Plane Waves

We studied waves with the transmission line theory. Now, we will study EM waves with the real field theory.

A major difference between the two is the following:

The transmission line theory defines local voltage \( v(z) \) & local current \( i(z) \).

Although the propagation of voltage & current waves are studied, \( v \) & \( i \) are scalars.

In the field theory, we study the \( \vec{E} \) & \( \vec{B} \) fields, which are vectors.

In many situations, we cannot define even local voltages and currents.
Now, we start to discuss waves from the field point of view.

In chapter 7, we talk about plane waves, which means the wave fronts are infinite planes. (1D)

Strict plane waves don’t exist in the physical world. We discuss them just for the sake of mathematical simplicity.

This discussion serves two purposes. First, we can illustrate some important concepts by using very simple math. The analysis of more physically realistic waves are more mathematically involved.

Second, it describes some physical wave phenomenon. For example, if we are very far from the source of a spherical wave, it can be very well approximated by a plane wave. We can regard sunlight as a plane wave. Another example is that we can view some guided waves as plane waves being bounced.

Read the introduction to Ch 7 before section 1 to get some feeling of the physical picture, which is more important in real life than the equations.

Then stop there, but remember to read 7-1 later.
Take good notes!

I'll teach you in a different way than the book does. I'll give you the picture first, and then the relatively simple math treatment for a simple case, and finally the general case, which is mathematically more complicated.

The book put the cart in front of the horse. It begins with the most general case, which is absolutely correct, but mathematically too complicated. You will be inundated by the math and will be lost if you follow this approach.

That's not the way our brains learn things. Once again, let's look at the big physical picture of EM waves, which I showed a couple of times already.

\[
\nabla \times \mathbf{E} = \mu \left( \nabla \times \mathbf{H} + \frac{\partial \mathbf{D}}{\partial t} \right) = \mu (\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t})
\]

\[
\nabla \times \mathbf{B} = -\frac{\partial \mathbf{D}}{\partial t}
\]

The source could be a capacitor, could be a wire.
Take-home messages:
1. The EM plane wave is transverse
2. $\mathbf{E} \perp \mathbf{H}$
We are talking about plane waves.
Let's call the propagation direction \( z \).
Then, by the definition of a plane wave, the \( \vec{E} \) and \( \vec{H} \) fields should be uniform in any \( x\text{-}y \) plane.

\[
\frac{\partial}{\partial x} \to 0, \quad \frac{\partial}{\partial y} \to 0, \quad \nabla \to \hat{z} \frac{\partial}{\partial z}, \quad \nabla^2 \to \frac{\partial^2}{\partial z^2}
\]

(Talk about two types of "1D")

Let's first work on this equation:

\[
\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (A \ \text{changing} \ \vec{E} \ \text{induces an} \ \vec{H})
\]

(What's the integral form?)

\[
\nabla \times \vec{H} = \hat{z} \frac{\partial}{\partial z} \times (\hat{x} H_x + \hat{y} H_y + \hat{z} H_z)
= \hat{z} \times \hat{x} \frac{\partial H_x}{\partial z} + \hat{z} \times \hat{y} \frac{\partial H_y}{\partial z}
= \hat{y} \frac{\partial H_x}{\partial z} - \hat{x} \frac{\partial H_y}{\partial z}
\]

Alternatively,

\[
\nabla \times \vec{H} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
0 & 0 & \frac{\partial}{\partial z} \\
H_x & H_y & H_z
\end{vmatrix} = \hat{x} \left( -\frac{\partial H_y}{\partial z} \right) - \hat{y} \left( -\frac{\partial H_x}{\partial z} \right)
\]

Anyway, we get \( (\nabla \times \vec{H}) \cdot \hat{z} = 0 \), i.e. \( \nabla \times \vec{H} \perp \hat{z} \)

If a field is constant in the \( xy \)-plane, any loop integral in the plane is zero.
\[ \frac{\partial \vec{E}}{\partial t} \cdot \hat{z} = 0 \], i.e. \[ \frac{\partial \vec{E}}{\partial t} \perp \hat{z} \]

\[ \frac{\partial E_z}{\partial t} = 0 \Rightarrow E_z = \text{Constant} \]

If there is \( E_z \), it does not change
\[ \Rightarrow \text{not part of the wave,} \]
\[ \text{just a dc background.} \]

Therefore, the plane wave \( \vec{E}(z, t) \) is a transverse wave.

Similarly, using \( \nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} \),
we get \[ \frac{\partial H_z}{\partial t} = 0 \Rightarrow H_z = \text{Constant} \].

\( \vec{H}(z, t) \) is also transverse.

\[ \therefore \text{The EM plane wave is a transverse wave.} \]

Not all EM waves are transverse.
(We have yet to prove $\vec{E} \perp \vec{H}$.)

Now that $\vec{E} \perp \hat{z}$, $\vec{H} \perp \hat{z}$, let's call the direction of $\vec{E}$ the $x$ direction, i.e., $\vec{E} = \hat{x} E_x \equiv \hat{x} \vec{E}$.

Then, $\frac{\partial \vec{E}}{\partial t} = \hat{x} \frac{\partial \vec{E}}{\partial t}$.

We know that

$$\varepsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} = -\hat{x} \frac{\partial H_y}{\partial z} + \hat{y} \frac{\partial H_x}{\partial z}$$

And,

$$\varepsilon \frac{\partial \vec{E}}{\partial t} = \varepsilon \hat{x} \frac{\partial \vec{E}}{\partial t}$$

$$\therefore \frac{\partial H_x}{\partial z} = 0 \Rightarrow H_x = 0$$

$$\vec{H} = \hat{y} H_y \equiv \hat{y} \vec{H}$$

$\vec{H} \perp \vec{E}$

Now we know $\vec{E} \perp \vec{H}$.

Similarly, from $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$, we get:

$$\frac{\partial \vec{E}}{\partial z} = -\mu \frac{\partial \vec{H}}{\partial t}$$ (2)
(1): \( \varepsilon \frac{\partial E}{\partial t} = -\frac{\partial H}{\partial z} \) \implies \begin{align*}
\varepsilon \frac{\partial^2 E}{\partial t^2} &= -\frac{\partial^2 H}{\partial z^2} \\
\varepsilon \frac{\partial^2 E}{\partial z \partial t} &= -\frac{\partial^2 H}{\partial z^2}
\end{align*}
(3)

(2): \( \mu \frac{\partial H}{\partial t} = -\frac{\partial E}{\partial z} \) \implies \begin{align*}
\mu \frac{\partial^2 H}{\partial t^2} &= -\frac{\partial^2 E}{\partial z^2} \\
\mu \frac{\partial^2 H}{\partial z \partial t} &= -\frac{\partial^2 E}{\partial z^2}
\end{align*}
(5)

(4) \implies \begin{align*}
\frac{\partial^2 E}{\partial t^2} &= \frac{1}{\varepsilon \mu} \frac{\partial^2 E}{\partial z^2} \equiv \nu_p^2 \frac{\partial^2 E}{\partial z^2}
\end{align*}
(6)

(5) \implies \begin{align*}
\frac{\partial^2 H}{\partial t^2} &= \frac{1}{\varepsilon \mu} \frac{\partial^2 H}{\partial z^2} \equiv \nu_p^2 \frac{\partial^2 H}{\partial z^2}
\end{align*}

\[ \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{\nu_p^2} \frac{\partial^2 E_x}{\partial t^2}, \quad \frac{\partial^2 H_y}{\partial z^2} = \frac{1}{\nu_p^2} \frac{\partial^2 H_y}{\partial t^2} \]

\( \nu_p = \sqrt{\frac{\varepsilon \mu}{c}} = \sqrt{\frac{\varepsilon_0 \mu_0 \varepsilon_r \mu_r}{c}} = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{c}{n} \)

Again, compare to transmission lines:

\[ \frac{\partial^2 v}{\partial z^2} = \frac{1}{\nu_p^2} \frac{\partial^2 v}{\partial t^2}, \quad \frac{\partial^2 i}{\partial z^2} = \frac{1}{\nu_p^2} \frac{\partial^2 i}{\partial t^2} \]
General EM wave equation in 3D

Read on your own (FYI):

Now, we use the mathematical language to describe this wave phenomenon.

$$\nabla \cdot \vec{D} = \rho \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow$$

$$\nabla \cdot \vec{B} = 0 \Rightarrow$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \Rightarrow$$

$$\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E} + \nabla \left( \nabla \cdot \vec{E} \right)$$

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times (\vec{H} \times \frac{\partial \vec{H}}{\partial t}) = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t} = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H})$$

$$= -\varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = -3 \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Let \( c = \frac{1}{\sqrt{\varepsilon \mu}} \).

Similarly, if we work on \( \nabla \times (\nabla \times \vec{H}) \), we will get

$$\nabla^2 \vec{H} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

Recall that \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)

In the 1-D case \( \frac{\partial^2}{\partial y^2} \rightarrow 0, \frac{\partial^2}{\partial z^2} \rightarrow 0 \), we have

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \frac{\partial^2 \vec{H}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

Compare to the transmission line:

$$\frac{\partial^2 \vec{u}}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 \vec{u}}{\partial t^2}, \quad \frac{\partial^2 \vec{i}}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 \vec{i}}{\partial t^2}$$
Back to our plane wave:

Now, the wave equations become

\[
\frac{\partial^2 E_x}{\partial z^2} = -\frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2}, \quad \frac{\partial^2 H_y}{\partial z^2} = -\frac{1}{v^2} \frac{\partial^2 H_y}{\partial t^2}
\]

\[
E_x = E_x(z, t) \text{ for the plane wave.}
\]

It's very easy to show that \( E_x = f(x - ct) \),

where \( f() \) is any function.

\[
f(x - ct_0)
\]

\[
f(x_0 - ct)
\]

By Fourier transform, a function can be decomposed to harmonic functions of different frequencies. Equivalently, \( f(x - ct_0) \) can be decomposed into different wavelengths of sinusoidal waves.

Therefore, we study time-harmonic fields of a single frequency \( \omega \).

We can always add up all the frequency components to get the total result.

Recall that we use the phasor to study time-

harmonic quantities

\[
E_x(z, t) = \Re \left[ \tilde{E}_x(z) e^{j\omega t} \right]
\]

\[
\frac{\partial}{\partial t} \to j\omega, \quad \frac{\partial^2}{\partial t^2} = -\omega^2.
\]

Convert differential eq's to algebra

eq's. → simple math.
\[
\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 \tilde{E}_x}{\partial x^2} \quad \Rightarrow \\
\frac{d^2 \tilde{E}_x}{dz^2} = -\frac{1}{v_p^2} \omega^2 \tilde{E}_x
\]

Let \( k = \frac{\omega}{v_p} \) — wave vector

\[
k = \frac{\omega}{v_p} = \omega \sqrt{\varepsilon \mu}
\]

\[
\frac{d^2 \tilde{E}_x}{dz^2} = -k^2 \tilde{E}_x
\]

At this point, you can see these are exactly the same wave eq's as we discussed for transmission lines.

Therefore, this eq should have the same solution

\[
\tilde{E}_x = E_{x_0} e^{-jkz} + E_{x_0} e^{jkz}
\]

waves in 2 directions

complex amplitudes

Now, let's look at the simplest case:

A plane wave of angular frequency \( \omega \), propagating toward the +z direction,

\[
\tilde{E}_x = E_{x_0} e^{-jkz}
\]

\[
\tilde{E}(z,t) = \text{Re} \left[ E_{x_0} e^{j(\omega t - k z)} \right]
\]
\[ \tilde{E}(z, t) = \hat{x}E_{x_0}^+ \cos(\omega t - k z) \]

\[ = \hat{x}E_{x_0}^+ \cos[k(z - \frac{\omega}{k}t)] \]

\[ \rightarrow f(z - v_p t); \quad v_p = \frac{\omega}{k} \]

Pay attention to notations:

phasors vs. real physical quantities;
scalars vs. vectors.
\[ \tilde{E}(z) = \hat{x}E_{x_0}^+ e^{-jkz} \] function of \( z \) only.
\[ \tilde{E}_x(z) = E_{x_0}^+ e^{-jkz} \]
\[ E_x(z, t) = |E_{x_0}^+| \cos(\omega t - k z + \phi_0) \]

Now, question: what is the magnetic field?

Recall that \[ \nabla \times \tilde{H} = \varepsilon \frac{\partial \tilde{E}}{\partial t} \],
and \[ \tilde{H} = \hat{y} H_y \] \[ \Rightarrow \nabla \times \tilde{H} = -\hat{x} \frac{\partial H_y}{\partial z} \]

\[ \therefore -\hat{x} \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial \tilde{E}}{\partial t} = \hat{x} \varepsilon \frac{\partial E_x}{\partial t} \]

\[ -\frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t} \]

Using phasors, let \[ \tilde{H}_y = H_{y_0}^+ e^{-jkz} \]
\[- \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t} \quad \Leftrightarrow \quad jk H_y^+ e^{-jks} = \varepsilon j \omega E_x^+ e^{-jks} \]

\[\frac{E_x^+}{H_y^+} = \frac{k}{\varepsilon \omega} = \frac{1}{\varepsilon \nu_p} = \frac{\sqrt{\varepsilon \mu}}{\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}} \]

(Recall that \( \nu_p = \frac{1}{\sqrt{\varepsilon \mu}} \))

Notice that \( E_x^+ \) & \( H_y^+ \) are "complex amplitudes" containing the phase.

\[\frac{E_x^+}{H_y^+} = \text{real} \quad \Rightarrow \quad E_x(3, t) \& H_y(3, t) \text{ are always in phase.} \]

And, their ratio is a constant

\[\frac{E(3, t)}{H(3, t)} = \sqrt{\frac{\mu}{\varepsilon}} = \eta : \text{wave impedance} \]

Just as for transmission lines,

\[\frac{v(3, t)}{i(3, t)} = Z_0 : \text{characteristic impedance} \]

Both are true for the quantities (E & H or v & i) associated w/ the wave propagating in one direction.
For a visual picture, again look at Fig. 7-5 in the textbook.

Notice that the wave impedance $\eta$ has the dimension of impedance:

$$\eta = \frac{E}{H} \quad \frac{V}{m} = \frac{V}{A} = \Omega$$

Another way to remember this relation:

$$\frac{E}{B} = \frac{E}{\mu H} = \frac{1}{\mu} \frac{1}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\varepsilon \mu}} = v_p$$

$$\therefore \quad E = v_p B$$

In free space, $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377.52$

$$\nu_p = c$$

$$E = cB$$

Mismatch $\implies$ reflection at interface between media

Relation between wave impedance and refractive index
The wave impedance is the intrinsic impedance of the medium, similar to the characteristic impedance of the transmission line.

In other words, to the plane wave, the medium feels like a load of resistance $\eta$.

For the free space, $\eta_0 = \sqrt{\mu_0 / \varepsilon_0} = 377 \Omega$

The free space feels like a resistor 377 $\Omega$.

We can define the wave vector $\mathbf{k} = \hat{z} k$, so $\mathbf{k} = \hat{z}$.

$\mathbf{E}$, $\mathbf{H}$, $\mathbf{B}$ follows the right hand rule.

We can also use the eqs in the book,

$\mathbf{H} = \frac{1}{\eta} \mathbf{k} \times \mathbf{E}$

$\mathbf{E} = -\eta \mathbf{k} \times \mathbf{H}$

In general, you don't need a coordinate.

When you review the content we discussed here, read the introduction before §7.1 first, as I said.

$\rightarrow$ §7.2 $\rightarrow$ §7.1.

Now we discuss the stuff in §7.1.

This way, we put the horse in front of the cart.

We just discussed wave propagation in either free space or a perfect insulator, where the EM field doesn't push any charge around, therefore no energy
is dissipated. Then, the propagation is lossless.

If the medium has a finite conductivity, the E field will push the charge back and forth. The charge being pushed will collide with things in its environment, giving up energy. Therefore, the wave propagation is lossy.

This is the physical picture of a lossy medium that I want you to have in your mind before we go into the math. I don’t want you to be inundated by the math and lose the physical picture.

Now, with this picture, we look at the math.

First, we repeat the treatment of the lossless case using phasors. We treat this case w/ differential eq’s. Now, we study the single frequency case using phasors.

\( \nabla \cdot \mathbf{E} = 0 \) (no charge)

\( \nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \) \[\Rightarrow\] \( \nabla \times \mathbf{E} = -j \omega \mu \mathbf{H} \)

\( \nabla \cdot \mathbf{H} = 0 \)

\( \nabla \times \mathbf{H} = j \omega \varepsilon \mathbf{E} \) \[\Rightarrow\] \( \nabla \times \mathbf{H} = j \omega \varepsilon \mathbf{E} \)

\( \nabla \times (\nabla \times \mathbf{E}) = -j \omega \mu \nabla \times \mathbf{H} = -j \omega \mu \varepsilon \mathbf{E} \)

Recall that \( \nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \)
\[ -\nabla^2 \vec{E} = \omega^2 \mu \varepsilon \vec{E} \]

\[ \nabla^2 \vec{E} = -\omega^2 \mu \varepsilon \vec{E} \]

This is the same as the result we got before,

\[ \nabla^2 \vec{E} = \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

because

\[ \frac{\partial}{\partial t} \rightarrow j\omega, \quad \frac{\partial^2}{\partial x^2} \rightarrow -\omega^2 \]

Recall that \( k = \omega \sqrt{\mu \varepsilon} \), so

\[ \nabla^2 \vec{E} = -k^2 \vec{E} \]

\[ \nabla^2 \vec{E} + k^2 \vec{E} = 0 \]

Now, we look at the lossy case.

Now, \( \nabla \times \vec{H} = \vec{J} + j\omega \mu \vec{E} \)

Recall N's law,

\[ \nabla \times \vec{H} = (\sigma + j\omega \varepsilon) \vec{E} \]

\[ = j\omega (\varepsilon - j\frac{\sigma}{\omega}) \vec{E} \]

We can regard the stuff in the parentheses as a "complex \( \varepsilon \)". Call it \( \varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} \).

So we have \( \nabla \times \vec{H} = j\omega \varepsilon_c \vec{E} \)

\[ \nabla \cdot (\nabla \times \vec{H}) = j\omega \varepsilon_c \nabla \cdot \vec{E} \]

We know \( \nabla \cdot (\nabla \times \vec{H}) = 0 \)

\[ \therefore \nabla \cdot \vec{E} = 0 \]
We only need to change the fourth eq. 
\( \nabla \times \vec{H} = j \omega \vec{E} \),
and it's formally unchanged.
The only difference is that \( \varepsilon_c \) is a complex #. Let's keep this in mind.

Similar to \( k = \omega \sqrt{\mu \varepsilon} \).

\[ \gamma = j \omega \sqrt{\mu \varepsilon_c}, \quad -\varepsilon_c \text{ complex} \]

\[ \gamma^2 = -\omega^2 / \varepsilon_c \]
\[ \gamma^2 \varepsilon_c - \gamma^2 \varepsilon_c = 0 \]

If \( \vec{E} = \hat{x} \vec{E}_x \).

\[
\frac{d^2 \vec{E}_x}{dx^2} - \gamma^2 \vec{E}_x = 0.
\]

Compare this to the lossy transmission line eq.
It's exactly the same equation, so the solution goes like this.

\[ \vec{E}_x = E_{x0} e^{-(\alpha+j\beta)z} = E_{x0} e^{-j\beta z} e^{-\alpha z} \]

\( \gamma = \alpha + j \beta \)

\[ \exp(-\gamma z) \leftrightarrow \exp(-jkz) \]

\( \gamma \) equivalent to \( k \).
How do we relate $\alpha$ (decay constant), $\beta$ to $\sigma$ (conductivity), $\varepsilon$, $\mu$?

$$\gamma = \alpha + j\beta = j\omega \sqrt{\mu\varepsilon},$$

$$\varepsilon_{c} = \varepsilon - j\frac{\sigma}{\omega}$$

$$\therefore \quad \gamma = \alpha + j\beta$$

$$= j\omega \sqrt{\varepsilon\mu} - j\frac{\sigma \mu}{\omega}$$

$$= j\omega \sqrt{\varepsilon\mu} \sqrt{1 - j\frac{\sigma}{2\omega\varepsilon}} \quad (\sigma \ll \omega \varepsilon \text{ for "good" insulators})$$

$$\approx j\omega \sqrt{\varepsilon\mu} \left(1 - j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$= \omega \sqrt{\varepsilon\mu} \cdot \frac{\sigma}{2\omega\varepsilon} + j\omega \sqrt{\varepsilon\mu}$$

$$= \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}} + j\omega \sqrt{\varepsilon\mu}$$

$$\therefore \quad \alpha = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}, \quad \beta = \omega \sqrt{\varepsilon\mu}$$

$$\sigma \ll \omega \varepsilon \Rightarrow \alpha \ll \beta$$

Compare $\beta = j\omega \sqrt{\varepsilon\mu}$ to $k = \omega \sqrt{\varepsilon\mu}$ for the lossless case.