Phasors

Handling the trigonometric functions can be tedious:

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

This is just a simple phase shift, say, \( \alpha = \omega t, \beta = \varphi_0 \).

We have Euler's identity: \( e^{ix} = \cos x + i \sin x \) (pronounced like oyler) that relates trigonometric functions to complex exponentials,

the math of which is a bit simpler:

\[ e^{j(\alpha + \beta)} = e^{j\alpha} e^{j\beta} \]

At least, phase shifting is much easier!

So, we use \( e^{j\alpha} \) to represent \( \cos \alpha \) for mathematical convenience.

But, why can we do this? What’s behind this?
\[ e^{j\omega t} = \cos\omega t + j\sin\omega t \]

The projections of circular motion are harmonic oscillations.

\[ \theta = \omega t + \phi \]

\[ \text{Re} \ e^{j\omega t} = \cos\omega t \]
\[ \text{Im} \ e^{j\omega t} = \sin\omega t \]

For every oscillation \( A\cos(\omega t + \phi) \), we add an "imaginary" partner \( jA\sin(\omega t + \phi) \)

\[ \Rightarrow \quad A e^{j(\omega t + \phi)} = A \cos(\omega t + \phi) + jA \sin(\omega t + \phi) \]

In stead of considering \( A\cos(\omega t + \phi) \), which is mathematically more complicated, we consider \( A e^{j(\omega t + \phi)} \), which is mathematically simpler.

Why can we do this?

— Linearity!

Both \( \cos\omega t \) & \( \sin\omega t \) are solutions of the Wave equation, so is their linear combo.
Based on Euler’s identity, we have a mathematical tool to more easily handle harmonic oscillations for linear systems or media.

It can be a pain to handle cos and sin funcions. So we add an “imaginary partner” to the cosine function:

Say, for a voltage

\[ v(t) = A\cos(\omega t + \phi_0) \rightarrow Ae^{j(\omega t + \phi_0)} \]

\[ = Ae^{j\phi_0}e^{j\omega t} \]

This “rotation” part is always there – a background. So leave it out.

The phasor, or “complex amplitude”

\[ \tilde{V} = Ae^{j\phi_0} = A \angle \phi_0 \]

carries two pieces of information:

- The real amplitude \( A \)
- The reference phase \( \phi_0 \)

Convert a phasor to a time domain function:

\[ \tilde{V} = Ae^{j\phi_0} = A \angle \phi_0 \]

\[ v(t) = \text{Re}(\tilde{V}e^{j\omega t}) \]

1. Put the rotation part, or background, back
2. Take the real part (i.e., throw away the “imaginary partner”)

Note: here we take a “cosine reference.” A different convention is the “sine reference,” where you add and throw away a “real partner.”
Use a phasor to represent a traveling wave

\[ v(x, t) = A \cos(\omega t - \beta x + \phi_0) \rightarrow Ae^{j(\omega t - \beta x + \phi_0)} \]

\[ = Ae^{j(-\beta x + \phi_0)}e^{j\omega t} \]

Leave out

The phasor, or complex amplitude at each position \( x \)

\[ \tilde{V}(x) = Ae^{j(-\beta x + \phi_0)} = A \angle (-\beta x + \phi_0) \]

Notice \( x \) dependence

\[ = (Ae^{j\phi_0})e^{-j\beta x} \]

Propagation factor

“complex amplitude” of wave

For both the time domain function \( v(t) = A \cos(\omega t + \phi_0) \) and the wave \( v(x, t) = A \cos(\omega t - \beta x + \phi_0) \), the time dependence is left out, since we are dealing with a single frequency.
Advantages — an example

\[ v(t) = V_0 \cos \omega t \]

Find \( i(t) \).

(This is a special case of the example of Fig. 1-20 in the textbook with \( \phi_0 = \frac{\pi}{2} \))

\[ \hat{V} = V_0 \]

\[ v(t) = \text{Re} (\hat{V} e^{j\omega t}) = \text{Re} (V_0 e^{j\omega t}) \]

\[ = V_0 \cos \omega t \]

Try to solve \( i(t) \) in the time domain:

\[ \frac{1}{C} \int i(t) \, dt + R i(t) = V_0 \cos \omega t \]

\[ \frac{1}{C} i(t) + R \frac{di(t)}{dt} = -\omega V_0 \sin \omega t \]

— need to solve the differential equation.
A good thing about complex exponentials:
\[
\frac{d}{dt} e^{jwt} = jw e^{jwt}, \quad \int e^{jwt} dt = \frac{1}{jw} e^{jwt}
\]
Differential equations are turned into algebra.
\[
\frac{1}{C} \int i(t) dt + R i(t) = V_0 \cos(wt)
\]
Adding "imaginary partners" to both \(i(t)\) & \(v(t)\):
\[
\frac{1}{C} \int \tilde{I} e^{jwt} dt + R \tilde{I} e^{jwt} = V_0 e^{jwt}
\]
Notice that \(\tilde{I}\) is a "constant" — no \(t\) dependence.
\[
\frac{1}{jwc} \tilde{I} e^{jwt} + R \tilde{I} e^{jwt} = V_0 e^{jwt} \quad \Rightarrow \quad \tilde{I} = \frac{V_0}{R + \frac{1}{jwc}}
\]
Now, relate the phasor back to the time domain:
\[
\frac{1}{R + \frac{1}{jwc}} = \frac{jwc}{1 + jwRC} = \frac{wC e^{j\frac{\pi}{2}}}{\sqrt{1 + w^2R^2C^2}} e^{j\phi_1}
\]
\[
= \frac{wC}{\sqrt{1 + w^2R^2C^2}} e^{j(\frac{\pi}{2} - \phi_1)},
\]
where \(\tan \phi_1 = wRC\), i.e. \(\phi_1 = \tan^{-1}(wRC) = \arctan(wRC)\)
\[ I = \frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}} \ e^{j(\frac{\pi}{2} - \phi)} \]

\[ i(t) = \text{Re} \left( \frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}} \ e^{j(\omega t + \frac{\pi}{2} - \phi)} \right) \]

\[ = \frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}} \ \cos (\omega t + \frac{\pi}{2} - \phi) \]

**Amplitude** \[\frac{\omega CV_0}{\sqrt{1 + \omega^2 R^2 C^2}}\]

**Phase** \[\phi\]

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**Important:**

Instantaneous value \(\leftrightarrow\) phasor

**Table 1-5** (in both 7/E & 6/E of the book)