Consider a pair of wires when \( f \) is low, \( \lambda \gg l \).

Quasi static.

But, when \( f \) is high, \( \lambda \sim \text{or} < l \).

Wave behavior.

You can actually get to this wave behavior by using circuit theory, w/o considering the details of the EM field!

The parallel plate capacitor is just the simplest case.

There's capacitance between any two pieces of conductors (two plates, two wires, co-ax cable...)

\[
C = C' \Delta z
\]

\( C' \): capacitance per length
A piece of wire is actually an inductor

$$B \propto i$$

When \( i \) changes with \( t \), so does \( B \)

$$\frac{dB}{dt} \rightarrow \frac{di}{dt}$$

$$\mathbf{v} \times \mathbf{E} \propto \frac{dB}{dt} \Rightarrow \mathbf{v} \propto \frac{di}{dt}$$

$$\mathbf{v} = L \frac{di}{dt}$$

A pair of wires coupled but similar,

$$L = L' \delta z$$

\( L' \) is the inductance per length

To make things simple, we first consider a pair of ideal wires:

No resistance, no shunt (leakage)
(Take good notes, different approach than in the textbook)

Now, zoom in on one segment

\[ i(z + \Delta z, t) \quad \rightarrow \quad \frac{L}{\Delta z} \quad \rightarrow \quad i(z + 3z, t) \]

\[ u(z, t) \quad \rightarrow \quad - \Delta u \quad + \quad c \Delta z \quad \rightarrow \quad u(z + 3z, t) \]

\[ \Delta z \]

**Inductor**

\[ \Delta u = u(z + \Delta z) - u(z, t) \]

\[ = -L' \Delta z \frac{\partial i}{\partial t} \]

\[ \frac{\partial u}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta u}{\Delta z} = -L' \frac{\partial i}{\partial t} \]

**Capacitor**

\[ \Delta i = -C' \Delta z \frac{\partial u}{\partial t} \]

\[ \frac{\partial i}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta i}{\Delta z} = -C' \frac{\partial u}{\partial t} \]

\[ \frac{\partial^2 u}{\partial z^2} = -L' \frac{\partial^2 i}{\partial t \partial z} \]

\[ \frac{\partial^2 u}{\partial z \partial t} = -L' \frac{\partial^2 i}{\partial t^2} \]

\[ \frac{\partial^2 i}{\partial z^2} = -C' \frac{\partial^2 u}{\partial t \partial z} \]

\[ \frac{\partial^2 i}{\partial z \partial t} = -C' \frac{\partial^2 u}{\partial t^2} \]

Partial differential equations
What equation do \( v \) & \( i \) follow?

Recall the wave equation from Physics 231?

\[
\frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 v}{\partial t^2}
\]

The wave equation!

Is this amazing?! You get the wave equation from the cat theory? Why this circuit approach works?

There's something more amazing.

We don't go to the EM field details now, but just give you the expressions for \( L' \times C' \); will revisit later. (Table 2-1 on pp. 53)

For a pair of wires:

\[
L' = \frac{\mu}{\pi} \ln \left[ \frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]
\]

distance between 2 wires

\[
C' = \frac{\pi \varepsilon}{\ln \left[ \frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]}
\]

\[v_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu \varepsilon}}, \text{ consistent w/ EMF theory!}\]

\( v = f(z - v_p t) \) is the general solution to this lossless, dispersionless wave equation.

Run the extra mile: Prove this. The shape \( v = f(z - v_p t) \) will propagate i.e. translate, as long as the medium is lossless and dispersionless.

pp. 54 in 7/E
You can check this off-line for other two-conductor transmission lines.

Can you guess the solution to these wave equations? (if the excitation is single-frequency)

\[ v(z, t) = V_0^+ \cos(\omega t - \beta z) \]
\[ i(z, t) = I_0^+ \cos(\omega t - \beta z) \]

\( \frac{\omega}{\beta} = \nu_p \)

A wave could also go the other way.

\[ v(z, t) = V_0^- \cos(\omega t + \beta z) \]
\[ i(z, t) = I_0^- \cos(\omega t + \beta z) \]

Of course, any linear combo is a solution:

\[ v(z, t) = V_0^+ \cos(\omega t - \beta z) + V_0^- \cos(\omega t + \beta z) \]

Same for \( i(z, t) \).

Recall that the sign represents the direction of propagation.

Remember we said last time that sometimes the \( \cos \) functions are a pain? We have a math tool — the phasor —
\[
v(z) \rightarrow V_0^+ e^{j(\omega t - \beta z)} + V_0^- e^{j(\omega t + \beta z)} = V_\beta e^{j\omega t}
\]
\[
V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}
\]
\[
I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z}
\]

\[
V_0^+ = |V_0^+| e^{j\phi_0}
\]

This tool will make our life a lot easier when we deal with more complicated situations.

Here's the more complicated situation:

There's no "ideal" wire. Any wire has some resistance.

\[ R = R' \Delta z \]

\[ R' : \text{resistance per length} \]

There's always some leakage between two conductors in a medium.

The shunt conductance

\[ G = G' \Delta z \]

\[ G' : \text{shunt conductance per length} \]

Notice that \( R' \) is the resistance of the conductors (or wires).

\( G' \) is the shunt conductance between the conductors (wires).

\[ R' \neq \frac{1}{G'} \]
Following the analysis in the book, which is similar to what we just did for the "ideal" case, you will get:

\[
\frac{d^2 \tilde{V}}{dz^2} - (R' + jωL') (G' + jωC') \tilde{V}(z) = 0
\]

Ordinary differential equation

Let's have a small digression back to the "ideal" case:

\[ R' = 0 \quad G' = 0 \]

\[
\frac{d^2 \tilde{V}}{dz^2} + ω^2 L' C' \tilde{V} = 0
\]

\[
\tilde{V}(z) = V_0 e^{±jω\sqrt{L'C'}z}
\]

Recall that

\[
\frac{ω}{β} = \frac{1}{\sqrt{L'C'}} \quad \Rightarrow \quad β = ω\sqrt{L'C'}
\]

\[
\therefore \tilde{V}(z) = V_0 e^{±jβz}
\]

No surprise, this is the result we just got.

Question: What's the difference between the two solutions?
Now, back to the more complicated, real case.

Let \( V^2 = (R' + jwL') (G' + jwC') \)

We then write

\[
\frac{d^2 \tilde{V}}{dz^2} - V^2 \tilde{V}(z) = 0
\]

\[\Rightarrow \tilde{V}(z) = V_0^+ e^{-\nu^+ z} + V_0^- e^{\nu^- z} \]

Similarly, \( \tilde{I}(z) = I_0^+ e^{-\nu^+ z} + I_0^- e^{\nu^- z} \)

\[\nu = \alpha + j\beta \]
\[\alpha = \text{Re} (\nu), \quad \beta = \text{Im} (\nu) \]

There are two \( \nu \)'s. Take the one with positive \( \alpha \).

\[\tilde{V}(z) = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \]

Talk about attenuation. \( R \& G \Rightarrow \text{loss} \).

Now, let's talk about a very important and interesting concept called characteristic impedance.

You may have heard about it. When you buy coax cables or flat lines, people say something like 50 \( \Omega \) or 75 \( \Omega \). What does that mean?

\[
\frac{d\tilde{V}}{dz} = -\nu V_0^+ e^{-\nu^+ z} + \nu V_0^- e^{\nu^- z} \]

\[-\frac{d\tilde{V}}{dz} = (R' + jwL') \tilde{I}(z)\]

\[= (R' + jwL') I_0^+ e^{-\nu^+ z} + (R' + jwL') I_0^- e^{\nu^- z} \]

From circuit,
For the identity to hold for all \( z \),

for the \( e^{\gamma z} \) term:

\[
\gamma V_0^+ = (R' + j\omega L')I_0^+
\]

\[
\Rightarrow \quad \frac{V_0^+}{I_0^+} = \frac{R' + j\omega L'}{\gamma} = \frac{R' + j\omega L'}{\sqrt{(R' + j\omega L')(G' + j\omega C')}} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}
\]

for the \( e^{-\gamma z} \) term:

\[
\gamma V_0^- = -(R' + j\omega L')I_0^-
\]

\[
\Rightarrow \quad \frac{V_0^-}{I_0^-} = -\frac{R' + j\omega L'}{\gamma} = -\frac{R' + j\omega L'}{\sqrt{(R' + j\omega L')(G' + j\omega C')}} = -\sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}
\]

Notice negative signs. Just because of sign convention (see circuit diagram)

Define \( Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \)

the characteristic impedance

Complex and explicitly dependent on frequency in the general (lossy) case.
For the wave traveling towards $+z$,

\[ \widetilde{V}^+ = V_0^+ e^{-\gamma z} \]
\[ \widetilde{I}^+ = I_0^+ e^{-\gamma z} \]

At any $z$,

\[ \frac{\widetilde{V}^+(z)}{\widetilde{I}^+(z)} = \frac{V_0^+}{I_0^+} = Z_0 \]

For the wave traveling towards $-z$,

\[ \widetilde{V}^- = V^- e^{\gamma z} \]
\[ \widetilde{I}^- = I_0^- e^{\gamma z} \]

At any $z$,

\[ \frac{\widetilde{V}^-(z)}{\widetilde{I}^-(z)} = \frac{V_0^-}{I_0^-} = -Z_0 \]

Again, notice this negative sign.

In general, $Z_0 \equiv \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}}$

is complex and explicitly dependent on frequency.

For the lossless transmission line, $R' = 0$ and $G' = 0$,

\[ Z_0 = \sqrt{\frac{L'}{C'}} \]

Real. No explicit frequency dependence.
**Take-home messages**

• Voltage $v$ and current $i$ follow the same differential equation
• Therefore same solution
• Therefore there is a constant ratio between their amplitudes and there is a constant shift between their phases for harmonic waves
• In the phasor form, the complex amplitude ratio is $\pm Z_0$
• Being a voltage/current ration, $Z_0$ has the dimension of impedance
• In general (lossy case), $Z_0$ is complex and explicitly depends on $\omega$
• In the lossless case, $Z_0$ is real w/o explicit frequency dependence

---

**Impedance match**

There is no way to tell the difference just by measuring $v$ and $i$.

Energy propagating away vs. energy dissipated

Analogy: laser beam going to infinity or hitting a totally black wall

The same as the infinitely long line!
If $Z_L = Z_0$, we say its impedance matched—the load gets all the energy from the wave.

What if $Z_L \neq Z_0$? What's gonna happen. The load doesn't get all the energy! Where does the rest of it go?

Consider a laser beam hitting a wall that's not totally black.

\[ V_L = \hat{V}(z=0) = V_0^+ + V_0^- \]
\[ I_L = \hat{I}(z=0) = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} \]

By definition,
\[ Z_L = \frac{\hat{V}_L}{\hat{I}_L} = \left( \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} \right) Z_0 \]

Solve this, and you'll get
The implication of this is...

One-to-one mapping between gamma and impedance.

\[ V_s = \frac{Z_i - Z_o}{Z_i + Z_o} \]

\[ \Gamma = \frac{V_s}{V_o} \]

\[ \Gamma = \frac{V_s}{V_o} = \frac{Z_i - Z_o}{Z_i + Z_o} \]

[Diagram]

Define voltage reflection coefficient.

Recall that for a lossless line, \( Z_o = \sqrt{\frac{V_o}{I_o}} \) is real.

So, in general, \( \Gamma \) is complex.

But \( \Gamma \) is usually complex.

Let's set our work to be...