Phasors

Handling the trigonometric functions can be tedious:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This is just a simple phase shift, say, $\alpha = \omega t$, $\beta = \varphi_0$.

i versus j. We'll talk about this later.

We have Euler's identity: $e^{ix} = \cos x + i\sin x$

(pronounced like oyler)

that relates trigonometric functions to complex exponentials,

the math of which is a bit simpler:

$$e^{j(\alpha+\beta)} = e^{j\alpha}e^{j\beta}$$

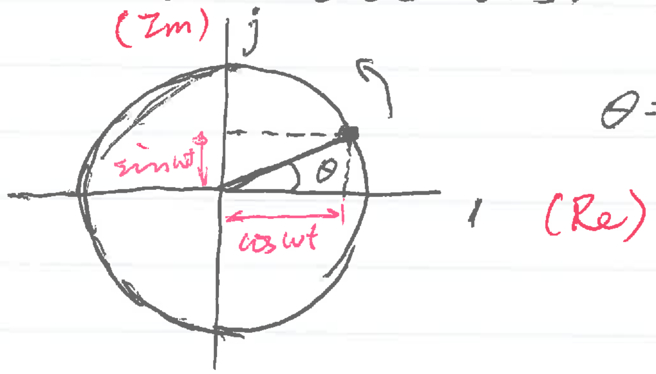
At least, phase shifting is much easier!

So, we use $e^{j\alpha}$ to represent $\cos\alpha$ for mathematical convenience.

But, why can we do this? What's behind this?

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

☆ The projections of circular motion are harmonic oscillations.



$$\theta = \omega t + \phi$$

$$\mathrm{Re}\, e^{j\omega t} = \cos\omega t$$
$$\mathrm{Im}\, e^{j\omega t} = \sin\omega t$$

For every oscillation $A\cos(\omega t + \phi)$, we add an "imaginary" partner $jA\sin(\omega t+\phi)$

$$\Rightarrow \qquad A e^{j(\omega t + \phi)}$$
$$= A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$$

Instead of considering $A\cos(\omega t+\phi)$, which is mathematically more complicated, we consider $Ae^{j(\omega t+\phi)}$, which is mathematically simpler.

Why can we do this?

— Linearity!

Both $\cos\omega t$ & $\sin\omega t$ are solutions of the wave equation, so is their linear combo.

Based on Euler's identity, we have a mathematical tool to more easily handle harmonic oscillations for linear systems or media

It can be a pain to handle cos and sin funcions.
So we add an "imaginary partner" to the cosine function:

Say, for a voltage

$$v(t) = A\cos(\omega t + \phi_0) \quad \rightarrow \quad Ae^{j(\omega t + \phi_0)}$$

$$= Ae^{j\phi_0} e^{j\omega t}$$

This "rotation" part is always there – a background.
So leave it out.

The phasor, or "complex amplitude"

$$\tilde{V} = Ae^{j\phi_0} = A\angle\phi_0$$

carries two pieces of information:

- The real amplitude A
- The reference phase ϕ_0

Convert a phasor to a time domain function:

$$\tilde{V} = Ae^{j\phi_0} = A\angle\phi_0$$

$$v(t) = \mathrm{Re}(\tilde{V} e^{j\omega t})$$

1. Put the rotation part, or background, back

2. Take the real part (i.e., throw away the "imaginary partner")

Note: here we take a "cosine reference." A different convention is the "sine reference," where you add and throw away a "real partner."

Use a phasor to represent a traveling wave

$$v(x, t) = A\cos(\omega t - \beta x + \phi_0) \quad \rightarrow \quad Ae^{j(\omega t - \beta x + \phi_0)}$$

$$= Ae^{j(-\beta x + \phi_0)} e^{j\omega t}$$

Leave out

The phasor, or complex amplitude at each position x

$$\tilde{V}(x) = Ae^{j(-\beta x + \phi_0)} = A\angle(-\beta x + \phi_0)$$

Notice x dependence

$$= (Ae^{j\phi_0}) e^{-j\beta x}$$

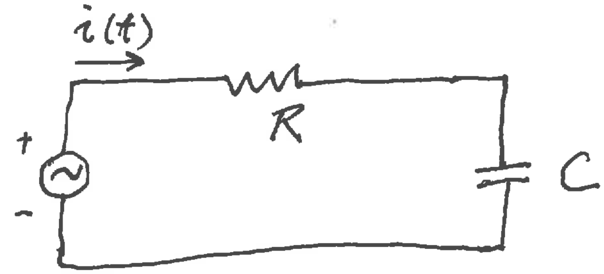
Propagation factor

"complex amplitude" of wave

For both the time domain function $v(t) = A\cos(\omega t + \phi_0)$ and the wave $v(x, t) = A\cos(\omega t - \beta x + \phi_0)$, the time dependence is left out, since we are dealing with a single frequency.

Advantages — an example

$v(t) = V_0 \cos \omega t$



Find $i(t)$.

(This is a special case of the example of Fig. 1-20 in the textbook with $\phi_0 = \frac{\pi}{2}$)

The ϕ_0 defined there, in a sine reference.

$$\boxed{\tilde{V} = V_0}$$

$$v(t) = Re\left(\tilde{V} e^{j\omega t}\right) = Re\left(V_0 e^{j\omega t}\right)$$

$$= V_0 \cos \omega t$$

Try to solve $i(t)$ in the time domain:

$$\frac{1}{C}\int i(t)\,dt + R\,i(t) = V_0 \cos \omega t$$

$$\frac{1}{C}\,i(t) + R\frac{di}{dt} = -\omega V_0 \sin \omega t$$

— need to solve the differential equation.

A good thing about complex exponentials:

$$\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t}, \qquad \int e^{j\omega t}\, dt = \frac{1}{j\omega} e^{j\omega t}$$

Differential equations are turned into algebra.

$$\frac{1}{C} \int i(t)\, dt + R i(t) = V_0 \cos\omega t$$

Adding "imaginary partners" to both $i(t)$ & $v(t)$:

$$\frac{1}{C} \int \tilde{I} e^{j\omega t}\, dt + R \tilde{I} e^{j\omega t} = V_0\, e^{j\omega t}$$

Notice that $\tilde{I}$ is a "constant" — no t dependence

$$\frac{1}{j\omega C} \tilde{I} \cancel{e^{j\omega t}} + R \tilde{I} \cancel{e^{j\omega t}} = V_0 \cancel{e^{j\omega t}} = \tilde{V} \cancel{e^{j\omega t}}$$

$$\boxed{\tilde{I} \left(\tilde{R} + \frac{1}{j\omega C} \right) = V_0 = \tilde{V}} \Rightarrow \tilde{I} = \frac{V_0}{R + \frac{1}{j\omega C}}$$

Now, relate the phasor back to the time domain:

$$\frac{1}{R + \frac{1}{j\omega C}} = \frac{j\omega C}{1 + j\omega RC} = \frac{\omega C e^{j\frac{\pi}{2}}}{\sqrt{1+\omega^2 R^2 C^2}\, e^{j\phi_1}}$$

$$= \frac{\omega C}{\sqrt{1+\omega^2 R^2 C^2}} e^{j\left(\frac{\pi}{2} - \phi_1\right)},$$

where $\tan\phi_1 = \omega RC$, i.e. $\phi_1 = \tan^{-1}(\omega RC) = \arctan(\omega RC)$

$$\therefore \; \tilde{I} = \frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\frac{\pi}{2} - \phi_1)}$$

$$i(t) = Re(\tilde{I} e^{j\omega t})$$

Put the $e^{j\omega t}$ back & take the real part

$$= Re\left[\frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}} e^{j(\omega t + \frac{\pi}{2} - \phi_1)}\right]$$

$$= \underbrace{\frac{\omega C V_0}{\sqrt{1+\omega^2 R^2 C^2}}}_{\text{Amplitude}} \cos(\underbrace{\omega t + \frac{\pi}{2} - \phi_1}_{\text{phase}})$$

Important:

Instantaneous value $\Longleftrightarrow$ phasor

Table 1-5 (in both 7/E & 6/E of the book)