

# Phasors

Handling the trigonometric functions can be tedious:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

This is just a simple phase shift, say,  $\alpha = \omega t$ ,  $\beta = \varphi_0$ .

(pronounced like oylar)

We have Euler's identity:  $e^{ix} = \cos x + i \sin x$  *i versus j. Will talk about this later.*

that relates trigonometric functions to complex exponentials,

the math of which is a bit simpler:

$$e^{j(\alpha+\beta)} = e^{j\alpha} e^{j\beta}$$

At least, phase shifting is much easier!

Wave propagation is phase shifting!  
Take  $\alpha$  as  $\omega t$  and  $\beta$  as  $\phi$ .

So, we use  $e^{j\alpha}$  to represent  $\cos \alpha$  for mathematical convenience.

But, why can we do this? What's behind this?

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

The projections of constant-speed circular motion are harmonic oscillations.

Conversely, the superposition of two harmonic oscillations with a  $\pi/2$  phase difference is a constant-speed circular motion.



$$\theta = \omega t + \phi$$

$$\begin{aligned} \text{Re } e^{j\omega t} &= \cos \omega t \\ \text{Im } e^{j\omega t} &= \sin \omega t \end{aligned}$$

For every oscillation  $A \cos(\omega t + \phi)$

we add an “imaginary partner”  $jA \sin(\omega t + \phi)$



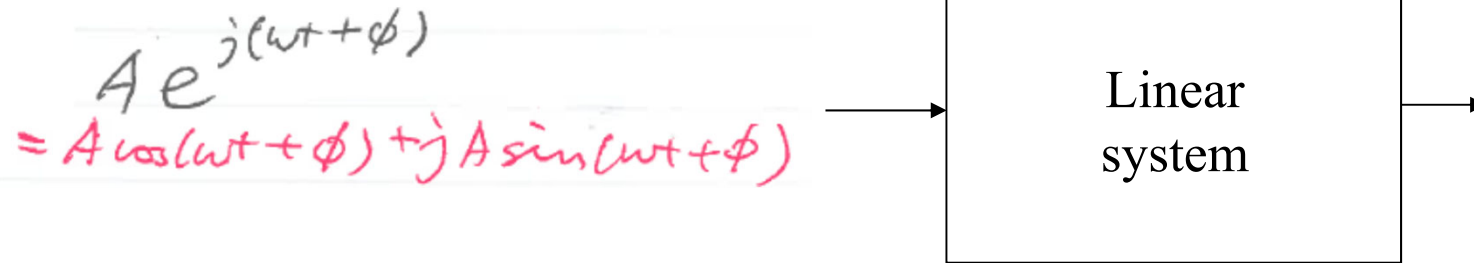
$$\begin{aligned} &A e^{j(\omega t + \phi)} \\ &= A \cos(\omega t + \phi) + j A \sin(\omega t + \phi) \end{aligned}$$

Instead of considering  $A \cos(\omega t + \phi)$ , which is mathematically more difficult to handle, we consider  $A e^{j(\omega t + \phi)}$ , which is easier.

After we solve the problem, we throw away the “imaginary partner,” keeping only the “real” part.

But why are we allowed to do this? What’s the justification?

The answer is “linearity.”



Response to a sum is the sum of responses.

Simple example of nonlinearity:  $f(x) = x^2$

$$(e^{j\omega t})^2 = ?$$

$$(\cos \omega t + j \sin \omega t)^2 = ?$$

$$(\cos \omega t + j \sin \omega t)^2 \neq \cos^2 \omega t + j \sin^2 \omega t$$

Based on Euler's identity, we have a mathematical tool to more easily handle harmonic oscillations for linear systems or **media**.

It can be a pain to handle cos and sin functions.

So we add an "imaginary partner" to the cosine function:

Say, for a voltage

$$\begin{aligned}v(t) = A\cos(\omega t + \phi_0) &\rightarrow Ae^{j(\omega t + \phi_0)} \\ &= Ae^{j\phi_0} e^{j\omega t}\end{aligned}$$

This "rotation" part is always there – a background.  
So leave it out.

The phasor, or "complex amplitude"

$$\tilde{V} = Ae^{j\phi_0} = A\angle\phi_0$$

carries two pieces of information:

- The real amplitude  $A$
- The reference phase  $\phi_0$

Convert a phasor to a time domain function:

$$\tilde{V} = Ae^{j\phi_0} = A\angle\phi_0$$

$$v(t) = \text{Re}(\tilde{V}e^{j\omega t})$$

1. Put the rotation part, or background, back

2. Take the real part (i.e., throw away the “imaginary partner”)

Note: here we take a “cosine reference.” A different convention is the “sine reference,” where you add and throw away a “real partner.”

Use a phasor to represent a traveling wave

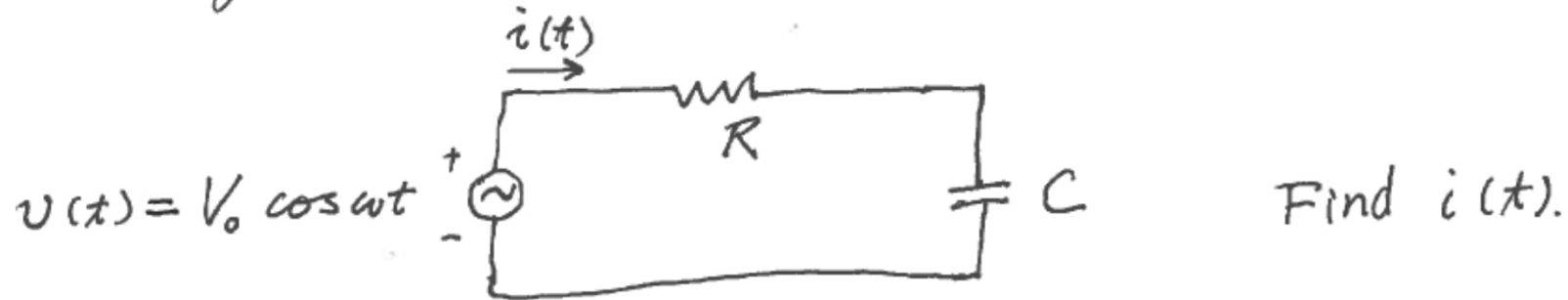
$$\begin{aligned}v(x, t) = A\cos(\omega t - \beta x + \phi_0) &\rightarrow Ae^{j(\omega t - \beta x + \phi_0)} \\ &= Ae^{j(-\beta x + \phi_0)} \underbrace{e^{j\omega t}}_{\text{Leave out}}\end{aligned}$$

The phasor, or complex amplitude at each position  $x$

$$\begin{aligned}\tilde{V}(x) &= Ae^{j(-\beta x + \phi_0)} = A\angle(-\beta x + \phi_0) \\ &\quad \swarrow \text{Notice } x \text{ dependence} \\ &= (Ae^{j\phi_0}) \underbrace{e^{-j\beta x}}_{\text{Propagation factor}} \\ &\quad \uparrow \text{“complex amplitude” of wave}\end{aligned}$$

For both the time domain function  $v(t) = A\cos(\omega t + \phi_0)$  and the wave  $v(x, t) = A\cos(\omega t - \beta x + \phi_0)$ , the time dependence is left out, since we are dealing with a single frequency.

## Advantages — an example



(This is a special case of the example of Fig. 1-20 in the textbook with  $\phi_0 = \frac{\pi}{2}$ )

$$\tilde{V} = V_0$$

↳ the  $\phi_0$  defined there, in a sine reference.

$$v(t) = \text{Re}(\tilde{V} e^{j\omega t}) = \text{Re}(V_0 e^{j\omega t}) = V_0 \cos \omega t$$

Try to solve  $i(t)$  in the time domain:

$$\frac{1}{C} \int i(t) dt + R i(t) = V_0 \cos \omega t$$

$$\frac{1}{C} i(t) + R \frac{di}{dt} = -\omega V_0 \sin \omega t$$

Need to solve an “ordinary” differential equation.



A good thing about complex exponentials:

$$\frac{d}{dt} e^{j\omega t} = j\omega e^{j\omega t}, \quad \int e^{j\omega t} dt = \frac{1}{j\omega} e^{j\omega t}$$

Differential equations are turned into algebra.

$$\frac{1}{C} \int i(t) dt + Ri(t) = V_0 \cos \omega t$$

Adding "imaginary partners" to both  $i(t)$  &  $v(t)$ :

$$\frac{1}{C} \int \tilde{I} e^{j\omega t} dt + R \tilde{I} e^{j\omega t} = V_0 e^{j\omega t}$$

$$\frac{1}{j\omega C} \tilde{I} e^{j\omega t} + R \tilde{I} e^{j\omega t} = V_0 e^{j\omega t} = \tilde{V} e^{j\omega t}$$

$$\frac{1}{j\omega C} \tilde{I} e^{j\omega t} + R \tilde{I} e^{j\omega t} = V_0 e^{j\omega t} = \tilde{V} e^{j\omega t}$$

$$\tilde{I} \left( R + \frac{1}{j\omega C} \right) = V_0 = \tilde{V} \Rightarrow \tilde{I} = \frac{V_0}{R + \frac{1}{j\omega C}}$$

Notice that  $\tilde{I}$  is a "constant" — no  $t$  dependence  
(complex amplitude)

Now, relate the phasor back to the time domain:

$$\frac{1}{R + \frac{1}{j\omega C}} = \frac{j\omega C}{1 + j\omega RC} = \frac{\omega C e^{j\frac{\pi}{2}}}{\sqrt{1 + \omega^2 R^2 C^2} e^{j\phi_1}}$$

$$= \frac{\omega C}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\frac{\pi}{2} - \phi_1)},$$

where  $\tan \phi_1 = \omega RC$ , i.e.  $\phi_1 = \tan^{-1}(\omega RC) = \arctan(\omega RC)$

$$\therefore \tilde{I} = \frac{\omega C V_0}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\frac{\pi}{2} - \phi_1)}$$

$$i(t) = \text{Re}(\tilde{I} e^{j\omega t})$$

*Put the  $e^{j\omega t}$  back & take the real part*

$$= \text{Re}\left[\frac{\omega C V_0}{\sqrt{1 + \omega^2 R^2 C^2}} e^{j(\omega t + \frac{\pi}{2} - \phi_1)}\right]$$

$$= \underbrace{\frac{\omega C V_0}{\sqrt{1 + \omega^2 R^2 C^2}}}_{\text{Amplitude}} \cos\left(\omega t + \underbrace{\frac{\pi}{2} - \phi_1}_{\text{phase}}\right)$$

Finish HW1 before next class meeting (Thu) starts.