ECE 341 Test 2, Spring 2019 Your Name_____ Tue 4/16/2019

Tested and to be learned:
Mathematical language. Vector decomposition and summation.
Coulomb's law. Field of electric dipole (see Example 4-7 in Textbook).
Image method.

Problem 1. Electrostatics

Total 28 + 5 (bonus)

One surface of an infinitely large ideal conductor plate is at the plane $x = 0$ of the Cartesian coordinate system, with the x-y plane being the plane of the paper and the z axis represented by the dot, as shown in the figure. The conductor plate is grounded (i.e. at a potential 0). A positive point charge Q is located at (d, 0, 0). Assuming free space (i.e. vacuum, with permittivity ε_0), find the following:

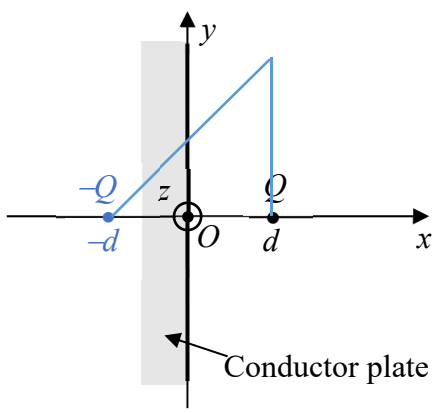
(1) The electric field and potential anywhere with $x < 0$, and

(2) The electric field and potential at point (d, $2d$, 0). Note: The filed is a vector while the potential is a scalar. You may express a field in the form of $E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$.

(3) The electric field and potential anywhere on the y axis.

(4) The potential anywhere in the plane $x = 0$.

(5) (Optional; for bonus) The electric field anywhere in the plane $x = 0$. Hint: For convenience, you may define your own polar coordinate in this plane $\mathbf{R} = y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$.



Solution:

3 (1) For $x < 0$, $\mathbf{E} = 0$ and $V = 0$.

(2) Image charge $-Q$ at ($-d$, 0, 0). 2

The field due to charge Q is

$$\vec{E}_1 = \hat{y} E_{1y}$$

$$E_{1y} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(2d)^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{4d^2}$$ 2

Let the field due to the image charge be $\mathbf{E}_2$.

$$|\vec{E}_2| = E_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{(2\sqrt{2}d)^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{8d^2}$$ 2

$$E_{2x} = -\frac{1}{\sqrt{2}}E_2 = \frac{-1}{4\pi\varepsilon_0}\frac{Q}{8\sqrt{2}d^2}$$ 2

$$E_{2y} = -\frac{1}{\sqrt{2}}E_2 = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{8\sqrt{2}d^2}$$

The total field $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$

$$E_x = E_{2x} = \frac{-Q}{32\pi\varepsilon_0 d^2}\cdot\frac{1}{\sqrt{2}}$$ 2

$$E_y = E_{1y} + E_{2y} = \frac{Q}{4\pi\varepsilon_0 d^2}\left(\frac{1}{4} - \frac{1}{8\sqrt{2}}\right) = \frac{Q}{32\pi\varepsilon_0 d^2}\left(2 - \frac{1}{\sqrt{2}}\right)$$ 2

$$\therefore\ \vec{E} = \frac{Q}{32\pi\varepsilon_0 d^2}\left[-\hat{x}\frac{1}{\sqrt{2}} + \hat{y}\left(2 - \frac{1}{\sqrt{2}}\right)\right]$$ 2 (for adding up)

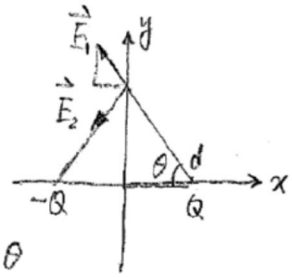
Question (2) is 16 points. Listed in the right column are partial credits for the intermediate results. Student gets full 16 points upon getting correct result in any sound manner, not necessarily in this procedure.

2 The potential V at $(d, 2d, 0)$ is the sum of the potentials due to Q and its image:

$$V = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{2d} - \frac{Q}{4\pi\varepsilon_0}\frac{1}{2\sqrt{2}d}$$

(3)

$$E_1(y) = E_2(y) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2+d^2}$$



$E_{1y}(y)$ and $E_{2y}(y)$ cancel each other.

The total field is

$$\vec{E}(0,y,0) = 2E_x(y)\hat{x} = \frac{-1}{2\pi\varepsilon_0}\frac{Q\hat{x}}{y^2+d^2}\cos\theta$$

4 $$= -\frac{1}{2\pi\varepsilon_0}\frac{Q\hat{x}}{y^2+d^2}\frac{d}{\sqrt{y^2+d^2}} = -\frac{Q\hat{x}}{2\pi\varepsilon_0}\frac{d}{(y^2+d^2)^{3/2}}$$

1 $V(0, y, 0) = 0$

Give partial credit of 2 points if student demonstrates general method of vector decomposition and summation.
3 points if result nearly correct, with the field in the -x direction.

(4)

2 $V(0, y, z) = 0$

(5)

5 (bonus) Using result of (3) and by symmetry,

$$\vec{E}(0,y,z) = -\frac{Q\hat{x}}{2\pi\varepsilon_0}\frac{d}{(y^2+z^2+d^2)^{3/2}}$$

Compare this in the far field limit y >> d & z >> d with the result Eq. 4.56 with theta = 90 deg.
Also see graphics in class note Magetostatics I, p. 16 (far field) and p. 17 (near field).

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Tested and to be learned:
Gauss's law.
Concept of electrostatic potential.
Boundary conditions

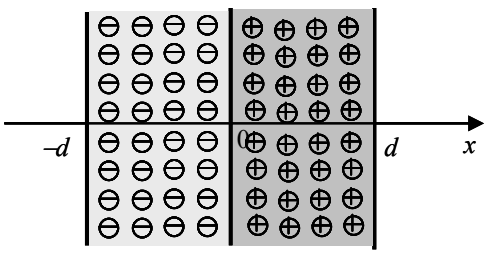
Practical implication:
Model of pn junction in the depletion approximation.

Total 27

Problem 2. Electrostatics

As shown in the figure below, an infinitely large slab of dielectric with a dielectric constant ε_r is joined at $x = 0$ to another infinitely large slab of dielectric, with the same dielectric constant ε_r. The dielectric slab on the right side ($x > 0$) has a uniform volume charge density ρ, while the one on the left side has a charge density $-\rho$. You may take ρ as positive. The thickness of both slabs is d. Find the electric field **and** potential as functions of x for **all** x. For convenience, you may set the potential at the interface between the two slabs to be zero, i.e. $V(0) = 0$. Visualize your results by sketching plots of potential V and field E as functions of x.

Note: The permittivity of vacuum is ε_0. You may write the permittivity of the dielectric as $\varepsilon = \varepsilon_r\varepsilon_0$. See class note Electrostatics II: Gauss's Law, p. 11; Electrostatics III: Potential, Current, & Ohm's Law, p. 8.



Method 1: the graphical way

Imagine a cylinder/prism with sidewall(s) parallel to the x axis, and the bottom and top surfaces at $x < -d$ and $x > d$, respectively. The net charge inside is 0.

3 Therefore, $\mathbf{E} = 0$ for $x < -d$ and $x > d$.

Now, leave the bottom surface at $x \le -d$, and move the top into the positively charged slab. The enclosed net charge is:

$$-\rho A d + \rho A x = -\rho A (d - x)$$

At the bottom surface, $\mathbf{E} = 0$. At the top surface,

$$\varepsilon E(x) A = -\rho A (d - x)$$

Give partial credit up to 3 points if student demonstrates understanding of the method but does not obtain correct result.

5 Therefore, for $0 < x < d$,

$$\boxed{E(x) = -\rho(d - x)/\varepsilon}$$

The direction of $\mathbf{E}$ is along the x axis, and the negative sign signifies that it points to $-x$.

Now, leave the top surface of the original cylinder/prism at $x \ge d$, and move the bottom into the negatively charged slab. The enclosed net charge is:

$$+\rho A x + \rho A d = \rho A (d + x)$$

At the top surface, $\mathbf{E} = 0$. At bottom the surface,

$$-E(x) A = \rho A (d + x)$$

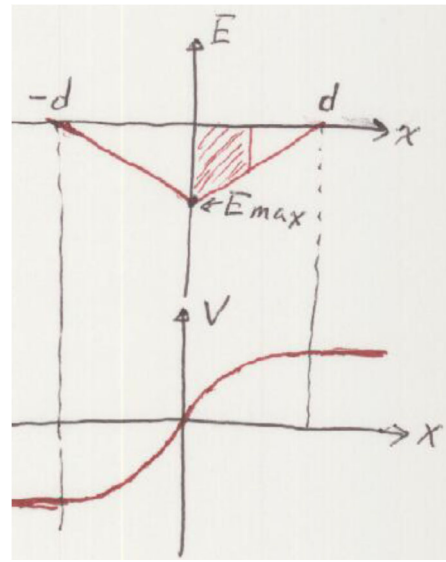
$$\therefore \boxed{E(x) = -\rho(d + x)/\varepsilon}$$

Give partial credit up to 3 points if student demonstrates understanding of the method but does not obtain correct result.

5 for $-d < x < 0$.

3 For $0 < x < d$, the potential

$$V(x) = V(0) - \int_0^x E(x)\,dx$$
$$= 0 + \frac{\rho}{\varepsilon}\int_0^x (d-x)\,dx$$
$$\boxed{= \frac{+\rho}{2\varepsilon}\,x(2d-x)}$$



2

(You may do the integral graphically: the shaded area.)

2 For $x \geq d$, the field is 0 therefore the potential is constant:

$$V(x) = V(d) = \frac{+\rho}{2\varepsilon}\,d^2$$

2

3 For $-d < x < 0$,

$$V(x) = V(0) - \int_0^x E(x)\,dx$$
$$= 0 - \frac{\rho}{\varepsilon}\int_0^x (d+x)\,dx = \boxed{\frac{\rho}{2\varepsilon}\,x(2d+x)}$$

2 For $x \leq -d$,

$$V(x) = V(-d) = \boxed{-\frac{\rho}{2\varepsilon}\,d^2}$$

Method 2: the mathematical way

Poisson equation $\nabla^2 V = -\frac{\rho(x)}{\varepsilon}$

In this particular one-dimensional situation:

$$\nabla^2 V = \frac{d^2 V}{dx^2}$$

Gauss's law: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$

In this particular one-dimensional situation:

$$\nabla \cdot \vec{E} = \frac{dE}{dx}$$

For $0 < d < x$,

$$\frac{dE}{dx} = \frac{\rho}{\varepsilon} \quad \Rightarrow \quad E(x) = \frac{\rho}{\varepsilon}x + \text{Const.}$$

Need a boundary condition to determine the integral constant.

No charge outside the slabs $\Rightarrow \boxed{E(x \geq d) = 0}$

$$\therefore \boxed{E(x) = \frac{\rho}{\varepsilon}(x-d) = -\frac{\rho}{\varepsilon}(d-x)}$$

For $-d < x < 0$.

$$\frac{dE}{dx} = \frac{-\rho}{\varepsilon} \Rightarrow E(x) = -\frac{\rho}{\varepsilon}x + \text{const}$$

Similarly, using the boundary condition $E(x \leq -d) = 0$

we get

$$\boxed{E(x) = \frac{-\rho}{\varepsilon}(d+x)}$$

Then take integrals to get $V(x)$.

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Tested and to be learned:
Dielectrics, dielectric breakdown.
Capacitors. Gauss's law. Energy stored in capacitors.
Conservativity of electrostatic fields. Boundary conditions.

Problem 3. Electrostatics

Total 21

(1) Two infinitely large, parallel, perfect conductor plates form a capacitor. The distance between the plates is d, and the permittivity of the dielectric between the two plates is ε. ***Derive*** an expression for the capacitance per area (simply writing down the result does not earn points). Hint: Refer to Figure (1) below, where the density of **E** field lines signifies the strength of the field, E. Since the plates are infinitely large, the field should be laterally uniform (i.e. no special locations).

(2) For a practical parallel-plate capacitor of plate area $A = ab$, where a and b are the sides of the rectangular plates, the capacitance C is approximated using the above derived expression, if $a >> d$ and $b >> d$. In doing this, we assume the field distribution at an edge is as shown in Figure (2) Left, but the actual distribution is as shown in Figure (2) Right. The "fringe effect" at edges is neglected in the approximate model. Show that the approximate model on the left violates some fundamental physical law.

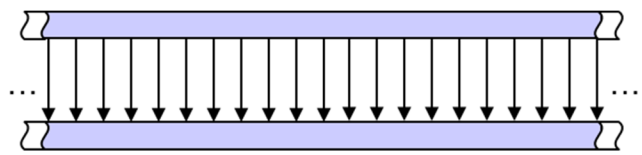

Fig. (1)

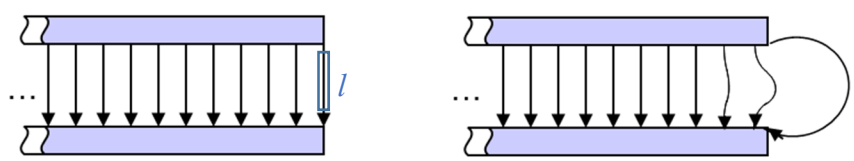

Fig. (2)

Solution:

3 (1) By Gauss's law,

$$E = \frac{\rho_s}{\varepsilon}$$

where $\pm\rho_s$ is the surface charge density of each plate.

Since the field is uniform, the voltage between the plates is $V = Ed$.

By definition, the capacitance of an area A is

$$C = \frac{Q}{V} = \frac{\rho_s A}{Ed} = \frac{\rho_s A}{\frac{\rho_s}{\varepsilon} d} = \varepsilon \frac{A}{d}$$

$$\therefore \frac{C}{A} = \frac{\varepsilon}{d}$$

See class note Electrostatics V: Capacitors, p. 2

Alternative proof: Find the field by adding up contributions from the two plates.

3 (2) For a rectangle loop shown in Fig. (2) Left, with length l and width 0,

$$\oint E\,dl = El \neq 0$$

The conservativity of the electrostatic field (a special case of Faraday's law) is violated.

See class note Electrostatics V: Capacitors, p. 1

Problem 3 continued HW 9, Problem 6

(3) As shown in Figure (3), a parallel-plate capacitor of plate distance d and area A is filled with two dielectrics, with permittivities ε_1 and ε_2, each in half of the plate area. Find the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two dielectrics if the voltage between the plates is V, ignoring the fringe effect.

(4) ***Derive*** an expression for the total capacitance (simply writing down the result does not earn points) of the capacitor in Figure (3).

(5) How much energy does this capacitor store when charged to voltage V?

(6) The breakdown fields are E_{b1} and E_{b2} for the two dielectrics, respectively. Given $E_{b1} > E_{b2}$, what is the maximum amount of energy that can be stored in this capacitor?

(7) Is there any fringe effect at the interface between the two dielectrics? Why?

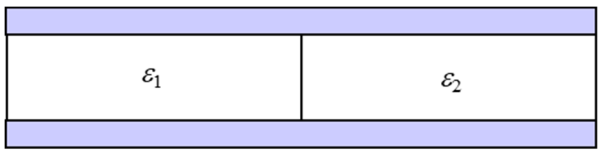

Fig. (3)

Solutions:

3 (3) $E_1 = E_2 = V/d$. Both $\mathbf{E}_1$ and $\mathbf{E}_2$ are perpendicular to the plates, pointing from the positively charged to the negatively charged plate. (Partial credit: 1 for E1 = E2, 1 for the value, 1 for the direction)

3 (4) The surface charge density of the two parts are

$$\boxed{\rho_{s1} = \varepsilon_1 E_1 = \varepsilon_1 \frac{V}{d}}, \quad \boxed{\rho_{s2} = \varepsilon_2 E_2 = \varepsilon_2 \frac{V}{d}}$$

Each of these 3 steps earns a partial credit of 1 point.

Therefore,

$$C = \frac{Q}{V} = \boxed{\frac{\rho_{s1} A/2 + \rho_{s2} A/2}{V}} = \frac{(\varepsilon_1 + \varepsilon_2) V/d}{2V} A = \frac{\varepsilon_1 + \varepsilon_2}{2} \frac{A}{d}$$

Student gets 2.5 points if all the three are done but the summing is wrong at the end.

3 (5) The total stored energy is

$$W = \frac{1}{2} C_1 V^2 + \frac{1}{2} C_2 V^2 = \frac{1}{2} \frac{\varepsilon_1 + \varepsilon_2}{2} \frac{A}{d} V^2 = \frac{\varepsilon_1 + \varepsilon_2}{4} \frac{A}{d} V^2$$

partial credit 2 points for this if summed incorresctly

3 (6) Since $E_{b1} > E_{b2}$,

$V_{max} = E_{b2} d$ Partial credit 1 point

$$W_{max} = \frac{\varepsilon_1 + \varepsilon_2}{4} \frac{A}{d} (E_{b2} d)^2 = \frac{\varepsilon_1 + \varepsilon_2}{4} A d E_{b2}^2$$

3 (7) No. partial credit 1 point for answering No.

$\because E_1 = E_2$

$\therefore \oint \vec{E} \cdot d\vec{l} = 0$ for any loop spanning both sides of the interface between dielectrics 1 and 2.

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Tested and to be learned:
Forces exerted by magnetic fields to moving charges.
Magnetic force does no work.
emf. Energy exchanges in electromechanical systems and circuits.

Problem 4. Magnetostatics

Total 14

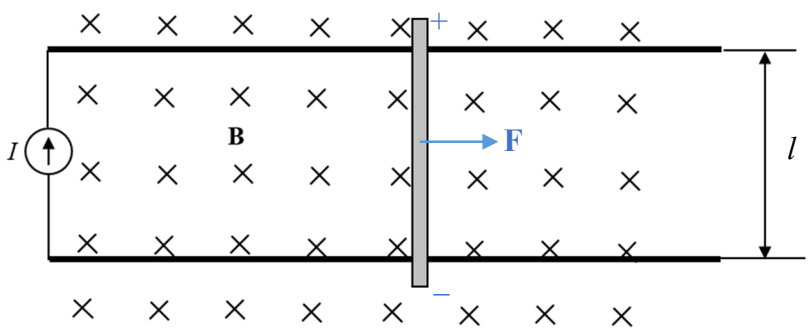
See the figure below. In a uniform magnetic field **B** (shown by crosses, going into the paper), a metal bar (considered a perfect conductor) is placed on a pair of parallel metal rails (also perfect conductors) a distance l apart. An ideal current source I is connected to the rails.

(1) Find the force **F** exerted by the magnetic field on the bar in terms of **B**, I, and l. Note: **F** is a vector. You need to specify its direction. You may indicate the direction on the figure.

(2) Driven by **F**, the bar slides, and it in turn drives a mechanical load. At the steady state, the bar moves at a constant velocity **v**, in the same direction as **F**. If we measure the voltage between the rails, what value will we get? Express your answers in terms of **v**, **B**, I, and l. Indicate the polarity of the voltage in the figure.

(3) If the magnetic field were turned off, what voltage would we measure between the rails?

(4) What is the electric power fed by the current source to the moving bar? Prove that energy is conserved and that the magnetic field does not do work.



Practical implications:
Foundation for understanding practical, more complicated motors.

Class notes Magnetostatics I, p. 1, 3, 6; Dynamic Fields, p. 15-18

Solutions:

3 (1) $F = IBl$. Direction of **F** shown in figure. 1.5 points for value, 1.5 for direction.

3 (2) $V = \text{emf} = vBl$. Polarity of V shown in figure. 1.5 points for value, 1.5 for polarity.

2 (3) $V = 0$ when $B = 0$, since all conductors are perfect.

6 (4) The electric power is $IV = IvBl$. partial credit 1 point.

Must demonstrate sound logic to receive full points.

With a finite **B**, the bar moving at velocity **v** pushes the load, doing work $\mathbf{F}\cdot\mathbf{v} = Fv = IBlv$.

Any mobile charge q in the bar, moving along with the bar at **v**, experiences a force $\mathbf{qv}\times\mathbf{B}$, giving rise to the emf vBl. To counter this emf, the electric power $IvBl$ must be done to push current I into the bar.

The electric power fed into the bar equals the mechanical power that the bar exerts to push the load. Therefore, energy is conserved and the net work done by magnetic forces is 0.

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Tested and to be learned:
The concept of magnetic flux. Dot product (projection).
Faraday's law.

Problem 5. Dynamic Fields

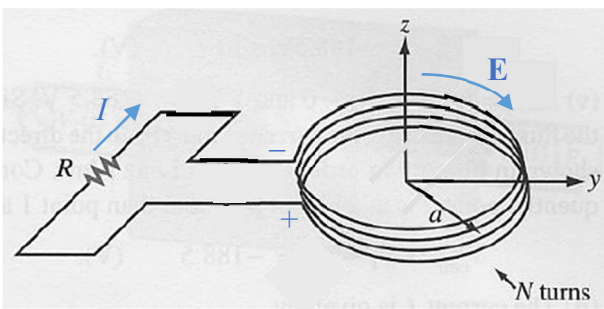
Total 10

An N-turn coil of a perfectly conducting wire is shown in the figure below. Each turn is circular with radius a, in the x-y plane with the center at the origin. The two ends are connected to a resistor R. In the presence of a magnetic field Class note Dynamic Fields, p. 13

$$\mathbf{B} = \hat{z}\,(B_0 + C_1 t) + \hat{y}\,C_2 t,$$

where t is time, find

1) An expression for the magnetic flux linking a single turn;
2) An expression for the induced emf;
3) The induced current in the circuit;
4) The direction of the current, assuming B_0, C_1, and C_2 are all positive. (Hint: Everybody is prone to mistakes. Describe your rationale, so that you can get partial points in case you get the conclusion wrong.)



Solution:

(1) The magnetic flux is

$$\Phi = B_z \pi a^2 = (B_0 + C_1 t)\pi a^2$$ 3

(2) The induced emf is:

$$|emf| = N\left|\frac{\partial \Phi}{\partial t}\right| = N C_1 \pi a^2$$ 3

The polarity of the emf and the direction of the induced electric field are shown in Figure.

(Polarity of emf not required to answer this question, although needed for the current direction in the next question.)

1 (3) The current is

$$I = \frac{N C_1 \pi a^2}{R}$$

Student receives this 1 point if showing I = emf / R but using a wrong answer to Question (2) for the emf.

3 (4) From the direction of the induced electric field that is obtained according to Faraday's law, the direction of the current is indicated by the arrow shown in the figure.

Partial credit 1 point if direction of the E field is correct.