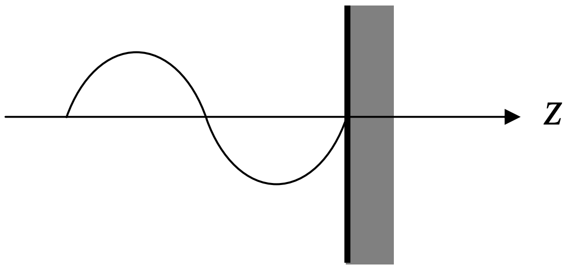
Standing Wave

Interference between the incident & reflected waves $\rightarrow$ Standing wave



A string with one end fixed on a wall

Incident: $y_1(z,t) = Y_0^+ \cos(\omega t - \beta z)$

$$\tilde{Y}_1(z) = Y_0^+ e^{-j\beta z}$$

(Set the incident wave's phase to be 0, i.e., Y_0^+ real & positive.)

Reflected: $y_2(z,t) = |Y_0^-| \cos(\omega t + \beta z + \phi)$

$$\tilde{Y}_2(z) = Y_0^- e^{j\beta z}, \text{ where } Y_0^- = |Y_0^-| e^{j\phi} = |Y_0^-| \angle\phi$$

The total displacement

$$\tilde{Y}(z) = \tilde{Y}_1(z) + \tilde{Y}_2(z) = Y_0^+ e^{-j\beta z} + Y_0^- e^{j\beta z}$$

We must have $\tilde{Y}(0) = 0 \quad \Rightarrow \quad Y_0^+ + Y_0^- = 0$ i.e. $Y_0^- = -Y_0^+$

$$\boxed{\tilde{Y}(z) = Y_0^+ (e^{-j\beta z} - e^{j\beta z})}$$

$$\tilde{Y}(z) = Y_0^+ (e^{-j\beta z} - e^{j\beta z})$$

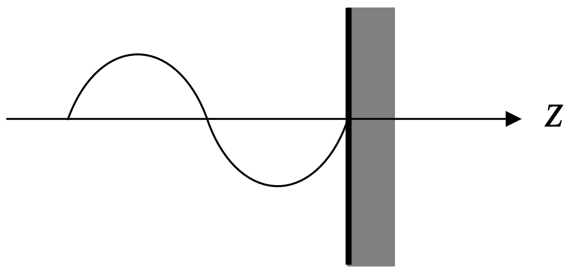
Recall that $\quad e^{j\theta} - e^{-j\theta} = 2j\sin\theta \quad \Rightarrow \quad \tilde{Y}(z) = -2jY_0^+ \sin(\beta z)$

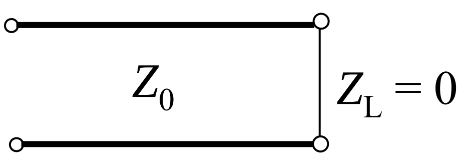
$$y(z,t) = \mathrm{Re}[-2jY_0^+ \sin(\beta z)] = 2Y_0^+ \sin(\beta z)\underline{\sin(\omega t)}$$

Why sin?

See Wikipedia Standing Wave animation to get visual picture:

Harmonic oscillation at each z, with amplitude following sin(βz)





$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = -1$$

Similarly, shorted transmission line:

By definition of "short circuit, $\tilde{V}(0) = 0 \Rightarrow V_0^+ + V_0^- = 0$

$$V_0^- = -V_0^+ \qquad \Gamma = -1$$

$$\tilde{V}(z) = -2jV_0^+ \sin(\beta z)$$

$$v(z,t) = \mathrm{Re}[-2jV_0^+ \sin(\beta z)] = 2V_0^+ \sin(\beta z)\sin(\omega t)$$

Like a mirror. What property of a mirror makes it a mirror?

But, $\quad I_0^- = I_0^+ \qquad \dfrac{I_0^+}{I_0^-} = -\Gamma = 1$

Find $\tilde{I}(z)$ and $i(z, t)$ on your own.

What if the transmission line is terminated in open circuit?

Note: open ended ≠ open circuit for high frequencies!
(You will see how to make an open circuit later.)

$$V_0^- = V_0^+ \qquad \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 1$$

$$\tilde{V}(z) = V_0^+(e^{-j\beta z} + e^{j\beta z}) = 2V_0^+\cos(\beta z)$$

Find $v(z, t)$ on your own.

$$I_0^- = -I_0^+ \qquad \text{(since total current is 0, by definition of open circuit)}$$

Find $\tilde{I}(z)$ and $i(z, t)$ on your own.

In all the above examples, $\Gamma = \pm 1$. Completely reflected.

At very high frequencies, we often can only measure the amplitude or power ($\propto$ amplitude squared), but not the instantaneous values or the waveform.

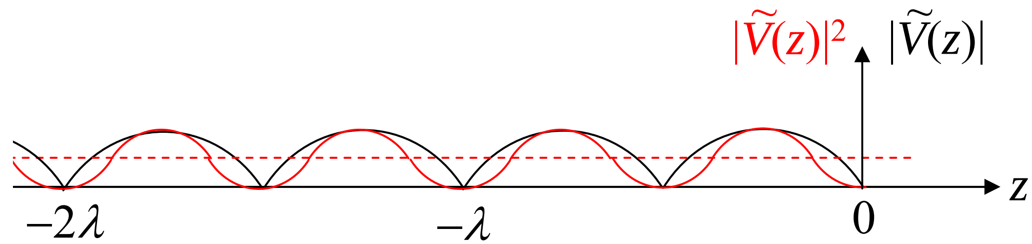
The following example is for the short circuit. The open circuit is similar (just with a shift of origin).

The amplitude of the voltage wave $v(z, t)$ at position z is

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}$$

The "complex amplitude" containing the phase

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}^*(z)}$$
$$= |-2jV_0^+ \sin(\beta z)|$$
$$= 2|V_0^+||\sin(\beta z)|$$



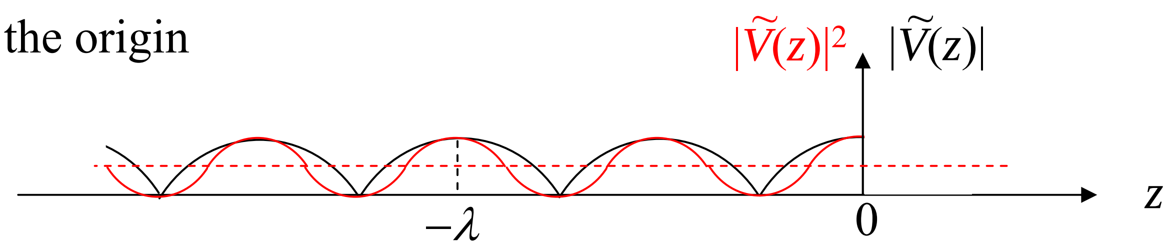
$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \sin^2(\beta z) = 2|V_0^+|^2 [1-2\cos(2\beta z)]$$

Question: What's the spatial period of the standing wave?

The open circuit: Just a shift of the origin

$$|\tilde{V}(z)| = 2|V_0^+||\cos(\beta z)|$$

$$|\tilde{V}(z)|^2 = 4|V_0^+|^2 \cos^2(\beta z)$$
$$= 2|V_0^+|^2 [1+2\cos(2\beta z)]$$



For both the short circuit (SC) and open circuit (OC),

$$\boxed{|\tilde{V}(z)|_{max} = 2|V_0^+|}$$

Constructive

$$\boxed{|\tilde{V}(z)|_{min} = 0}$$

Destructive

$$\boxed{|\Gamma| = 1}$$

Complete reflection. Completely a standing wave.

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$.
Also complete reflection. We'll talk about those cases later.

What if $|\Gamma| \neq 1$?

Partially standing, partially traveling.

Now, let's look at the maxima and minima of this combination of a standing wave and a traveling wave.
Recall that, in general, Γ is a complex number:

$$\Gamma = |\Gamma| e^{j\theta_r}$$

$$\tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{Incident}} + \underbrace{V_0^- e^{j\beta z}}_{\text{Reflected}} = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z} = V_0^+ e^{-j\beta z} + |\Gamma| e^{j\theta_r} V_0^+ e^{j\beta z}$$

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\ \tilde{V}^*(z)}$$
$$= \sqrt{V_0^+\left(e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}\right)\ (V_0^+)^*\left(e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z}\right)}$$

Notice that $V_0^+ (V_0^+)^* = |V_0^+|^2$

V_0^+ is a complex amplitude

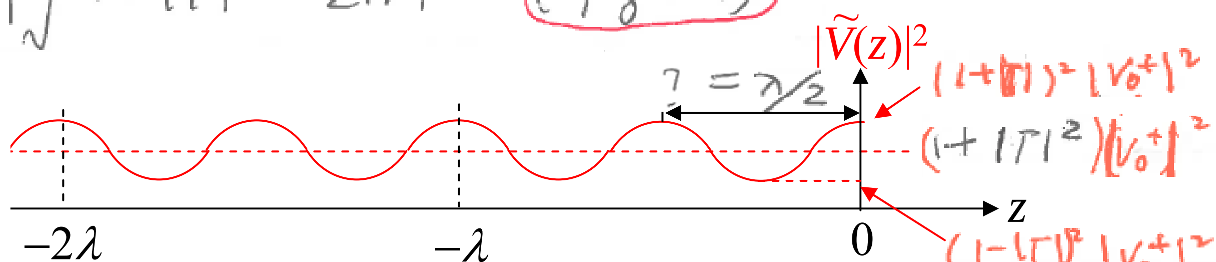
$$\therefore |\tilde{V}(z)| = |V_0^+| \sqrt{(\cdots)(\cdots)}$$
$$= |V_0^+| \sqrt{1 + |\Gamma|^2 + \underbrace{2|\Gamma| \cos(2\beta z + \theta_r)}_{\text{Interference term}}}$$

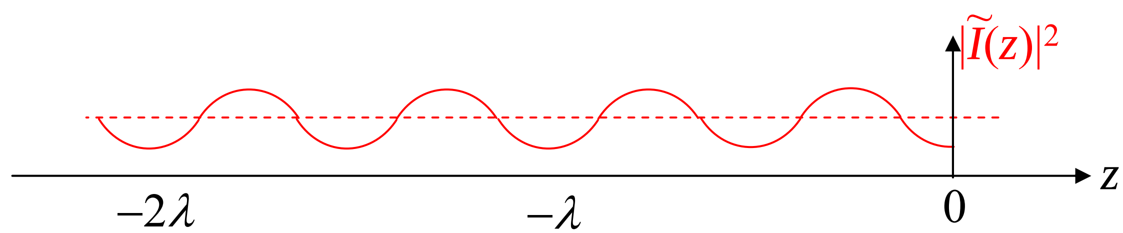
Similarly, $|\tilde{I}(z)| = |I_0^+| \sqrt{1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta z + \theta_r)}$

It's more convenient to plot $|\tilde{V}(z)|^2$ and $|\tilde{I}(z)|^2$ than the amplitudes.

$$2\beta\, ? = 2\pi$$

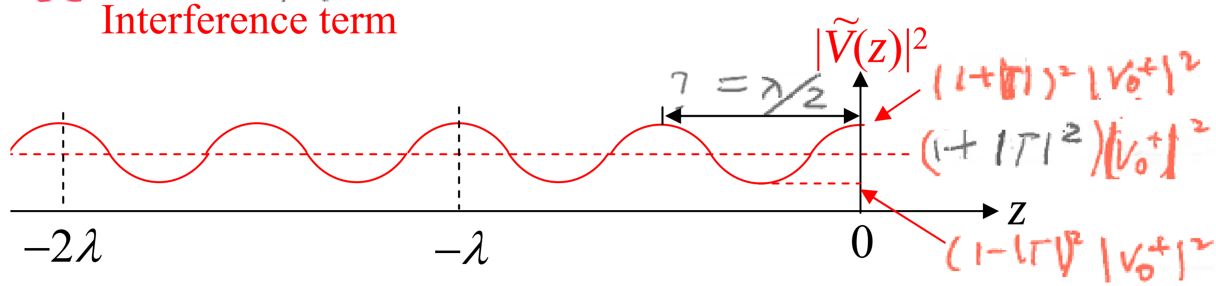
$$\therefore \boxed{? = \frac{2\pi}{2\beta} = \frac{1}{2}\lambda}$$





$$|\tilde{V}(z)| = |V_0^+|\sqrt{(\cdots)(\cdots)}$$

$$= |V_0^+|\sqrt{1+|\Gamma|^2 + \underbrace{2|\Gamma|\cos(2\beta z+\theta_r)}_{\text{Interference term}}}$$



Constructive interference

$$|\tilde{V}|_{max} = |V_0^+|\sqrt{1+|\Gamma|^2+2|\Gamma|}$$
$$= |V_0^+|(1+|\Gamma|)$$

Destructive interference

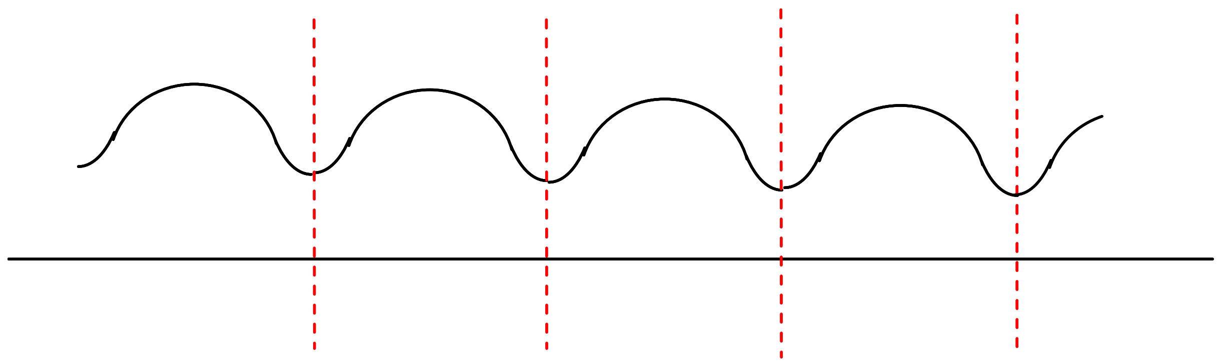
$$|\tilde{V}|_{min} = |V_0^+|\sqrt{1+|\Gamma|^2-2|\Gamma|}$$
$$= |V_0^+|(1-|\Gamma|)$$

Pay attention to the max, min, and average values

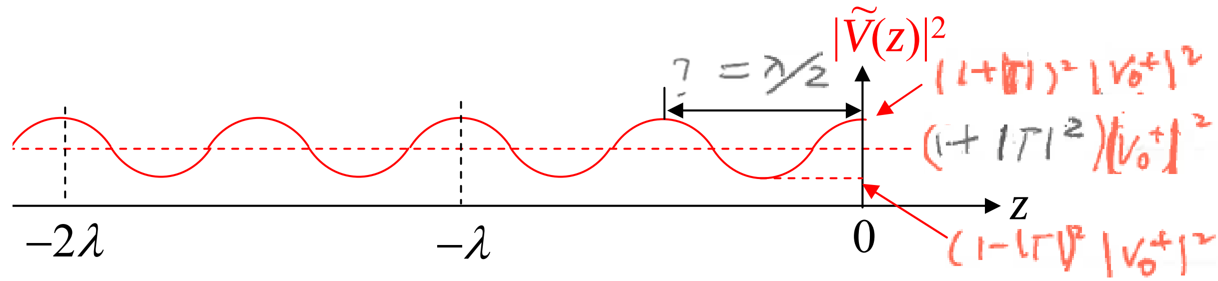
In this plot, we have assumed a special case $\theta_r = 0$.
Can you think of a kind of load that leads to $\theta_r = 0$?
Question: In general, what's the condition for $\theta_r = 0$?

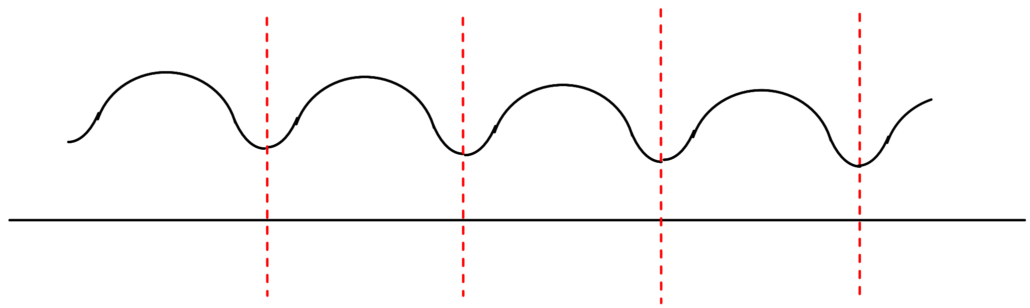
We stated that it's more convenient to plot the amplitudes squared than the amplitudes themselves.
But how does the plot of $|\tilde{V}(z)|$ look?

It looks like this:



Work out the max and min values.
Notice the important difference between its shape and that of $|\tilde{V}(z)|$





$$|\tilde{V}|_{max} = |V_0^+|\sqrt{1+|\Gamma|^2+2|\Gamma|} = |V_0^+|\,(1+|\Gamma|)$$

$$|\tilde{V}|_{min} = |V_0^+|\sqrt{1+|\Gamma|^2-2|\Gamma|} = |V_0^+|\,(1-|\Gamma|)$$

Now we define the voltage standing wave ratio (VSWR), or simply standing wave ratio (SWR)

$$\boxed{S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

Special (extreme) cases:

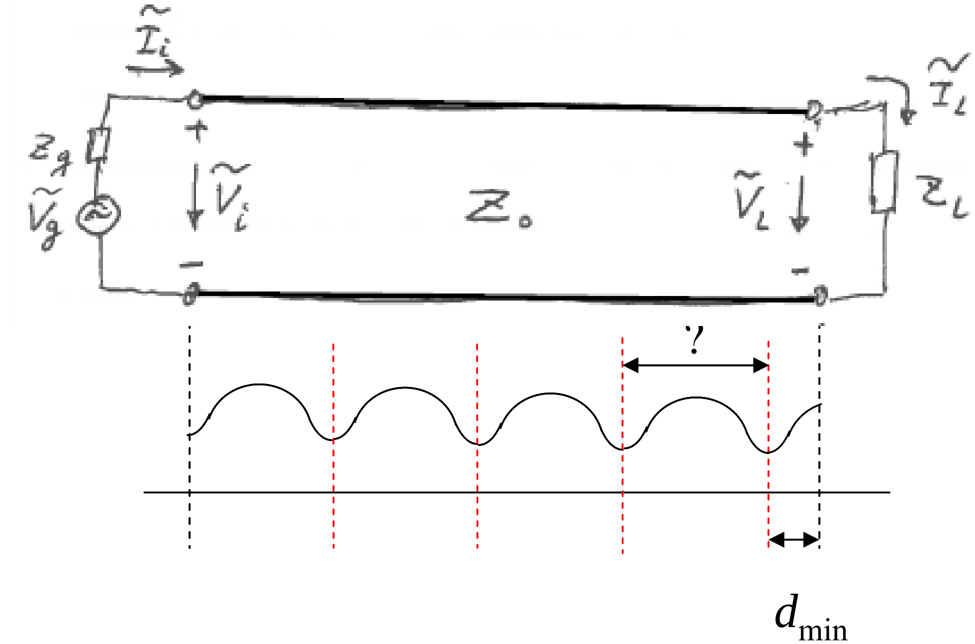
$|\Gamma| = 1 \quad S = \infty \;\Rightarrow$ All standing wave. $|\tilde{V}|_{min} = 0$
(Recall short & open. Other such cases to be discussed)

$|\Gamma| = 0, \quad S = 1 \;\Rightarrow$ All traveling wave. No reflection.
(What's the condition for this? How does the plot look?)

Slotted line

A tool to measure impedance. See in the textbook, Fig. 2-16 (pp. 74 in 7/E, pp. 73 in 6/E, or pp. 60 in 5/E). Based on the one-to-one mapping between z_L and Γ.

The detector measures the local field (proportional to voltage) as a function of longitudinal position z.



Sliding the detector, you find the voltage maxima and minima.

The distance between adjacent minima is ______

You also get the max/min ratio

$$S = \frac{|\tilde{V}|_{max}}{|\tilde{V}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

(You only care about the ratio, not the actual values.)

Solving $S = \dfrac{1+|\Gamma|}{1-|\Gamma|}$, you get $|\Gamma|$. But this is not Γ yet!

The hope is: If you know Γ, you get z_L using the one-to-one mapping between the two. (Recall that.) You know Z_0, thus you can find Z_L.

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$$\Gamma = |\Gamma| e^{j\theta_r}$$

We already know $|\Gamma|$. Just need to find θ_r.

We know $\frac{\lambda}{2} \xRightarrow{\beta = \frac{2\pi}{\lambda}} \beta$

We know $z_{min} = -d_{min}$

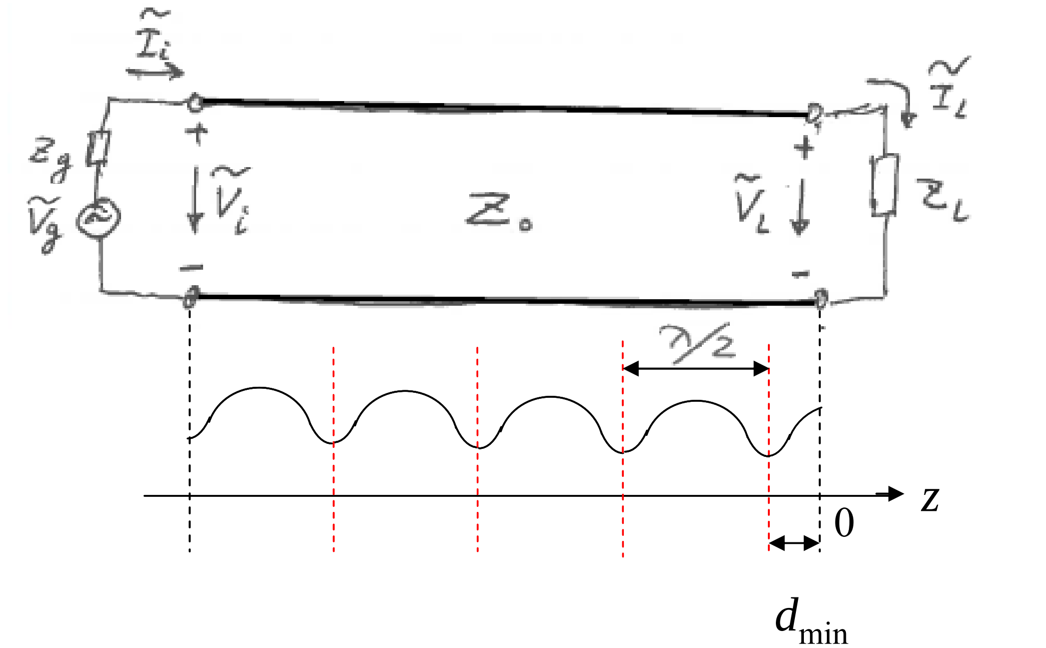
$$2\beta z_{min} + \theta_r = -\pi$$

$$-2\beta d_{min} + \theta_r = -\pi$$

$$2\beta d_{min} - \theta_r = \pi$$

So you find θ_r.

$$\Gamma = |\Gamma| e^{j\theta_r} \Rightarrow z_L \Rightarrow Z_L$$



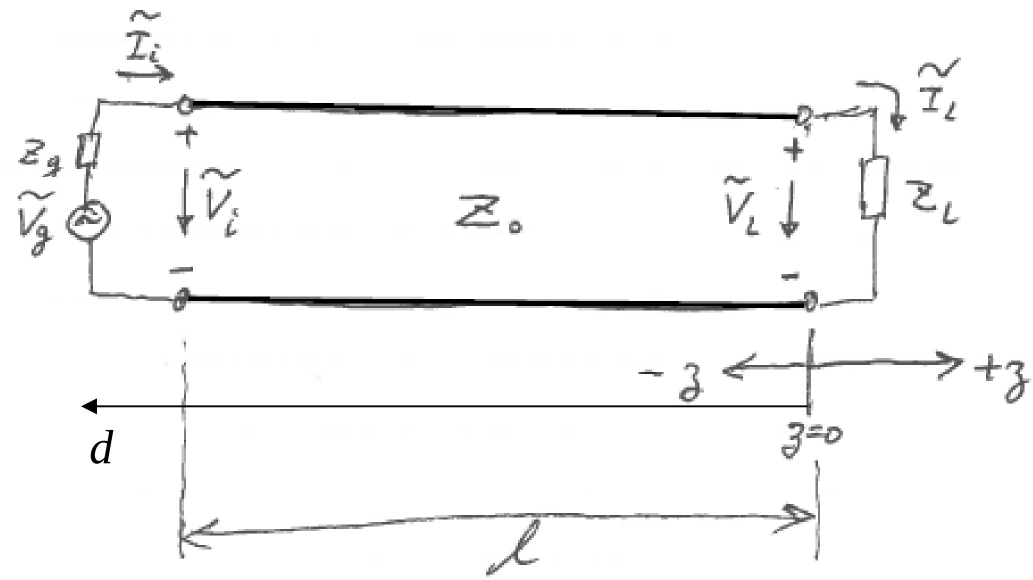
$$|\tilde{V}(z)| = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta z + \theta_r)}$$

Question:
In principle, we can also obtain the result by measuring positions of maxima.
But we prefer minima. Why?

We have so far always dealt with negative z, because we draw the transmission line to the left of the load.

We don't like to always carry the negative sign.

So we define $d = -z$, the distance from the load.



$$\tilde{V}(z) = \underbrace{V_0^+ e^{-j\beta z}}_{\text{Incident}} + \underbrace{\Gamma V_0^+ e^{j\beta z}}_{\text{Reflected}}$$

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{j\beta z}$$

$$\Rightarrow \tilde{V}(d) = V_0^+ e^{j\beta d} + \Gamma V_0^+ e^{-j\beta d}$$

$\boxed{\text{Pay attention to signs.}}$

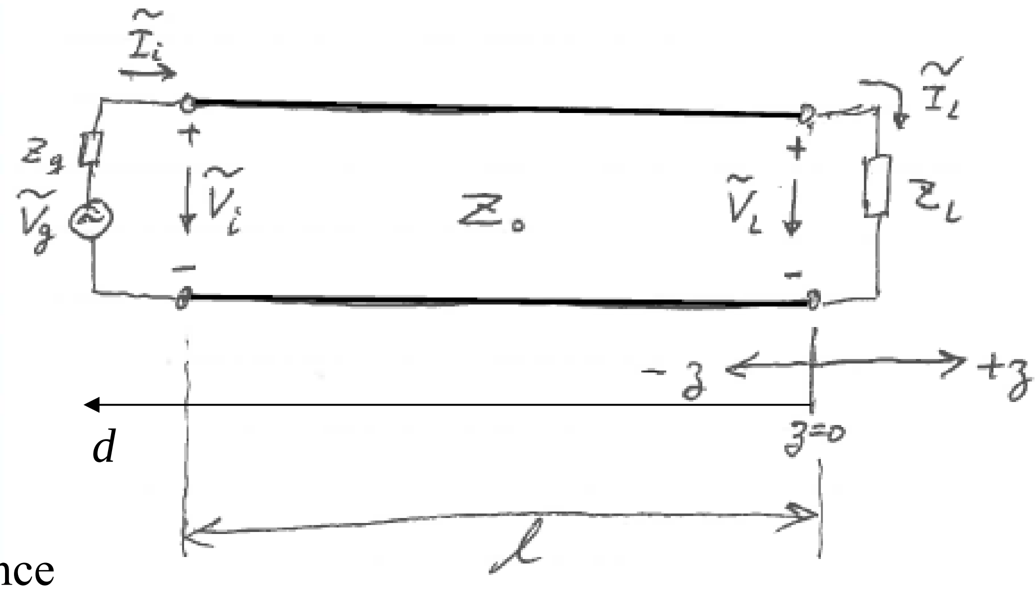
$$\tilde{I}(z) = I_0^+ e^{-j\beta z} - \Gamma I_0^+ e^{j\beta z}$$

$$\Rightarrow \tilde{I}(d) = I_0^+ e^{j\beta d} - \Gamma I_0^+ e^{-j\beta d}$$

$$\tilde{V}(d) = V_0^+ e^{j\beta d} + \Gamma V_0^+ e^{-j\beta d}$$

$$\tilde{I}(d) = I_0^+ e^{j\beta d} - \Gamma I_0^+ e^{-j\beta d}$$

$$\frac{\tilde{V}^+(d)}{\tilde{I}^+(d)} = \frac{V_0^+}{I_0^+} = Z_0$$



Now let's consider the equivalent impedance looking into the transmission line at a distance d from the load:

$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{\cancel{V_0^+}(e^{j\beta d} + \Gamma e^{-j\beta d})}{\cancel{V_0^+}(e^{j\beta d} - \Gamma e^{-j\beta d})} Z_0$$

$$= Z_0 \cdot \frac{1 + \Gamma e^{-2j\beta d}}{1 - \Gamma e^{-2j\beta d}} \equiv Z_0 \cdot \frac{1 + \Gamma_d}{1 - \Gamma_d}$$

$$\Gamma_d = \Gamma e^{-2j\beta d}$$

Compare to $\dfrac{Z_L}{Z_0} = \dfrac{1+\Gamma}{1-\Gamma}$, thus the definition.

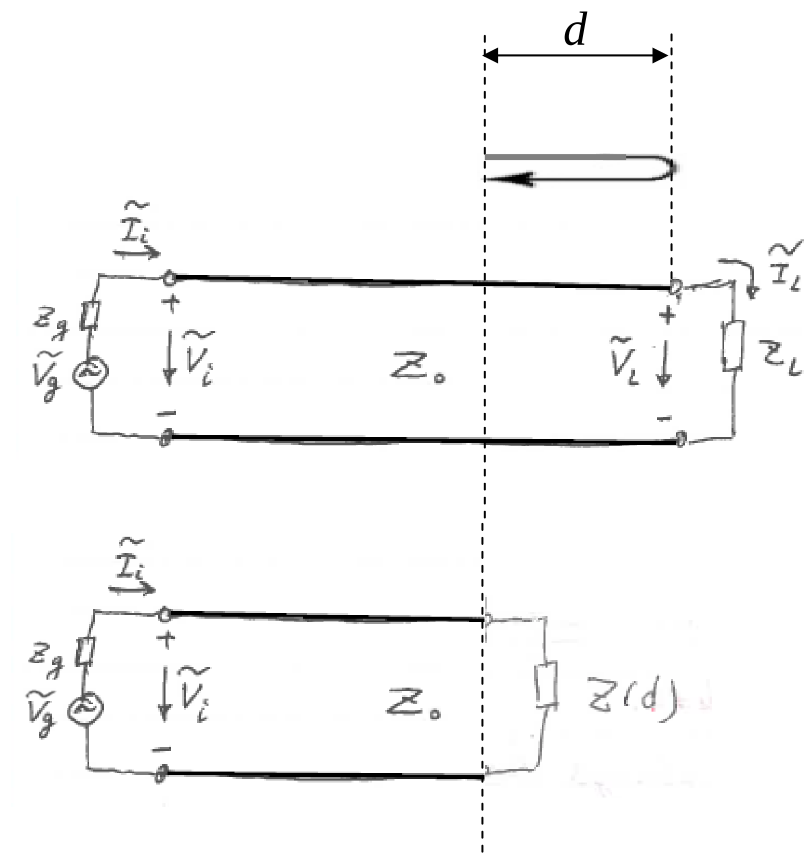
$$Z(d) = Z_0 \cdot \frac{1+\Gamma_d}{1-\Gamma_d}, \quad \Gamma_d = \Gamma e^{-2j\beta d}$$ (equivalent reflection coefficient at *d*)

How to interpret this?

Say, the incident wave voltage is $\tilde{V}(d)$ at *d*.

When it arrives at the load, it becomes $\tilde{V}(d)e^{-j\beta d}$

-- just a phase shift.



At the load, the reflection wave is $\tilde{V}(d)\Gamma e^{-j\beta d}$

This reflection wave travels a distance *d* to back.

At distance d, it becomes

$$\tilde{V}(d)\Gamma e^{-j\beta d} e^{-j\beta d} = \tilde{V}(d)\Gamma e^{-2j\beta d}$$

-- just another phase shift.

Thus the equivalent reflection coefficient at *d* is

$$\Gamma_d = \Gamma e^{-2j\beta d}$$

-- just imagine the interface is at *d*.

$Z(d)$ corresponds to Γ_d in exactly the same manner as any Z_L to Γ.

Therefore the $\boxed{\text{equivalent circuit.}}$

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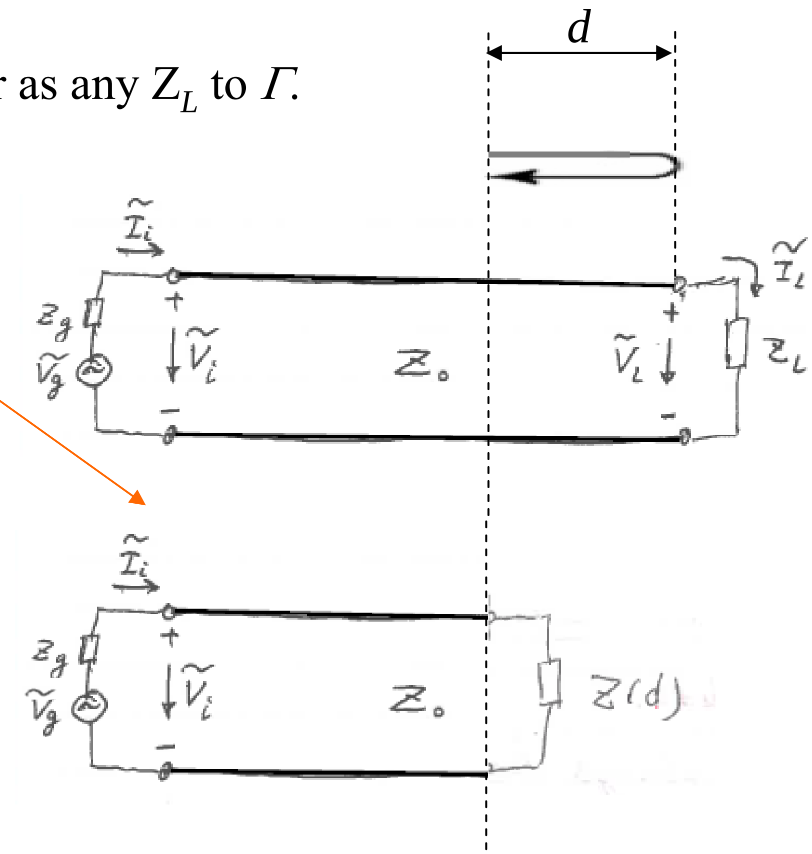
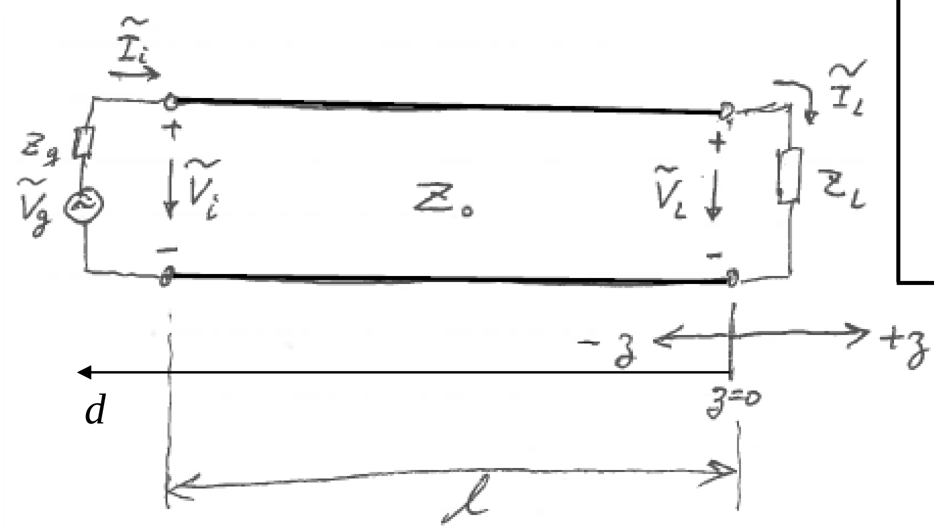
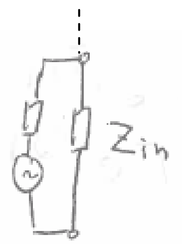


You can have such an equivalent circuit at any *d*, all the way up to *l* for the entire transmission line:





At the input end of the transmission line,

$$Z_{in} = \underset{d=\ell}{Z(\ell)} = Z_0 \frac{1+\Gamma_\ell}{1-\Gamma_\ell}$$

This way, you turn the transmission line problem in to a simple circuit problem.

Question: Given $\tilde{V}_g$ and Z_g, how do you find $\tilde{V}_i$?

All cases of $|\Gamma| = 1$

There are cases where $|\Gamma| = 1$ but $\Gamma \neq \pm 1$. Also complete reflection, but neither short nor open.

We mentioned that, a quarter wavelength away from a short, the equivalent circuit is an open.

What is the equivalent impedance anywhere in between?

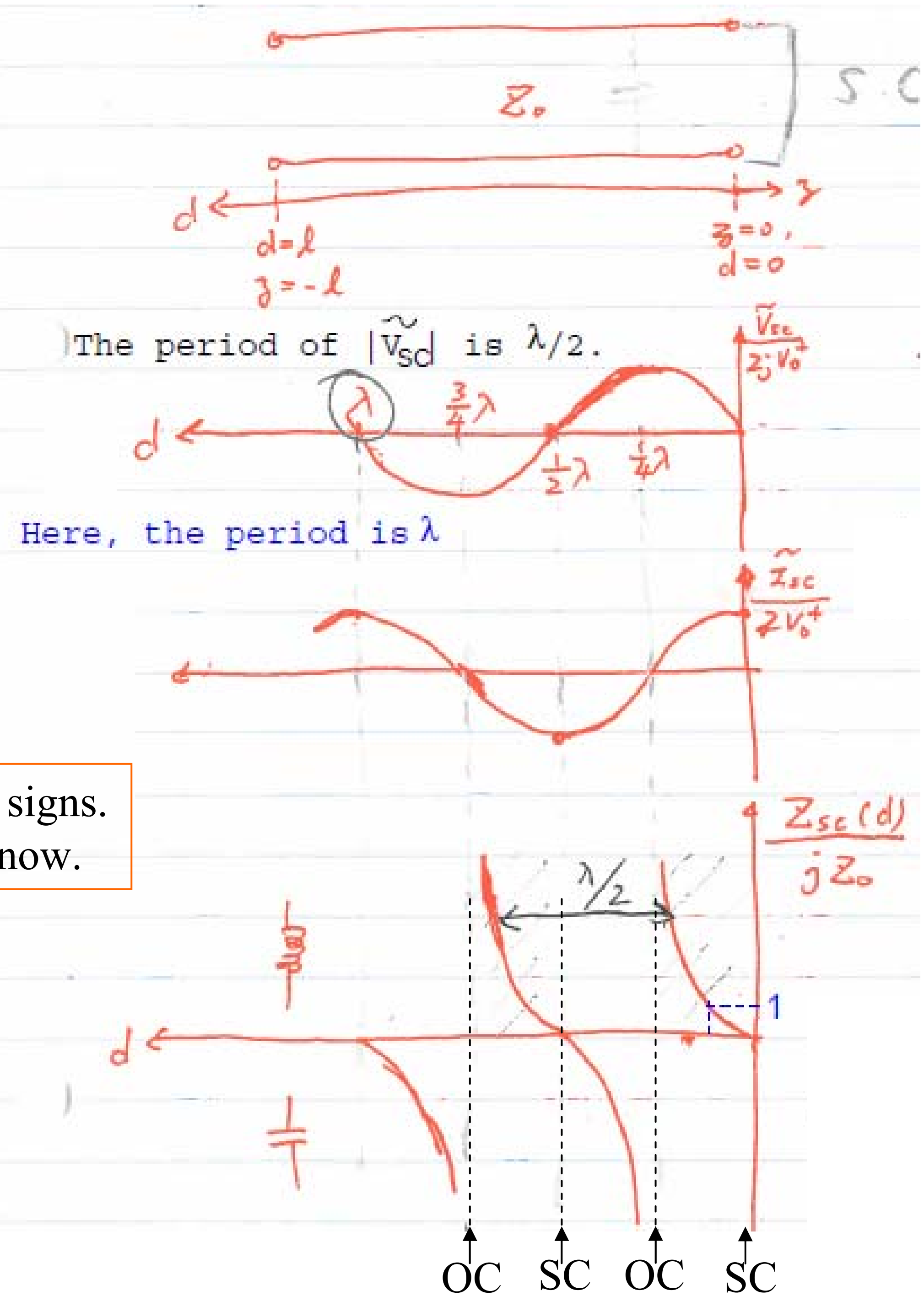
Let's have a closer look at the short circuit.

$$\tilde{V}_{sc}(d) = V_0^+\left(e^{j\beta d} - e^{-j\beta d}\right)$$
$$= 2jV_0^+ \sin\beta d$$

$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}\left(e^{j\beta d} + e^{-j\beta d}\right) = \frac{2V_0^+}{Z_0}\cos\beta d$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)}$$
$$= jZ_0 \tan\beta d$$

Pay attention to signs.
We are using d now.



$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} = j Z_0 \tan \beta d$$

Understand this from a physics point of view:

Reactive loads don't dissipate power.
Thus complete reflection.
The difference is just in the phase.

Equivalent impedance

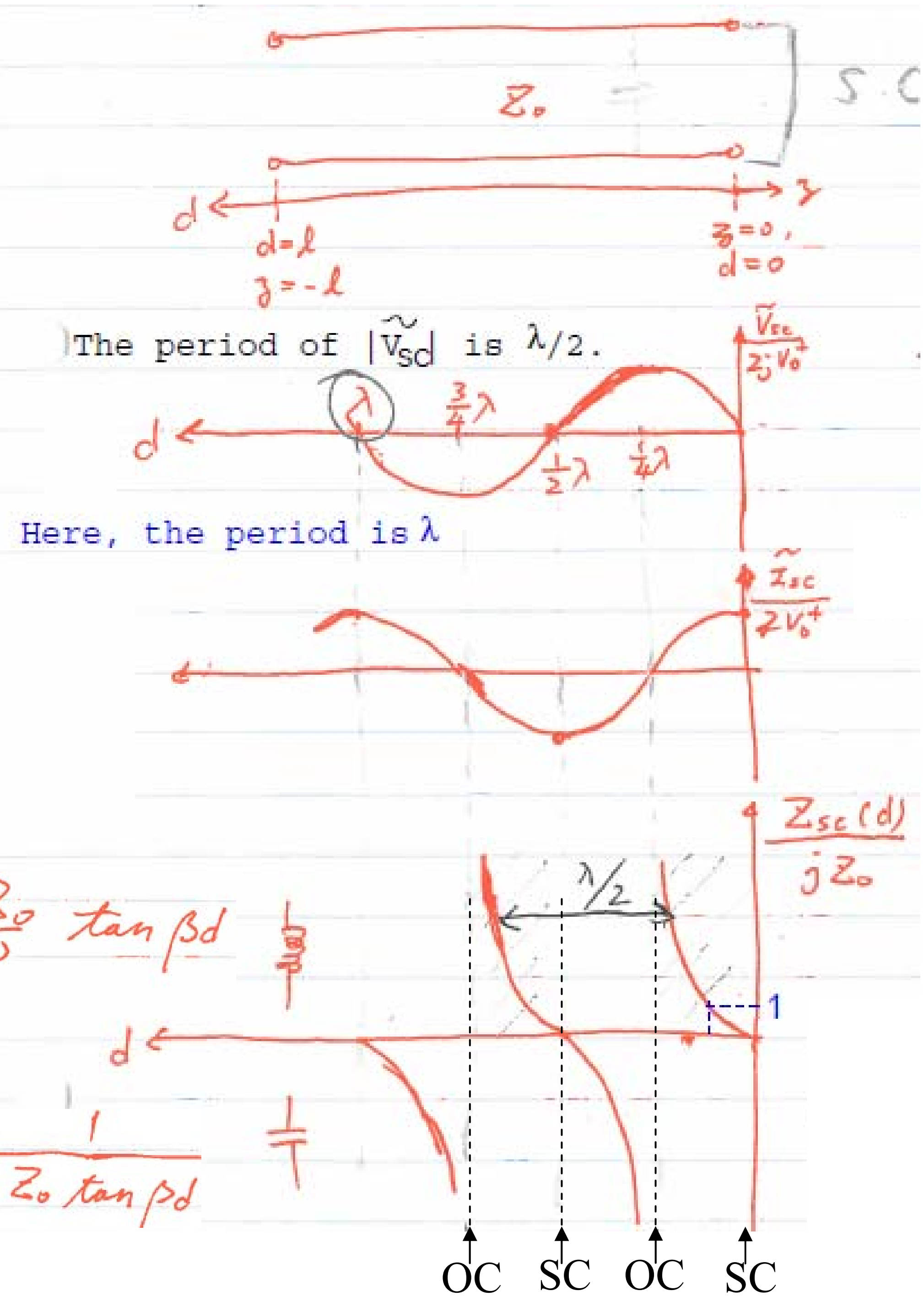
For $\tan \beta d > 0$

$$j\omega L_{eq} = j Z_0 \tan \beta d \Rightarrow L_{eq} = \frac{Z_0}{\omega} \tan \beta d$$

For $\tan \beta d < 0$

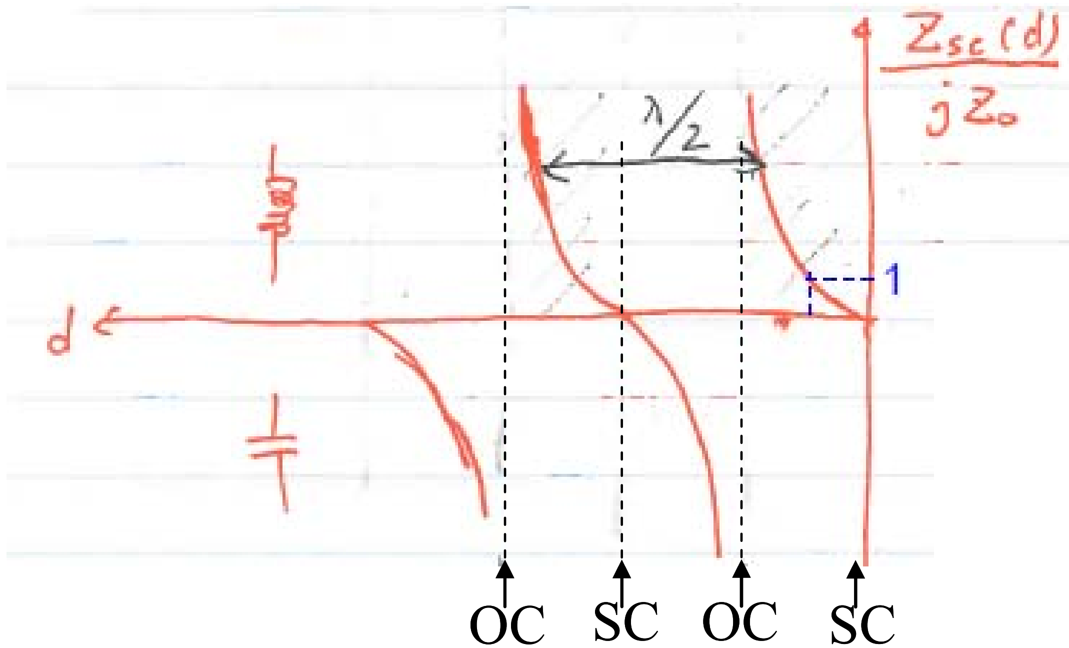
$$\frac{1}{j\omega C_{eq}} = j Z_0 \tan \beta d \Rightarrow C_{eq} = -\frac{1}{\omega Z_0 \tan \beta d}$$

Notice frequency dependence.



The case of open circuit termination

Now that we already know the case of short circuit termination, what's the easiest way to work out the open circuit termination case?



With a short circuit, you can make an open circuit.

(For complete solution, see Fig. 2-21 in textbook, pp. 81 in 7/E or pp. 82 in 6/E)

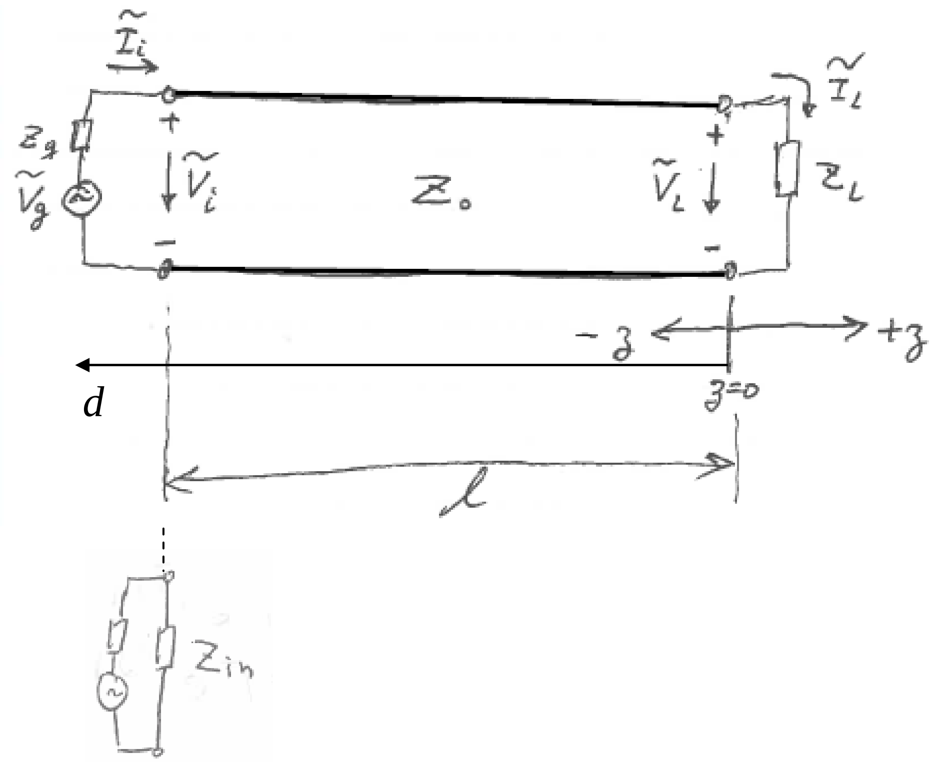
Now let’s go back to the general case and look at the equivalent input impedance.

$$Z_{in} = Z(d=l) = Z_0 \cdot \frac{e^{j\beta l} + \Gamma e^{-j\beta l}}{e^{j\beta l} - \Gamma e^{-j\beta l}}$$

(make sure you understand how this is arrived at)

Insert

$$\begin{cases} \Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} \\ e^{j\beta l} = \cos\beta l + j\sin\beta l \\ e^{-j\beta l} = \cos\beta l - j\sin\beta l \end{cases}$$



and you get

$$Z_{in} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} Z_0$$

or in the normalized form:

$$z_{in} = \frac{z_L \cos\beta l + j\sin\beta l}{\cos\beta l + jz_L \sin\beta l}$$

Recall that $z_L = \dfrac{Z_L}{Z_0}$

Question: What is the unit of z_L?

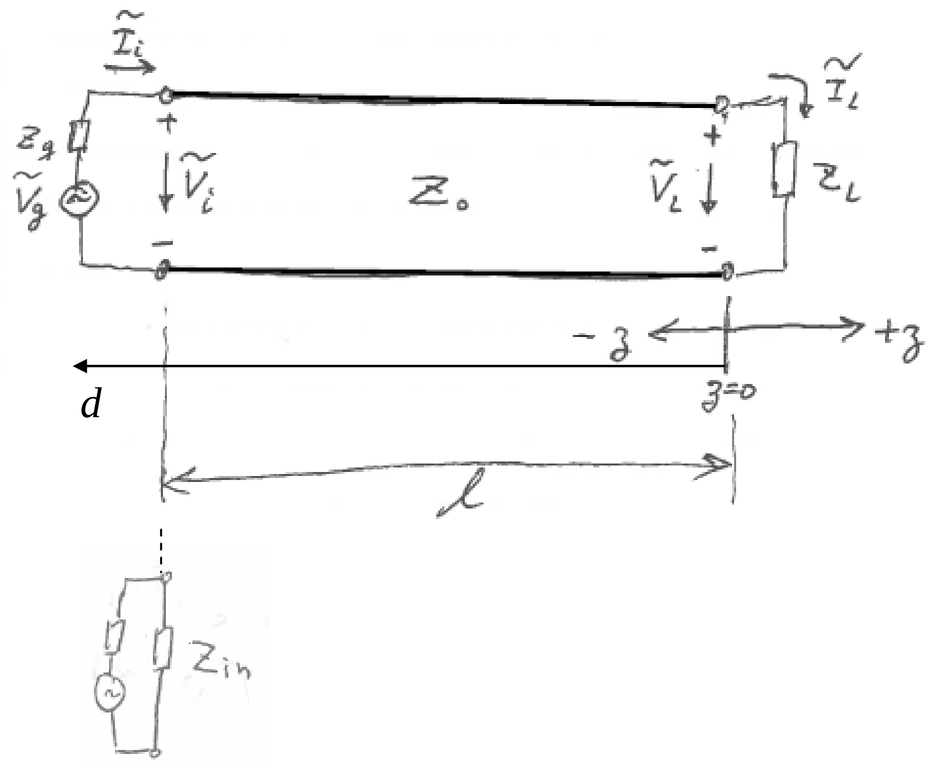
Do not confuse “input” with “incident”

$\tilde{V}_i$ or $\tilde{V}_{in}$ is the voltage at the input end. It is the sum of incident and reflected waves **there**.

$$\tilde{V}_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} = V_0^+ \Big(\underbrace{e^{j\beta\ell}}_{\text{incident}} + \underbrace{\Gamma e^{-j\beta\ell}}_{\text{reflected}}\Big)$$

The incident wave voltage at the input end is

$$\tilde{V}_{inc}(\ell) = V_0^+ e^{j\beta\ell}$$



The quarter wavelength magic

$$Z_{in} = \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} Z_0$$

For $\beta l = n\pi$,

i.e. $l = n \cdot \frac{\lambda}{2}$

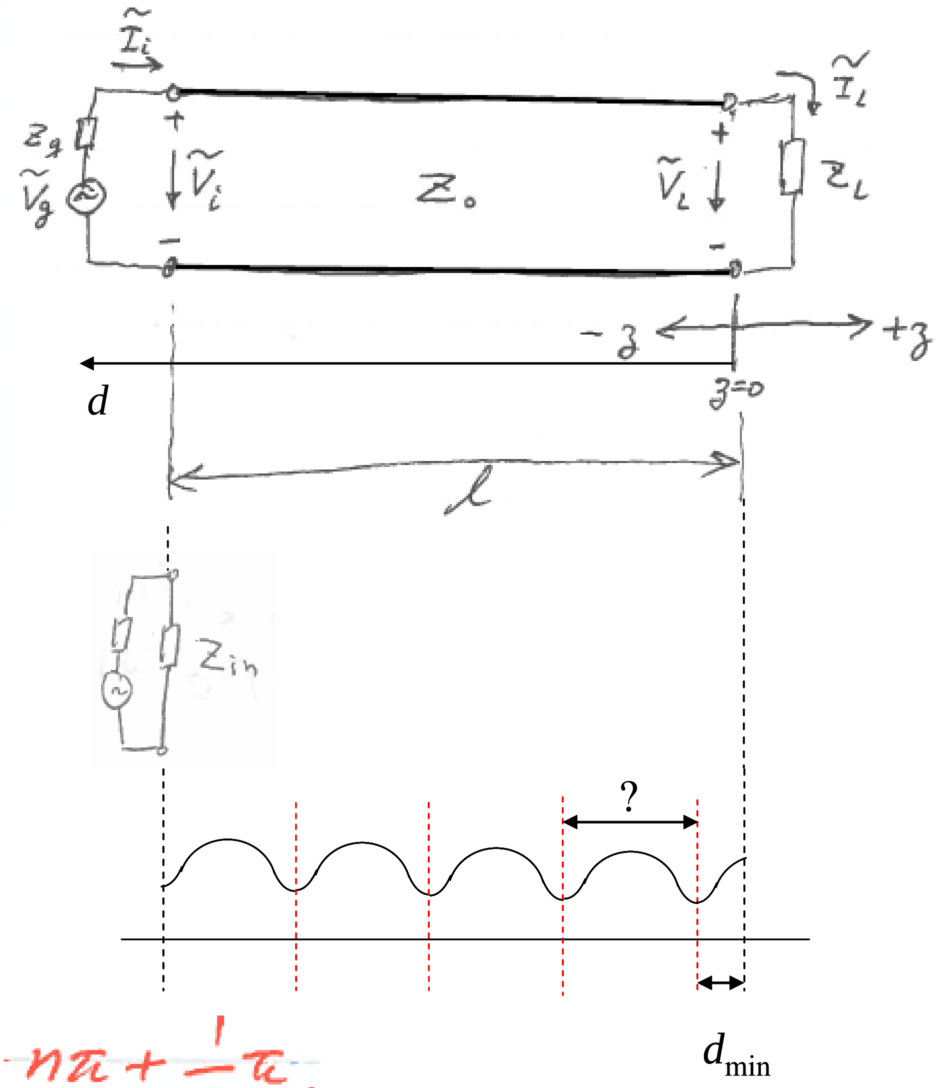
$$\boxed{Z_{in} = Z_L}$$

Periodic. This is for generic Z_L.
For the special case of purely reactive loads, see slide 17.

For $\begin{cases} \cos\beta l = 0 \\ \sin\beta l = \pm 1 \end{cases}$ i.e. $\beta l = n\pi + \frac{1}{2}\pi$.

i.e. $l = n \cdot \frac{\lambda}{2} + \frac{\lambda}{4}$,

$$\boxed{Z_{in} = \frac{Z_0^2}{Z_L}}$$



For $\ell = n \cdot \frac{\lambda}{2} + \frac{\lambda}{4}$,

$$\boxed{Z_{in} = \frac{Z_0^2}{Z_L}}$$

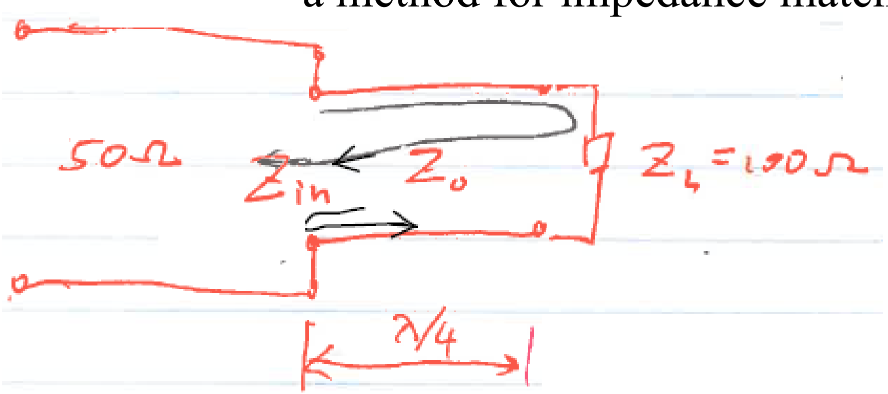
or

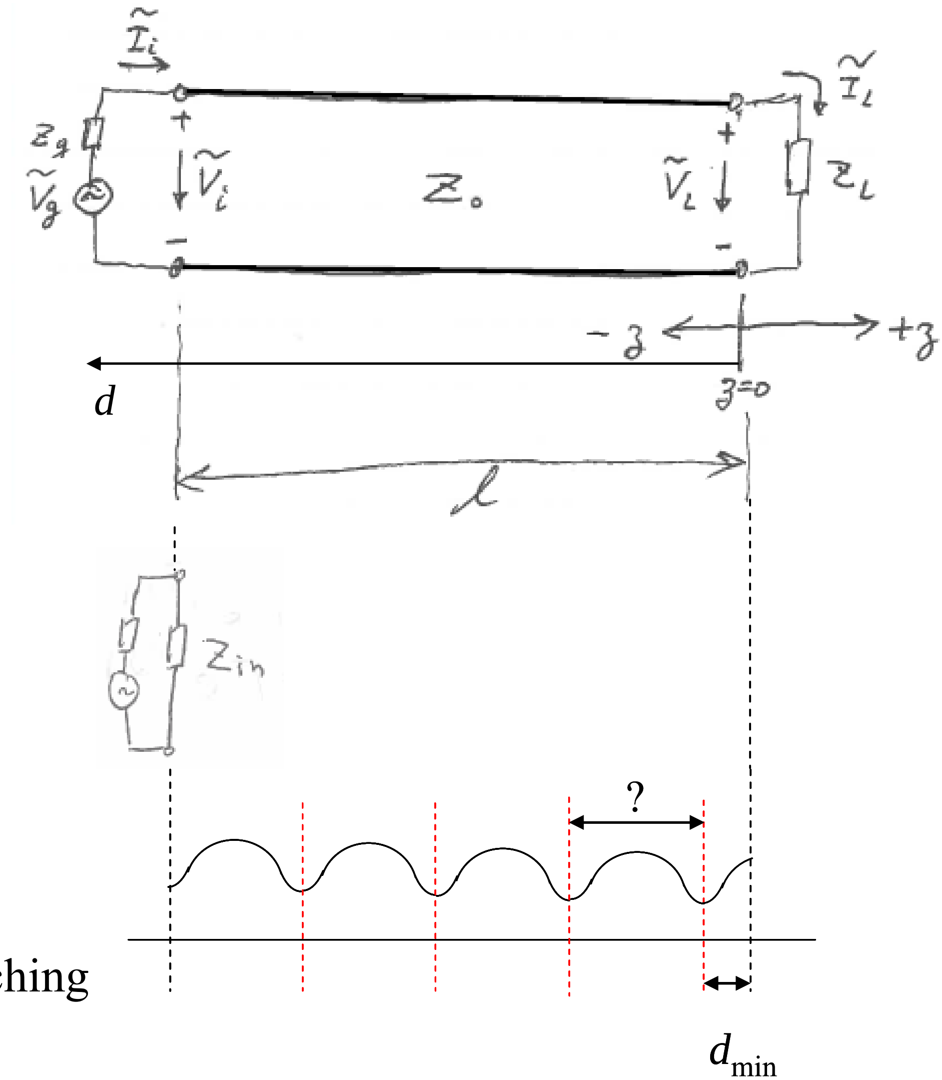
$$\boxed{Z_0 = \sqrt{Z_{in} \cdot Z_L}}$$

Question: What are the equivalent "normalized" forms?

The quarter wavelength transformer

-- a method for impedance matching

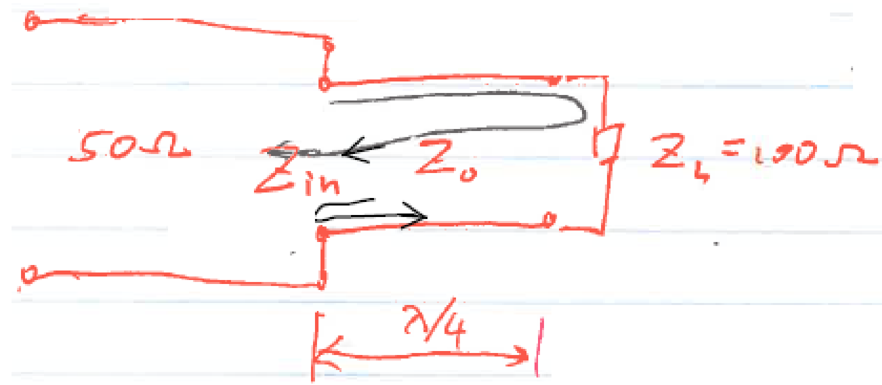




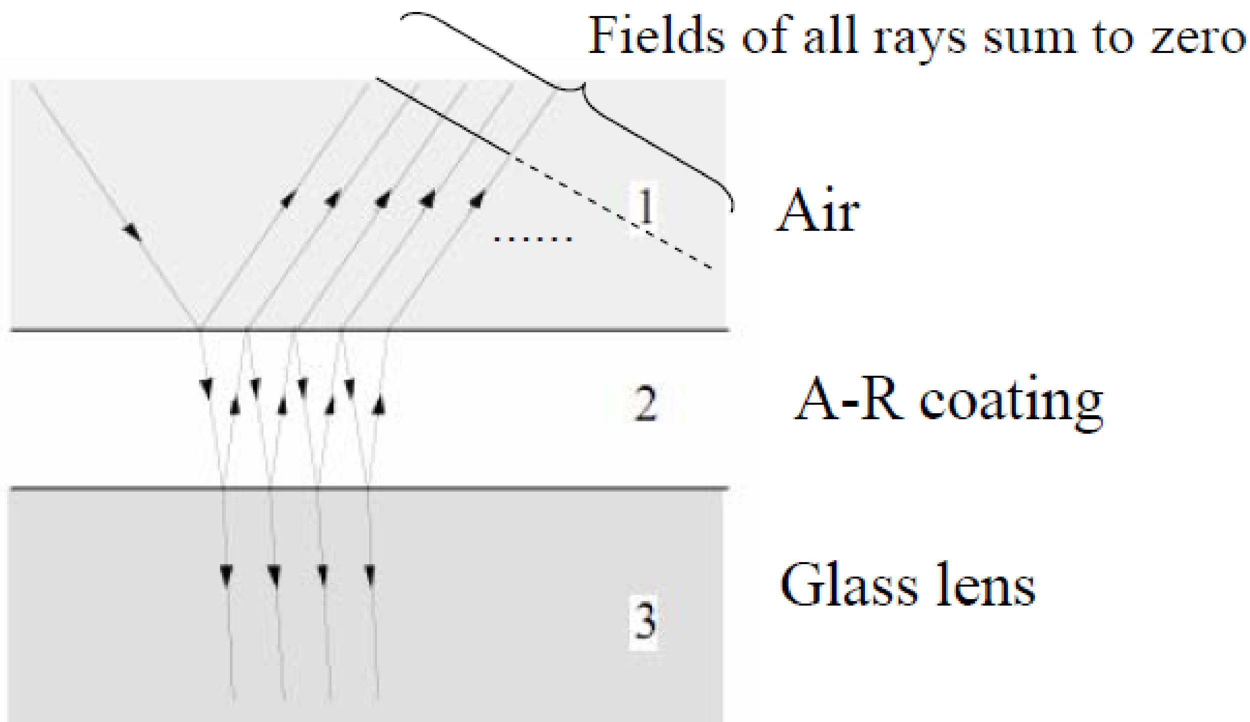
The quarter wavelength transformer

To better understand why it works, let's look at its optical analog.

WITHOUT ANTI-REFLECTIVE WITH ANTI-REFLECTIVE

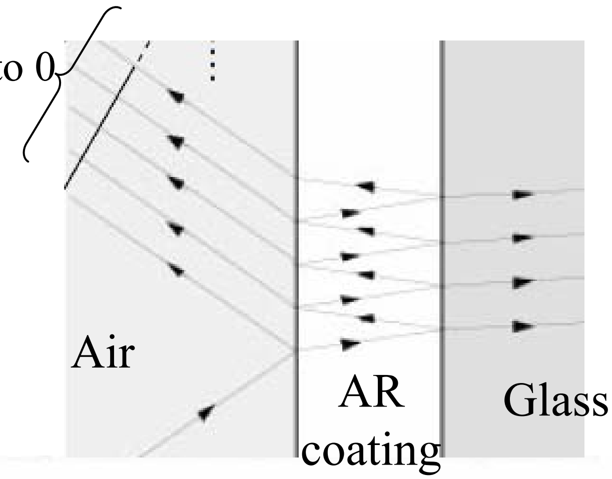


Anti-reflection coating



The quarter wavelength magic explained in the multiple reflection point of view

Optical analog: the AR coating



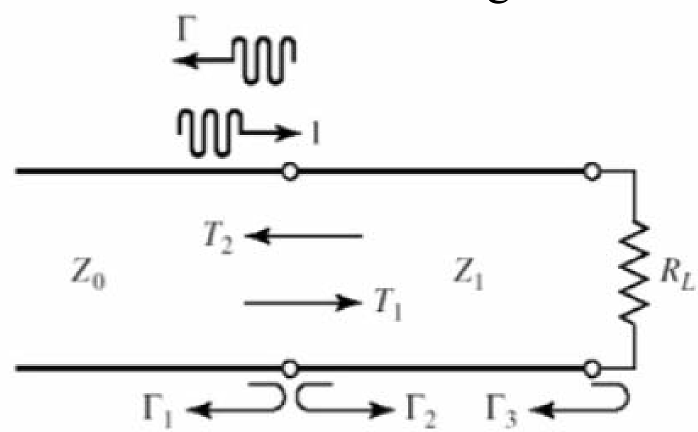
$$\Gamma_1 = \frac{Z_1 - Z_o}{Z_1 + Z_o}$$

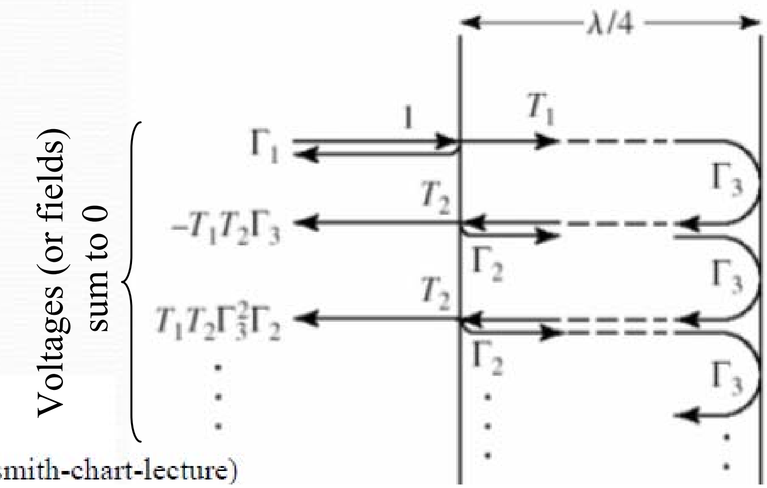
$$\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1}$$

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$

$$T_1 = \frac{2Z_1}{Z_1 + Z_o}$$

$$T_2 = \frac{2Z_o}{Z_1 + Z_o}$$





(Adapted from: Naveed Ramzan, http://www.slideshare.net/nramzan19/smith-chart-lecture)

Take-home messages

- Standing waves are simply due to interference between incident & reflected waves.
- The variations of real positive amplitudes (of voltage & current) and modulus squares of amplitudes ($|\tilde{V}(d)|^2$ and $|\tilde{I}(d)|^2$) with position (i.e. distance from load) are periodic, analogous to interference stripes in optics and are indeed one-dimensional interference patterns.
- The period of the patterns are half wavelength.
- The reflection wave amplitude is a fraction (≤ 1) of the incident, and its phase is shifted relative to the incident, right upon reflection. Thus the reflection coefficient is complex.
- In general voltage and current of a transmission line are combinations of a traveling wave and a standing wave.
- When the load is purely reactive (including short and open) complete reflection. These are the same in terms of absence of energy dissipation. Thus the you can obtain any desired reactance value by terminating a transmission line in any reactive component; you only need to have the right distance from the load.
- At any distance d from the load, you have an equivalent impedance $Z(d)$, such that you feel as if the transmission line is terminated in $Z(d)$ right there.
- $Z(\lambda/4)$ and Z_L [or more generally $Z(d + \lambda/4)$ and $Z(d)$] have a special relation, which is used as a method for impedance matching. This method eliminates reflection because multiple reflections sum up to 0.