Consider a pair of wires (ideal wires)

Length $<< \lambda$

Length $>> \lambda$, say, infinitely long

Voltage along a cable can vary!

Voltage $v(z) \sim E(z)D$

We can actually get to this wave behavior by using circuit theory, w/o going into details of the EM fields!
There is capacitance between any two pieces of conductors. A pair of plates, wires, etc., or the core and shields of a co-ax cable.

\[ C = C' \Delta z \]

\( C' \): capacitance per length

A piece of wire is actually an inductor

\[ B \times i \]

When \( i \) changes with \( t \), so does \( B \)

\[ \frac{dB}{dt} \rightarrow \frac{di}{dt} \]

\[ v \times E = \frac{\partial B}{\partial t} \Rightarrow v \times \frac{di}{dt} \]

\[ v = L \frac{di}{dt} \]

A pair of wires, coupled, but similar.

\[ L = L' \Delta z \]

\( L' \) is the inductance per length
To make things simple, we first consider a pair of *ideal* wires. No resistance, no shunt (leakage).

Pay close attention. We take a different approach than does the book.

Now, zoom in on one segment:
Inductor

\[ \Delta v = v(z + \Delta z) - v(z, t) \]
\[ = -L' \Delta z \frac{\partial i}{\partial t} \]

\[ \frac{\partial v}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta v}{\Delta z} = -L' \frac{\partial i}{\partial t} \]

Capacitor

\[ \Delta i = c' \Delta z \frac{\partial v}{\partial t} \]

\[ \frac{\partial i}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta i}{\Delta z} = -c' \frac{\partial v}{\partial t} \]

Take derivatives with regard to \( z, t \)
Let \( v_p = \frac{1}{\sqrt{L'c'}} \), we have
\[
\frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 v}{\partial x^2}
\]

\( v = f(v_p t - z) \) is the general solution to this equation.

Do it on your own: verify this.
Let \( v_p = \frac{1}{\sqrt{L'C'}} \), we have

\[
\frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 v}{\partial x^2}.
\]

\( v = f(v_p t - z) \) is the general solution to this equation.

So, this is the wave equation!
More strictly, the lossless, dispersionless, linear wave equation.
Assume: no resistance, no leakage; \( v_p \) independent of frequency;
\( v_p \) independent of \( v \)

Is this amazing?
We arrived at the wave equation from circuit theory, regardless of frequency.
Why does this approach work?

One more agreement:

\[
L' = \frac{\mu}{\pi} \ln \left[ \frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]
\]

\[
C' = \frac{\pi \varepsilon}{\ln \left[ \frac{D}{d} + \sqrt{\left(\frac{D}{d}\right)^2 - 1} \right]}
\]

\[
\therefore \quad v_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu\varepsilon}}
\]

Consistent with EM theory! Check this offline for other transmission lines.

See Table 2-1, pp. 45 in 7/E, pp.53 in 6/E
\[ \frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2} \frac{\partial^2 v}{\partial x^2} \] (the equation for \(i\) is in the same form – formally the same.)

The wave equation.
More strictly, the lossless, dispersionless, linear wave equation.
Assume: no resistance, no leakage; \(v_p\) independent of frequency; \(v_p\) independent of \(v\)

\[ v = f(v_p t - z) \] is the general solution to this equation.

What are the single frequency, harmonic solutions?
\[ \frac{\partial^2 \nu}{\partial z^2} = \frac{1}{\nu_p^2} \frac{\partial^2 \nu}{\partial x^2} \]

(the equation for \( i \) is in the same form – formally the same.)

\( \nu = f(\nu_p t - z) \) is the general solution to this equation.

Single frequency, harmonic solutions:

\[ \nu(z,t) = |V_0^+| \cos(\omega t - \beta z + \phi_0^+) \]

\[ i(z,t) = |I_0^+| \cos(\omega t - \beta z + \phi_0^+) \]

Here, \( V_0^+ \) and \( I_0^+ \) are “complex amplitudes” that we will talk about later. For the waves, they are not the phasors of the waves. We will talk about the distinction.

\[ |V_0^+| \text{ and } |I_0^+| \] are the real amplitudes, or simply amplitudes.

We have not shown the voltage and current waves are in phase. But they are. You can take this as a conclusion for now. Or, if you are interested, read the proof next page.
Here we show that the voltage and current waves are in phase with each other:

\[ v(z, t) = |V_0^+| \cos(\omega t - \beta z + \phi_{v0}^+) \]

\[ i(z, t) = |I_0^+| \cos(\omega t - \beta z + \phi_{i0}^+) \]

\[ \frac{\partial v}{\partial z} = \beta |V_0^+| \sin(\omega t - \beta z + \phi_{v0}^+) \]

\[ \frac{\partial i}{\partial t} = -\omega |I_0^+| \sin(\omega t - \beta z + \phi_{i0}^+) \]

Recall that

\[ \lim_{\Delta z \to 0} \frac{\Delta v}{\Delta z} = -L' \frac{\partial i}{\partial t} \]

For this to hold for any arbitrary \( z \), we must have \( \phi_{v0}^+ = \phi_{i0}^+ = \phi_0^+ \)

So, in phase!

We also get a by-product:

\[ \frac{|V_0^+|}{|I_0^+|} = \frac{\omega L'}{\beta} = v_p L' \]

Unit: \( \frac{m \cdot H}{s \cdot m} = H/s = \Omega \)

These conclusions are important!

Anywhere, any time \( v(z,t)/i(z,t) = \text{constant} \)
Define \[ \left| \frac{V_0^+}{I_0^+} \right| = \frac{\omega L'}{\beta} = v_p L' \equiv Z_0 \quad (Z_0 \text{ is real, i.e., purely resistive}) \]

Consider:

\[ Z_0 \quad \text{......} \quad \text{vs.} \quad Z_0 \]

Infinitely long

There is no way to tell the difference just by measuring \( v \) and \( i \).

Energy propagating away vs. energy dissipated

Analogy: laser beam going to infinity or hitting a totally black wall

You may also use

\[ \frac{\partial i}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta i}{\Delta z} = -C' \frac{\partial v}{\partial x} \]

Doing the derivatives in a similar way as in last page, you will also see the voltage and current waves are in phase.

You will have a similar “by-product” about the \( v/i \) ratio.

It may look different, but you should be able to show they are equal.

Do it on your own. Hint: use \[ v_p = \frac{1}{\sqrt{L'C'}} \]
\[ v^+(z, t) = |V_0^+| \cos(\omega t - \beta z + \phi_0^+) \]

Rewrite:
\[ i^+(z, t) = |I_0^+| \cos(\omega t - \beta z + \phi_0^+) \]

In what direction do these waves propagate?
Waves propagating the other way are also solutions to the same equations:

\[ v^-(z,t) = |V_0^-| \cos(\omega t + \beta z + \phi^-_0) \]
\[ i^-(z,t) = |I_0^-| \cos(\omega t + \beta z + \phi^-_0) \]

Of course, any linear combinations of waves in opposite directions are also solutions:

\[ v(z,t) = |V_0^+| \cos(\omega t - \beta z + \phi^+_0) + |V_0^-| \cos(\omega t + \beta z + \phi^-_0) \]
\[ i(z,t) = |I_0^+| \cos(\omega t - \beta z + \phi^+_0) + |I_0^-| \cos(\omega t + \beta z + \phi^-_0) \]

They may represent the combinations of incident and reflected waves.

Recall that we have a mathematical tool to
1. Avoid the pain of dealing trigonometric functions, and
2. Turn partial differential equations to ordinary differential equations by putting aside the known time variation.
Express waves with phasors

\[ v(z, t) = |V_0^+| \cos(\omega t - \beta z + \phi_0^+) + |V_0^-| \cos(\omega t + \beta z + \phi_0^-) \]

\[ \tilde{V} = \tilde{V}^+ + \tilde{V}^- = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \]

Where do the phases go?

Recall the details how we convert a time-domain function to a phasor:

By adding an “imaginary partner”

\[ v(z, t) \rightarrow |V_0^+| e^{j(\omega t - \beta z + \phi_0^+)} + |V_0^-| e^{j(\omega t + \beta z + \phi_0^-)} = (|V_0^+| e^{j\phi_0^+}) e^{-j\beta z} + (|V_0^-| e^{j\phi_0^-}) e^{j\beta z} e^{j\omega t} \]

This is not the phasor yet.

Throw away the known time variation \( e^{j\omega t} \)

and define the complex amplitudes \( V_0^+ = |V_0^+| e^{j\phi_0^+} \) and \( V_0^- = |V_0^-| e^{j\phi_0^-} \)

We get the phasor:

\[ \tilde{V} = \tilde{V}^+ + \tilde{V}^- = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \]

Positive going \hspace{1cm} Negative going

Notice sign and direction
Express waves with phasors

The current wave is similar

\[ i(z, t) = |I_0^+| \cos(\omega t - \beta z + \phi_0^+) + |I_0^-| \cos(\omega t + \beta z + \phi_0^-) \]

phasor

\[ \tilde{I} = \tilde{I}^+ + \tilde{I}^- = I_0^+ e^{-j\beta z} + I_0^- e^{j\beta z} \]

This tool makes our life much easier when we deal with a more complicated situation.

Here’s the more complicated situation:

No wires are ideal.
Any wire has some resistance.

\[ R = R' \triangle \delta \]

Resistance per length

There is always some shunt conductance between two wires

\[ G = G' \triangle \delta \]

Shunt conductance per length

Notice that \( R' \) and \( G' \) describes two different things. \( R' \neq 1/G' \)
Analyze the circuit in the phasor way

\[
\frac{d\tilde{V}}{dz} = \lim_{\Delta z \to 0} \frac{\Delta \tilde{V}}{\Delta z} = -(R' + j\omega L')\tilde{I}
\]

\[
\frac{d\tilde{I}}{dz} = \lim_{\Delta z \to 0} \frac{\Delta \tilde{I}}{\Delta z} = -(G' + j\omega C')\tilde{V}
\]

Take derivatives on one equation and insert it into the other, you get

\[
\frac{d^2\tilde{V}}{dz^2} - (R' + j\omega L') (G' + j\omega C') \tilde{V}(z) = 0
\]

and a formally same equation for the current.

This is an ordinary differential equation. Because we used phasors.

We have arrived at this equation just by circuit analysis using phasors. You could also first to the circuit analysis in the time domain, arriving at partial differential equations, and then convert quantities to phasors and arrive at the same ordinary differential equations, as done in the book (Sections 2-3 & 2-4).
The partial differential equations for the general, more complicated situation are, well, too complicated. We don’t even bother to tackle them. Let’s look at the simpler ordinary differential equation:

\[
\frac{d^2 \hat{V}}{d\tilde{z}^2} - (R' + j\omega L') (G' + j\omega C') \hat{V}(\tilde{z}) = 0
\]

Before solving this equation, let’s first have a digression back to the ideal case

\[
R' = 0 \quad G' = 0
\]

\[
\frac{d^2 \hat{\bar{V}}}{d\tilde{z}^2} + \omega^2 L' C' \hat{\bar{V}} = 0 \quad \Rightarrow \quad \hat{\bar{V}}(\tilde{z}) = \hat{V}_0^{\pm} e^{\pm j\omega\sqrt{L'C'}}
\]

Recall that

\[
\frac{\omega}{\beta} = v_p = \frac{1}{\sqrt{L'C'}} \quad \Rightarrow \quad \beta = \omega \sqrt{L'C'}
\]

we have

\[
\hat{V} = \hat{V}_0^{\pm} e^{\pm j\beta z}
\]

Notice that these are actually two solutions. What are the difference between the two solutions?
Waves propagating in two opposite directions: \[ \vec{V} = \vec{V}^+ + \vec{V}^- = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \]

No surprise. The solutions we got earlier for the ideal case.

Now, back to the more complicated, general case

\[ \frac{d^2 \vec{V}}{dz^2} - (R' + j\omega L') (G' + j\omega C') \vec{V}(z) = 0 \]

Let \[ \gamma^2 = (R' + j\omega L') (G' + j\omega C') \]

we can write \[ \frac{d^2 \vec{V}}{dz^2} - \gamma^2 \vec{V}(z) = 0 \]

Compare this to the ideal case \[ \frac{d^2 \vec{V}}{dz^2} + \omega^2 L' C' \vec{V} = 0 \]

These two equations are “formally” the same, except \(-\gamma^2\) is complex.

So, the solutions are \[ \vec{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \]

What kind of waves are they?
\[ \tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \]

What kind of waves are they?

Let \( \gamma = \alpha + j \beta \) (we are doing nothing. Any complex number can be written as this), we have the first solution

\[ \tilde{V}^+ = V_0^+ e^{-\gamma z} = V_0^+ e^{-(\alpha + j \beta)z} = V_0^+ e^{-\alpha z} e^{-j \beta z} \]

What is this?

Similarly, \[ \tilde{I}^+ = I_0^+ e^{-\gamma z} = I_0^+ e^{-(\alpha + j \beta)z} = I_0^+ e^{-\alpha z} e^{-j \beta z} \]

Why do the waves attenuate when there is resistance or shunt leakage? (Why is there no attenuation when the wire is made of a perfect conductor and the medium between them is a perfect insulator)?

Recall that, in circuit theory, reactive versus resistive...,
Again, there can be waves going the other way.

\[ \tilde{V} = V_0^+ e^{-\alpha z} e^{-j\beta z} + V_0^- e^{\alpha z} e^{j\beta z} \quad \tilde{I} = I_0^+ e^{-\alpha z} e^{-j\beta z} + I_0^- e^{\alpha z} e^{j\beta z} \]

Now, we discuss the most important concept of the first half of the semester:

\[ \tilde{V} = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{Take derivatives} \quad \frac{d\tilde{V}}{dz} = -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} \]

From circuit analysis

\[ \frac{d\tilde{V}}{dz} = \lim_{\Delta z \to 0} \frac{\Delta \tilde{V}}{\Delta z} = -(R' + j\omega L')\tilde{I} = -(R' + j\omega L')I_0^+ e^{-\alpha z} e^{-j\beta z} - (R' + j\omega L')I_0^- e^{\alpha z} e^{j\beta z} \]

For the identity to hold for all \( z \), we must have the following:

For the \( e^{-\gamma z} \) term:

\[ \gamma V_0^+ = (R' + j\omega L')I_0^+ \]

\[ \Rightarrow \quad \frac{V_0^+}{I_0^+} = \frac{R' + j\omega L'}{\gamma} = \frac{R' + j\omega L'}{\sqrt{(R' + j\omega L')(G' + j\omega C')}} = \frac{\sqrt{R' + j\omega L'}}{G' + j\omega C'} \]
For the $e^{j\omega t}$ term:  
\[ \gamma V_0^- = -(R' + j\omega L') I_0^- \]

\[ \Rightarrow \quad \frac{V_0^-}{I_0^-} = -\frac{R' + j\omega L'}{\gamma} = -\frac{R' + j\omega L'}{\sqrt{(R' + j\omega L')(G' + j\omega C')}} = -\frac{R' + j\omega L'}{\sqrt{G' + j\omega C'}} \]

Notice negative signs. Just because of sign convention (see circuit diagram)

Define  
\[ Z_0 \equiv \frac{\sqrt{R' + j\omega L'}}{\sqrt{G' + j\omega C'}} \]

the characteristic impedance

Complex and explicitly dependent on frequency in the general (lossy) case.
For the wave traveling towards $+z$,

$\tilde{V}^+ = V_0^+ e^{-\gamma z}$

$\tilde{I}^+ = I_0^+ e^{-\gamma z}$

At any $z$, $\frac{\tilde{V}^+(z)}{\tilde{I}^+(z)} = \frac{V_0^+}{I_0^+} = Z_0$

For the wave traveling towards $-z$,

$\tilde{V}^- = V^- e^{\gamma z}$

$\tilde{I}^- = I^- e^{\gamma z}$

At any $z$, $\frac{\tilde{V}^-(z)}{\tilde{I}^-(z)} = \frac{V^-}{I^-} = -Z_0$

Again, notice this negative sign.

In general, $Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$ is complex and explicitly dependent on frequency.

For the lossless transmission line, $R' = 0$ and $G' = 0$,

$Z_0 = \sqrt{\frac{L'}{C'}}$

Real. No explicit frequency dependence.
Take-home messages

- Voltage $v$ and current $i$ follow the same differential equation
- Therefore same solution
- Therefore there is a constant ratio between their amplitudes and there is a constant shift between their phases for harmonic waves going in one direction
- In the phasor form, the complex amplitude ratio is $\pm Z_0$
- Being a voltage/current ration, $Z_0$ has the dimension of impedance
- In general (lossy case), $Z_0$ is complex and explicitly depends on $\omega$
- In the lossless case, $Z_0$ is real w/o explicit frequency dependence
- $Z_0$ being real means the voltage and the current are in phase.
There is no way to tell the difference just by measuring $v$ and $i$.

Energy propagating away vs. energy dissipated

Analogy: laser beam going to infinity or hitting a totally black wall

**Impedance match**

The same as the infinitely long line!

We want impedance match! (Reasons?)
Now, let’s look at a transmission line with a source and a load.

If $Z_L = Z_0$, impedance matched. All energy delivered to load. Good!

What if $Z_L = Z_0$?

The load says $\frac{\tilde{V}(0)}{\tilde{I}(0)} = \frac{\tilde{V}_L}{\tilde{I}_L} = Z_L$

If there is only the incident wave, $\frac{\tilde{V}^+(0)}{\tilde{I}^+(0)} = \frac{V_0^+}{I_0^+} = Z_0$

Something has to happen to resolve this “conflict.” That something is the reflection.

$\tilde{V}_L = \tilde{V}(z=0) = V_0^+ + V_0^-$

$\tilde{I}_L = \tilde{I}(z=0) = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$

By definition, $Z_{rl} = \frac{\tilde{V}_L}{\tilde{I}_L} = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0$

Solve it and we have $V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$
If $Z_L = Z_0$, there has to be a reflection wave.

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

The load does not get all energy carried by the incident wave.

**Where does the rest of the energy go?**

Consider analogy: laser beam hitting wall not totally black/dark.

Define the voltage reflection coefficient

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L}{Z_0} - 1$$

$$\frac{Z_L}{Z_0} = \frac{1 + \Gamma}{1 + \Gamma}$$

- One-to-one mapping between $\Gamma$ and $Z_L/Z_0$
- The ratio $Z_L/Z_0$ more important than $Z_L$ itself
Therefore we define the normalized load impedance \( z_L = \frac{Z_L}{Z_0} \).

Thus,
\[
\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{z_L - 1}{z_L + 1}
\]
\[
z_L = \frac{1 + \Gamma}{1 - \Gamma}
\]

Notice the one-to-one mapping. This is very important!

For the current
\[
\frac{I_0^-}{I_0^+} = - \frac{V_0^-}{V_0^+} = -\Gamma
\]

Where does the negative sign come from?

\[
Z_0 = \sqrt{\frac{L'}{C'}} \quad \text{is real for a lossless line, but}
\]
\[
Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \text{is complex in general.}
\]

\( Z_L \) is complex in general. Thus, \( \Gamma \) is complex in general.