{\vec{E}}(z) = \hat{x} E_{x_0}^+ e^{-jkz}$$

$$\tilde{E}_x(z) = E_{x_0}^+ e^{-jkz}$$

$$\vec{E}(z,t) = \hat{x}\left|E_{x_0}^+\right| \cos(\omega t - kz + \phi_0)$$

$$E_x(z,t) = \left|E_{x_0}^+\right| \cos(\omega t - kz + \phi_0)$$

Phasors are functions of *z* only.

Of course, the solution for magnetic field *H* is formally the same. $\tilde{H}_y = H_{y_0}^+ e^{-jkz}$

Recall that for the wave in one direction along a transmission line there is a relation between the voltage and current.

Similarly, there is also a definitive relation between *E* and *H*.

$$\left.\begin{array}{l}\nabla \to \hat{z}\dfrac{\partial}{\partial z} \\ \vec{H} = \hat{y}\, H_y\end{array}\right\} \Rightarrow \left.\begin{array}{l}\nabla \times \vec{H} = \varepsilon \dfrac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{H} = -\hat{x}\dfrac{\partial H_y}{\partial z}\end{array}\right\} \Rightarrow -\hat{x}\dfrac{\partial H_y}{\partial z} = \varepsilon \dfrac{\partial \vec{E}}{\partial t} = \hat{x}\,\varepsilon \dfrac{\partial E_x}{\partial t}$$

$$\Rightarrow \left.\begin{array}{l}-\dfrac{\partial H_y}{\partial z} = \varepsilon \dfrac{\partial E_x}{\partial t} \\ \tilde{H}_y = H_{y0}^{+} e^{-jkz}\end{array}\right\} \Rightarrow \cancel{j}k H_{y0}^{+} \cancel{e^{-jkz}} = \varepsilon \cancel{j}\omega E_{x0}^{+} \cancel{e^{-jkz}} \Rightarrow$$

$$\frac{E_{x0}^{+}}{H_{y0}^{+}} = \frac{k}{\varepsilon \omega} = \frac{1}{\varepsilon v_p} = \frac{\sqrt{\varepsilon \mu}}{\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

(Recall that $v_p = \dfrac{1}{\sqrt{\varepsilon \mu}}$)

$$\boxed{\frac{E_{xo}^{+}}{H_{yo}^{+}} = \frac{k}{\varepsilon\omega} = \frac{1}{\varepsilon v_p} = \frac{\sqrt{\varepsilon\mu}}{\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} \qquad \left(\text{Recall that } v_p = \frac{1}{\sqrt{\varepsilon\mu}}\right)}$$

Notice that E_{xo}^{+} and H_{yo}^{+} are "complex amplitude" containing the phase.

$\dfrac{E_{xo}^{+}}{H_{yo}^{+}}$ is real $\Rightarrow$ $E_x(z,t)$ & $H_y(z,t)$ are always in phase.

(for the lossless case; will talk about the lossy case)

And, their ratio is a constant:

$$\boxed{\frac{E(z,t)}{H(z,t)} = \sqrt{\frac{\mu}{\varepsilon}} \equiv \eta}$$

wave impedance

just as in transmission lines: $\dfrac{v(z,t)}{i(z,t)} = Z_o$ characteristic impedance

Both are for a traveling wave going in one direction.

For a visual picture, again look at Figure 7-5 in the textbook:

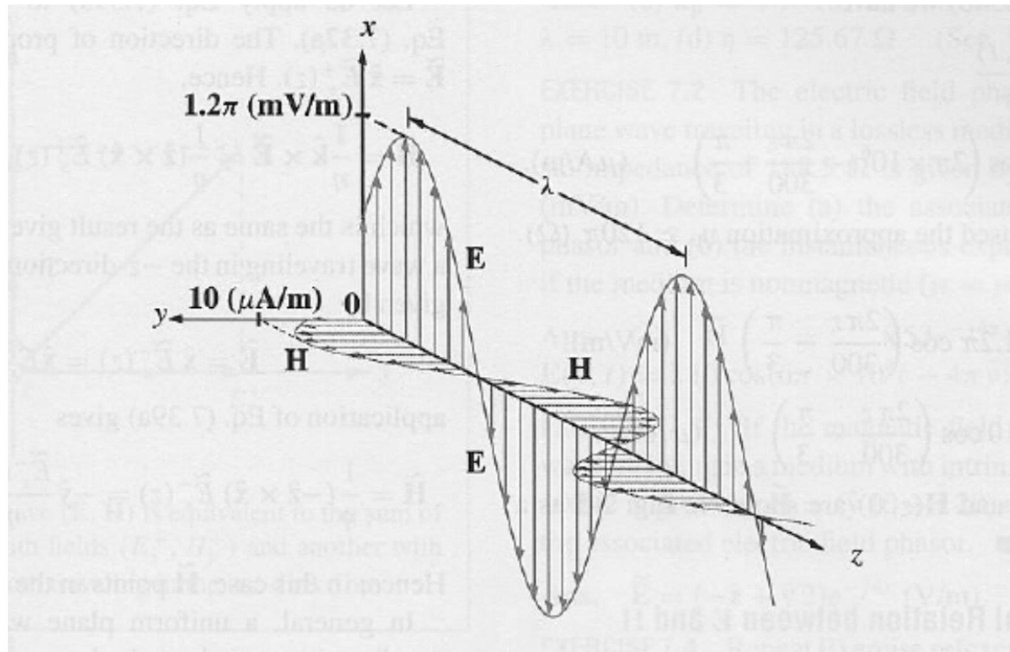


Figure 7-5: Spatial variations of **E** and **H** at $t = 0$ for the plane wave of Example 7-1.

Notice that the wave impedance has the dimension of impedance:

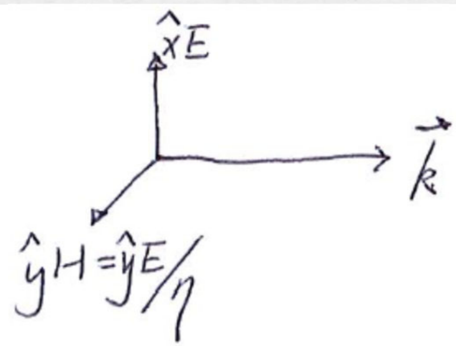
$$\eta = \frac{E}{H}$$

$$\frac{V/m}{A/m} = \frac{V}{A} = \Omega$$

Another way to remember this relation:

$$\frac{E}{B} = \frac{E}{\mu H} = \frac{1}{\mu}\sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{\sqrt{\varepsilon\mu}} = v_p = \frac{c}{n}$$

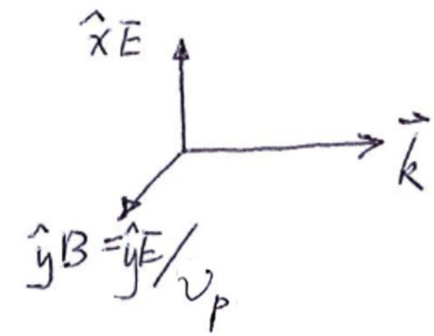
$$\therefore \boxed{E = v_p B}$$



$$1/n = \frac{1}{\sqrt{\varepsilon_r \mu_r}} = \frac{1}{\sqrt{\varepsilon_r}}$$

(for non-magnetic materials, $\mu_r = 1$)

$$n = \sqrt{\varepsilon_r \mu_r} = \sqrt{\varepsilon_r}$$



In free space, $\eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega$

$$v_p = c$$

$$E = cB$$

In a medium,

$$\eta = \sqrt{\frac{\mu_r}{\varepsilon_r}}\,\eta_0 = \sqrt{\frac{1}{\varepsilon_r}}\,\eta_0$$

(for non-magnetic materials, $\mu_r = 1$)

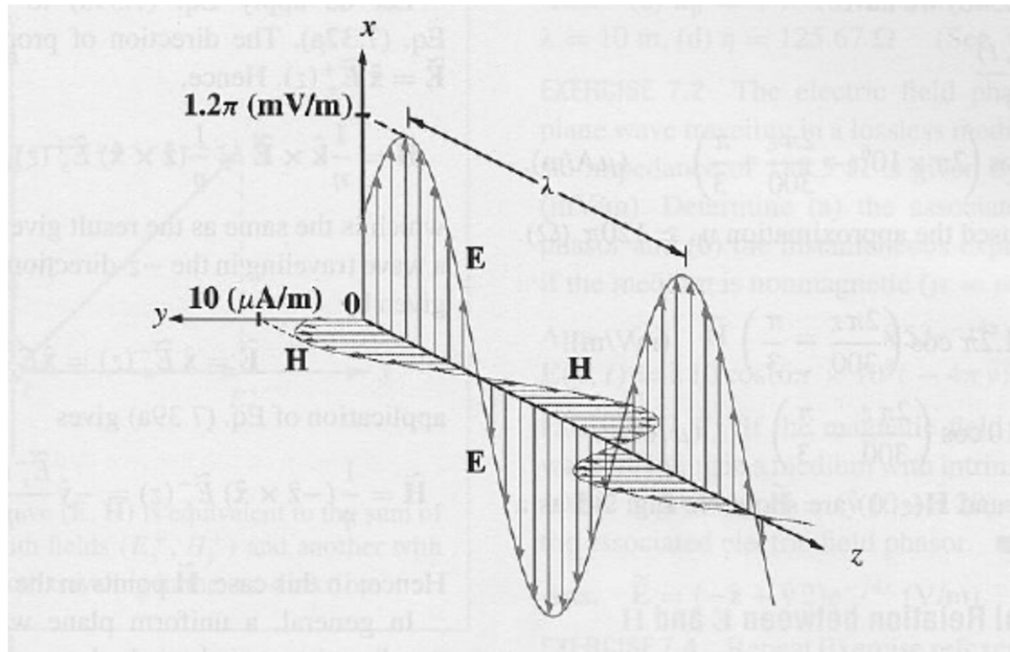
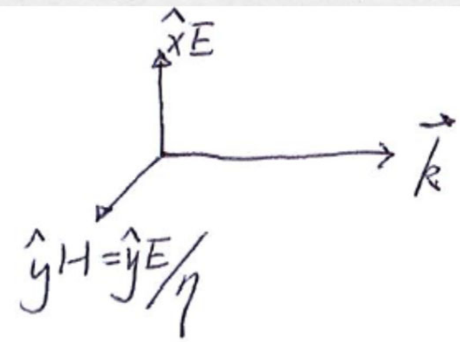


Figure 7-5: Spatial variations of E and H at $t = 0$ for the plane wave of Example 7-1.

The wave impedance is the intrinsic impedance of the medium, similar to the characteristic impedance of the transmission line.

Mismatch ➔ reflection at interface between media



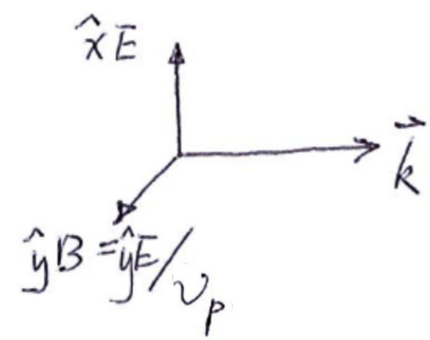
In microwave engineering, we talk about wave impedance.

In optics, we talk about refractive index.

$$1/n = \frac{1}{\sqrt{\varepsilon_r \mu_r}} = \frac{1}{\sqrt{\varepsilon_r}}$$

(for non-magnetic materials, $\mu_r = 1$)

$$n = \sqrt{\varepsilon_r \mu_r} = \sqrt{\varepsilon_r}$$



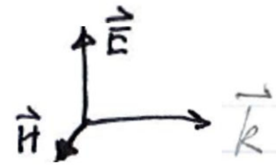
Now you see the relation between wave impedance and refractive index.

$$\eta = \sqrt{\frac{\mu_r}{\varepsilon_r}}\,\eta_0 = \sqrt{\frac{1}{\varepsilon_r}}\,\eta_0 \qquad 1/n = \frac{1}{\sqrt{\varepsilon_r \mu_r}} = \frac{1}{\sqrt{\varepsilon_r}} \qquad n = \sqrt{\varepsilon_r \mu_r} = \sqrt{\varepsilon_r}$$

(for non-magnetic materials, $\mu_r = 1$)

Mismatch ➔ reflection at interface between media

The relation between directions of **E**, **H**, and **k** is independent of the coordinate system. **k** does not have to be in the *z* direction.



Now we also know the ratios between them:

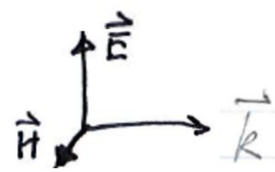
$$\boxed{\frac{E(z,t)}{H(z,t)} = \sqrt{\frac{\mu}{\varepsilon}} \equiv \eta}$$

You can use these equations in the textbook to remember these relations:

$$\tilde{\vec{H}} = \frac{1}{\eta}\,\hat{k} \times \tilde{\vec{E}}$$

$$\tilde{\vec{E}} = -\eta\,\hat{k} \times \tilde{\vec{H}}$$

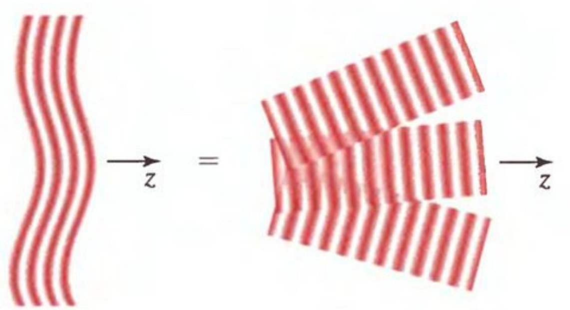
But I find it easier to just remember the right hand rule for directions and the *E/H* ratio being the wave impedance.



We explained why we study time-harmonic waves in transmission lines.

The concept of Fourier transform also applies to the space domain and the "k domain".
k is the spatial equivalent of ω.

While time is 1D, space is 3D.
$\mathbf{k}$ is a vector.



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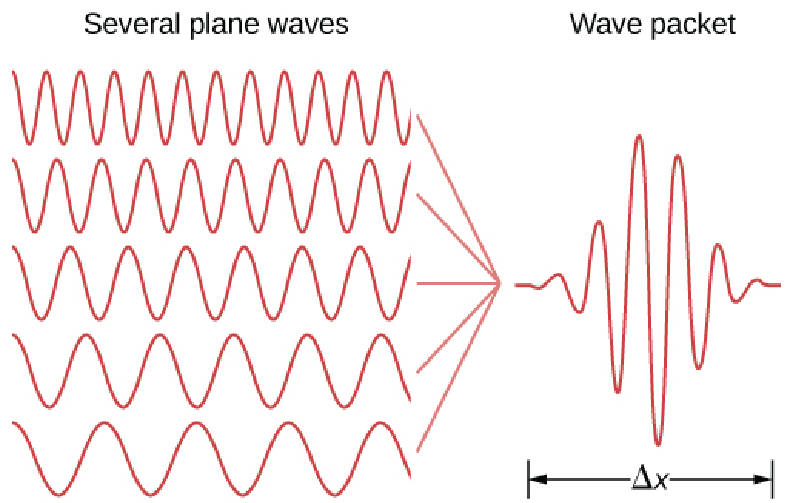
An arbitrary wave in space can be analyzed as superposition of plane waves



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Therefore, any arbitrary wave can be viewed as a superposition of time-harmonic plane waves of various frequencies (wavelengths) propagating in various directions.

Another reason to study plane waves: they are a good approximation in many cases (e.g. sunlight, laser beam).

This familiar picture is the relation between **E**, **H**, and **k** in a lossless medium. It does not apply to all media.

Unlike the textbook, we discuss this simplest case first, and then move on to the more complicated lossy case.

Review these notes, and the introduction of Chapter 7, then Section 7-2.
(The general case in Section 7-1 will be discussed next.)
Do Homework 12 Problems 1, 2.

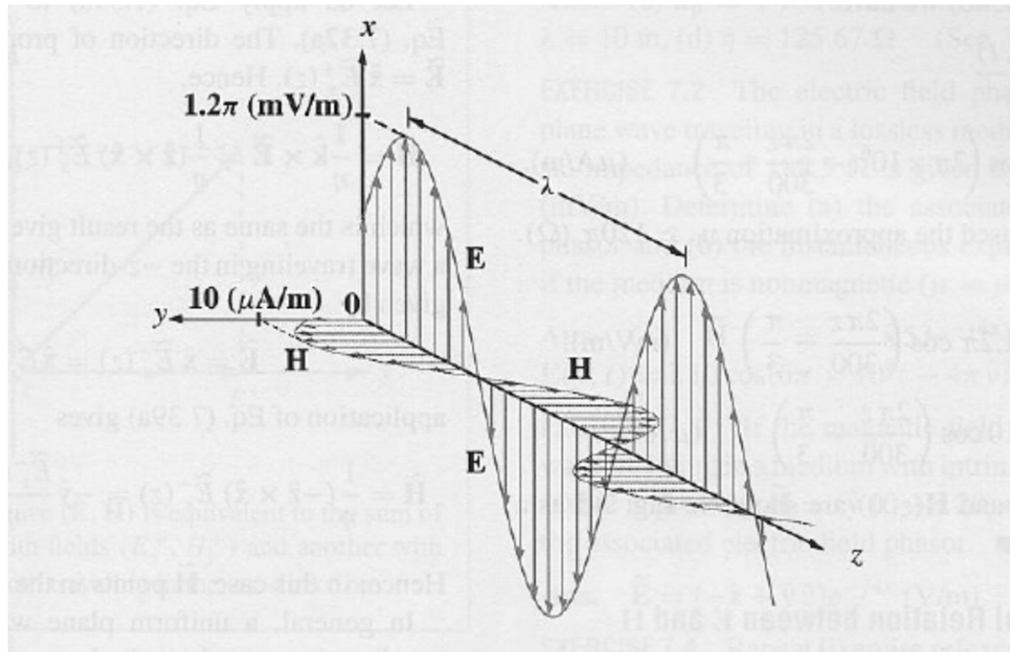
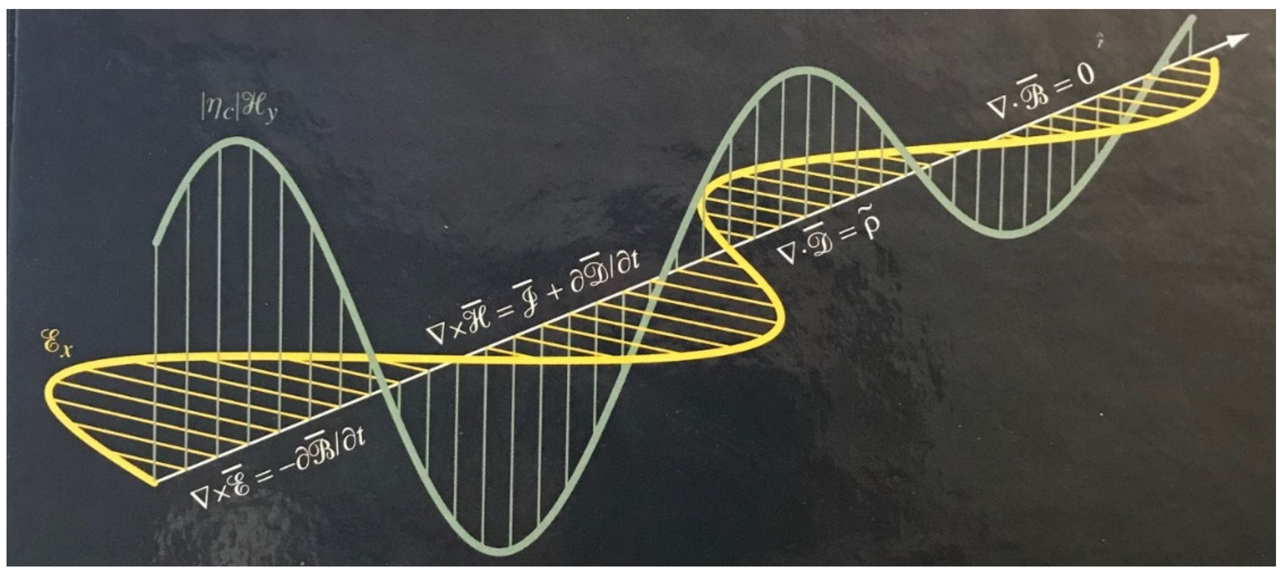


Figure 7-5: Spatial variations of **E** and **H** at $t = 0$ for the plane wave of Example 7-1.

Recall that v and i are not in phase in a lossy transmission line.

Also recall that in the Introduction we showed this (unfamiliar) picture, where E and H of a plane wave are not in phase in a lossy medium.



What causes loss?

Any finite conductivity leads to loss.

For AC, a closed circuit is not necessary.
Damping to dipole oscillation causes loss. (Bound electrons)

The wave in a lossy medium loses a certain percentage of its energy per distance propagated, and therefore decays. In what trend?

Lossless	Lossy
$\begin{cases} \nabla\times\vec{E} = -\mu\dfrac{\partial\vec{H}}{\partial t} \\ \nabla\times\vec{H} = \varepsilon\dfrac{\partial\vec{E}}{\partial t} \end{cases}$	$\begin{cases} \nabla\times\vec{E} = -\mu\dfrac{\partial\vec{H}}{\partial t} \\ \begin{cases} \nabla\times\vec{H} = \vec{J} + \varepsilon\dfrac{\partial\vec{E}}{\partial t} \\ \vec{J} = \sigma\vec{E} \end{cases} \end{cases}$
$\dfrac{\partial}{\partial t} \to j\omega$	$\dfrac{\partial}{\partial t} \to j\omega\ ,\ \tilde{\vec{J}} = \sigma\tilde{\vec{E}}$
	$\nabla\times\tilde{H} = (\sigma + j\omega\varepsilon)\tilde{\vec{E}} = j\omega\left(\varepsilon - j\dfrac{\sigma}{\omega}\right)\vec{E}$
	$\varepsilon_c \equiv \varepsilon - j\dfrac{\sigma}{\omega}$
$\nabla\times\tilde{H} = j\omega\varepsilon\tilde{\vec{E}}$	$\nabla\times\tilde{H} = j\omega\varepsilon_c\tilde{\vec{E}}$

(No Change. In this course we ignore magnetic loss even when considering lossy media)

Real current plus displacement current

Replace ε with ε_c, and you get the lossy case. Everything is "formally" the same. Just keep in mind that ε_c is complex.

Lossless	Lossy
$$\frac{d^2\tilde{E}_x}{dz^2} = -k^2\tilde{E}_x$$	$$\frac{d^2\tilde{E}_x}{dz^2} - \gamma^2\tilde{E}_x = 0$$
$$k = \frac{\omega}{v_p} = \omega\sqrt{\varepsilon\mu} \qquad k^2 = \omega^2\varepsilon\mu$$	$$\gamma \equiv j\omega\sqrt{\mu\varepsilon_c} \qquad \gamma^2 = -\omega^2\mu\varepsilon_c$$ γ is the equivalent of jk; γ^2 the equivalent of $-k^2$
$$\tilde{E}_x = E_{x_0}^+ e^{-jkz} + E_{x_0}^- e^{jkz}$$	$$\tilde{E}_x = E_{x_0}^+ e^{-\gamma z} + E_{x_0}^- e^{\gamma z}$$

For the lossy case, let $\gamma = \alpha + j\beta$

$$\tilde{E}_x = E_{x_0}^+ e^{-\gamma z} + E_{x_0}^- e^{\gamma z}$$

$$\Rightarrow \quad \tilde{E}_x = E_{x_0}^+ e^{-\alpha z} e^{-j\beta z} + E_{x_0}^- e^{\alpha z} e^{j\beta z}$$

What do these two terms mean?

Now we look at the wave propagating in one direction: $\tilde{E}_x = E^+_{x_0} e^{-\alpha z} e^{-j\beta z}$

Recall the wave impedance $\dfrac{E^+_{x_0}}{H^+_{y_0}} = \dfrac{k}{\varepsilon\omega} = \dfrac{\sqrt{\varepsilon\mu}}{\varepsilon} = \sqrt{\dfrac{\mu}{\varepsilon}}$

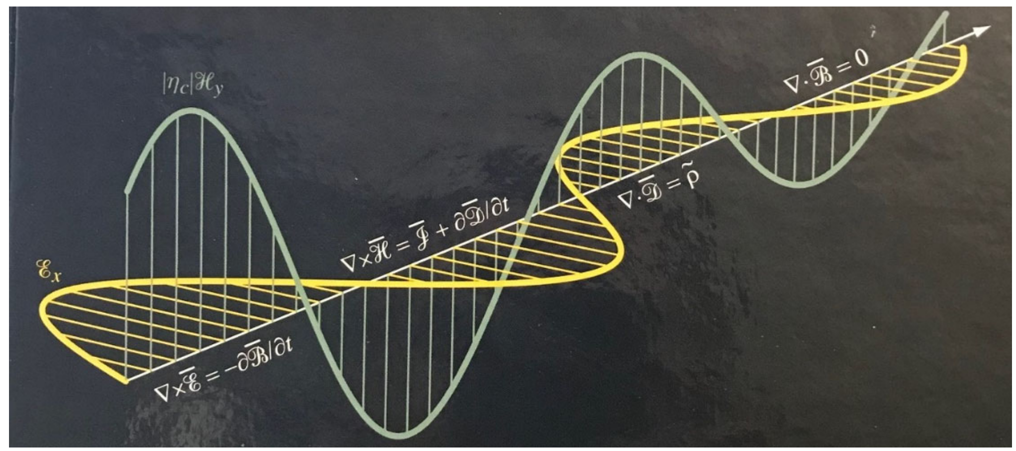
μ and ε are both real and positive $\Rightarrow$

$\dfrac{E^+_{x_0}}{H^+_{y_0}}$ is real $\Rightarrow$ $E_x(z,t)$ & $H_y(z,t)$ are in phase in the lossless case.

In a lossy medium, with ε_c replacing ε,

$E_x(z,t)$ & $H_y(z,t)$ are not in phase.

Keep in mind that ε_c is complex.



Recall the characteristic impedances of lossless and lossy transmission lines.

Origin of the difference between lossless and lossy:

$$\nabla\times\vec{H} = \vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \vec{J} = \sigma\vec{E}$$

$$\nabla\times\tilde{\vec{H}} = (\sigma + j\omega\varepsilon)\tilde{\vec{E}} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\vec{E}$$

Displacement current (always present) is $\pi/2$ out of phase with **E** field.
A $\pi/2$ of phase shift from Faraday's law.
Thus **E** & **H** in phase in lossless case.
Real current (only in lossy media) is in phase with **E**.
Thus **E** & **H** not in phase in lossy case.

We now relate the medium properties σ, μ, and ε to γ

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

$$\gamma = \alpha + j\beta \equiv j\omega\sqrt{\mu\varepsilon_c}$$

$$= j\omega\sqrt{\varepsilon\mu - j\frac{\sigma\mu}{\omega}}$$

$$= j\omega\sqrt{\varepsilon\mu}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

($\sigma \ll \omega\varepsilon$ for good insulators)

$$\approx j\omega\sqrt{\varepsilon\mu}\left(1 - j\frac{\sigma}{2\omega\varepsilon}\right) = \cancel{\omega}\sqrt{\varepsilon\mu}\cdot\frac{\sigma}{2\cancel{\omega}\varepsilon} + j\omega\sqrt{\varepsilon\mu}$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\varepsilon\mu}$$

$$\therefore \boxed{\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}, \quad \beta = \omega\sqrt{\varepsilon\mu}}$$

$$\sigma\uparrow \Rightarrow \alpha\uparrow$$

Compare $\begin{cases} \beta = \omega\sqrt{\varepsilon\mu} & \text{Lossy} \\ k = \omega\sqrt{\varepsilon\mu} & \text{Lossless} \end{cases}$

$$\frac{E_{xo}^{+}}{H_{yo}^{+}} = \frac{k}{\varepsilon \omega} = \frac{\sqrt{\varepsilon\mu}}{\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

Since ε_c is complex, $E_x(z,t)$ & $H_y(z,t)$ are not in phase , with ε_c replacing ε.

$$\varepsilon_c \equiv \varepsilon - j\frac{\sigma}{\omega}$$

In a lossy medium, what is the phase difference between $E_x(z,t)$ & $H_y(z,t)$?

Review these notes, along with textbook Section 7-1.
Finish Homework 12.
Compare the **E** & **H** waves here with the *v* and *i* waves in transmission lines. By doing this, you will gain a good understanding of waves.

Take-home Messages

- Electromagnectic plane waves are transverse waves
  - Not all EM waves are transverse.
- $\mathbf{E} \perp \mathbf{H}$
- $\mathbf{E} \perp \hat{\boldsymbol{k}}$ and $\mathbf{H} \perp \hat{\boldsymbol{k}}$, $\mathbf{k}$ being the wave vector representing the propagation direction

  $k = \omega\sqrt{\varepsilon\mu}$ in lossless media

  This relation independent of choice of coordinate

- Constant ratio between $\widetilde{E}$ and $\widetilde{H}$: wave impedance
- Wave impedance mismatch results in reflection
- Wave impedance real in lossless media, thus E and H in phase
- Wave decays in lossy media; loss due to finite conductivity
- "Complex dielectric constant" used to treat loss; simple expression of decay constant and propagation constant for good insulators
- Due to complex dielectric constant (resulting from real current that is in phase with $\mathbf{E}$), wave impedance of a loss medium is complex
  - Thus E and H not in phase in a lossy medium.

Limitations: Our discussions limited to homogeneous, isotropic, dispersionless, and non-magnetic ($\mu_r = 1$) media.

End of Semester

- Review all notes, listed textbook sections. Review homework
- Review the first ppt – Introduction
- Review the course contents as a whole, and relate to other subjects
- Strive to gain true understanding (necessary condition for an A in this course)
- Answer questions I raised in class but did not answer (all in notes)
- Finals: EM field theory (contents after Test 1) weighs much more, but there will be transmission line problems. Transmission line problems will not involve detailed work; they test your understanding of the most basic essence.
- Work on the Project. Get something out of it (beyond earning the points)

Thank you!