Figure P1.2: (a) Pressure wave as a function of distance at \( t = 0 \) and (b) pressure wave as a function of time at \( x = 0 \).

**Problem 1.3** A harmonic wave traveling along a string is generated by an oscillator that completes 180 vibrations per minute. If it is observed that a given crest, or maximum, travels 300 cm in 10 s, what is the wavelength?

**Solution:**

\[
\begin{align*}
  f &= \frac{180}{60} = 3 \text{ Hz.} \\
  \nu_p &= \frac{300 \text{ cm}}{10 \text{ s}} = 0.3 \text{ m/s.} \\
  \lambda &= \frac{\nu_p}{f} = \frac{0.3}{3} = 0.1 \text{ m} = 10 \text{ cm}.
\end{align*}
\]

**HW1:P2**

Two waves, \( y_1(t) \) and \( y_2(t) \), have identical amplitudes and oscillate at the same frequency, but \( y_2(t) \) leads \( y_1(t) \) by a phase angle of 60°. If \( y_1(t) = 4 \cos(2\pi \times 10^3 t) \), write down the expression appropriate for \( y_2(t) \) and plot both functions over the time span from 0 to 2 ms.

**Solution:**

\[
y_2(t) = 4 \cos(2\pi \times 10^3 t + 60^\circ).
\]
The height of an ocean wave is described by the function

\[ y(x, t) = 1.5 \sin(0.5t - 0.6x) \text{ (m)}. \]

Determine the phase velocity and the wavelength and then sketch \( y(x, t) \) at \( t = 2 \) s over the range from \( x = 0 \) to \( x = 2\lambda \).

**Solution:** The given wave may be rewritten as a cosine function:

\[ y(x, t) = 1.5 \cos(0.5t - 0.6x - \pi/2). \]

By comparison of this wave with Eq. (1.32),

\[ y(x, t) = A \cos(\omega x - \beta x + \phi_0), \]

we deduce that

\[ \omega = 2\pi f = 0.5 \text{ rad/s}, \quad \beta = \frac{2\pi}{\lambda} = 0.6 \text{ rad/m}, \]

\[ u_p = \frac{\omega}{\beta} = \frac{0.5}{0.6} = 0.83 \text{ m/s}, \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.6} = 10.47 \text{ m}. \]
At $t = 2$ s, $y(x, 2) = 1.5 \sin(1 - 0.6x)$ (m), with the argument of the cosine function given in radians. Plot is shown in Fig. P1.5.

A wave traveling along a string in the $+x$-direction is given by

$$y_1(x, t) = A \cos(\omega t - \beta x),$$

where $x = 0$ is the end of the string, which is tied rigidly to a wall, as shown in Fig. 1-21 (P1.6). When wave $y_1(x, t)$ arrives at the wall, a reflected wave $y_2(x, t)$ is generated. Hence, at any location on the string, the vertical displacement $y_s$ will be the sum of the incident and reflected waves:

$$y_s(x, t) = y_1(x, t) + y_2(x, t).$$

(a) Write down an expression for $y_2(x, t)$, keeping in mind its direction of travel and the fact that the end of the string cannot move.

(b) Generate plots of $y_1(x, t)$, $y_2(x, t)$ and $y_s(x, t)$ versus $x$ over the range $-2\lambda \leq x \leq 0$ at $\omega t = \pi/4$ and at $\omega t = \pi/2$.

Solution:

(a) Since wave $y_2(x, t)$ was caused by wave $y_1(x, t)$, the two waves must have the same angular frequency $\omega$, and since $y_2(x, t)$ is traveling on the same string as $y_1(x, t)$,
the two waves must have the same phase constant $\beta$. Hence, with its direction being in the negative $x$-direction, $\gamma_2(x,t)$ is given by the general form

$$\gamma_2(x,t) = B\cos(\omega t + \beta x + \phi_0),$$

(1)

where $B$ and $\phi_0$ are yet-to-be-determined constants. The total displacement is

$$\gamma_s(x,t) = \gamma_1(x,t) + \gamma_2(x,t) = A\cos(\omega t - \beta x) + B\cos(\omega t + \beta x + \phi_0).$$

Since the string cannot move at $x = 0$, the point at which it is attached to the wall, $\gamma_s(0,t) = 0$ for all $t$. Thus,

$$\gamma_s(0,t) = A\cos(\omega t) + B\cos(\omega t + \phi_0) = 0.$$  

(2)

(i) Easy Solution: The physics of the problem suggests that a possible solution for (2) is $B = -A$ and $\phi_0 = 0$, in which case we have

$$\gamma_2(x,t) = -A\cos(\omega t + \beta x).$$

(3)

(ii) Rigorous Solution: By expanding the second term in (2), we have

$$A\cos(\omega t) + B(\cos(\omega t \cos \phi_0 - \sin \omega t \sin \phi_0)) = 0,$$

or

$$(A + B \cos \phi_0) \cos \omega t - (B \sin \phi_0) \sin \omega t = 0.$$  

(4)

This equation has to be satisfied for all values of $t$. At $t = 0$, it gives

$$A + B \cos \phi_0 = 0,$$

(5)
and at $\omega t = \pi / 2$, (4) gives

$$B \sin \phi_0 = 0. \quad (6)$$

Equations (5) and (6) can be satisfied simultaneously only if

$$A = B = 0 \quad (7)$$

or

$$A = -B \quad \text{and} \quad \phi_0 = 0. \quad (8)$$

Clearly (7) is not an acceptable solution because it means that $y_1(x, t) = 0$, which is contrary to the statement of the problem. The solution given by (8) leads to (3).

(b) At $\omega t = \pi / 4$,

$$y_1(x, t) = A \cos(\pi / 4 - \beta x) = A \cos \left( \frac{\pi}{4} - \frac{2\pi x}{\lambda} \right),$$

$$y_2(x, t) = -A \cos(\omega t + \beta x) = -A \cos \left( \frac{\pi}{4} + \frac{2\pi x}{\lambda} \right).$$

Plots of $y_1$, $y_2$, and $y_5$ are shown in Fig. P1.6(b).

![Figure P1.6: (b) Plots of $y_1$, $y_2$, and $y_5$ versus $x$ at $\omega t = \pi / 4$.](image)

At $\omega t = \pi / 2$,

$$y_1(x, t) = A \cos(\pi / 2 - \beta x) = A \sin \beta x = A \sin \frac{2\pi x}{\lambda},$$
CHAPTER 1

\[ y_2(x, t) = -A \cos(\pi/2 + \beta x) = A \sin \beta x = A \sin \frac{2\pi x}{\lambda}. \]

Plots of \( y_1, y_2, \) and \( y_3 \) are shown in Fig. P1.6(c).

\[ \omega t = \pi/2 \]

Figure P1.6: (c) Plots of \( y_1, y_2, \) and \( y_3 \) versus \( x \) at \( \omega t = \pi/2. \)

Problem 1.7 Two waves on a string are given by the following functions:

\[ y_1(x, t) = 4 \cos(20t - 30x) \quad \text{(cm)}, \]
\[ y_2(x, t) = -4 \cos(20t + 30x) \quad \text{(cm)}, \]

where \( x \) is in centimeters. The waves are said to interfere constructively when their superposition \( |y_s| = |y_1 + y_2| \) is a maximum and they interfere destructively when \( |y_s| \) is a minimum.

(a) What are the directions of propagation of waves \( y_1(x, t) \) and \( y_2(x, t) \)?

(b) At \( t = (\pi/50) \) s, at what location \( x \) do the two waves interfere constructively, and what is the corresponding value of \( |y_s| \)?

(c) At \( t = (\pi/50) \) s, at what location \( x \) do the two waves interfere destructively, and what is the corresponding value of \( |y_s| \)?

Solution:

(a) \( y_1(x, t) \) is traveling in positive \( x \)-direction. \( y_2(x, t) \) is traveling in negative \( x \)-direction.
(b) At \( t = (\pi/50) \) s, \( y_s = y_1 + y_2 = 4[\cos(0.4\pi - 30x) - \cos(0.4\pi + 3x)] \). Using the formulas from Appendix C,
\[
2\sin x \sin y = \cos(x - y) - (\cos x + y),
\]
we have
\[
y_s = 8\sin(0.4\pi) \sin 30x = 7.61 \sin 30x.
\]
Hence,
\[
|y_s|_{\text{max}} = 7.61
\]
and it occurs when \( \sin 30x = 1 \), or \( 30x = \frac{\pi}{2} + 2n\pi \), or \( x = \left( \frac{\pi}{60} + \frac{2n\pi}{30} \right) \) cm, where \( n = 0, 1, 2, \ldots \).

(c) \( |y_s|_{\text{min}} = 0 \) and it occurs when \( 30x = n\pi \), or \( x = \frac{n\pi}{30} \) cm.

**Problem 1.8**  Give expressions for \( y(x, t) \) for a sinusoidal wave traveling along a string in the negative \( x \)-direction, given that \( y_{\text{max}} = 40 \) cm, \( \lambda = 30 \) cm, \( f = 10 \) Hz, and

(a) \( y(x, 0) = 0 \) at \( x = 0 \),

(b) \( y(x, 0) = 0 \) at \( x = 7.5 \) cm.

**Solution:** For a wave traveling in the negative \( x \)-direction, we use Eq. (1.17) with \( \omega = 2\pi f = 20\pi \) (rad/s), \( \beta = 2\pi/\lambda = 2\pi/0.3 = 20\pi/3 \) (rad/s), \( A = 40 \) cm, and \( x \) assigned a positive sign:
\[
y(x, t) = 40\cos \left( 20\pi t + \frac{20\pi}{3}x + \phi_0 \right) \text{ (cm)},
\]
with \( x \) in meters.

(a) \( y(0, 0) = 0 = 40\cos \phi_0 \). Hence, \( \phi_0 = \pm \pi/2 \), and
\[
y(x, t) = 40\cos \left( 20\pi t + \frac{20\pi}{3}x \pm \frac{\pi}{2} \right)
= \begin{cases} 
-40\sin \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = \pi/2, \\
40\sin \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = -\pi/2.
\end{cases}
\]

(b) At \( x = 7.5 \) cm = \( 7.5 \times 10^{-2} \) m, \( y = 0 = 40\cos(\pi/2 + \phi_0) \). Hence, \( \phi_0 = 0 \) or \( \pi \), and
\[
y(x, t) = \begin{cases} 
40\cos \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = 0, \\
-40\cos \left( 20\pi t + \frac{20\pi}{3}x \right) \text{ (cm)}, & \text{if } \phi_0 = \pi.
\end{cases}
\]
Hence, \( y_2(t) \) lags \( y_1(t) \) by 54°.

**Problem 1.12** The voltage of an electromagnetic wave traveling on a transmission line is given by \( v(z,t) = 5e^{-\alpha x} \sin(4\pi \times 10^9 t - 20\pi z) \) (V), where \( z \) is the distance in meters from the generator.

(a) Find the frequency, wavelength, and phase velocity of the wave.

(b) At \( z = 2 \) m, the amplitude of the wave was measured to be 1 V. Find \( \alpha \).

**Solution:**

(a) This equation is similar to that of Eq. (1.28) with \( \omega = 4\pi \times 10^9 \) rad/s and \( \beta = 20\pi \) rad/m. From Eq. (1.29a), \( f = \omega/2\pi = 2 \times 10^9 \) Hz = 2 GHz; from Eq. (1.29b), \( \lambda = 2\pi/\beta = 0.1 \) m. From Eq. (1.30),

\[ u_p = \omega/\beta = 2 \times 10^8 \text{ m/s}. \]

(b) Using just the amplitude of the wave,

\[ 1 = 5e^{-\alpha z}, \quad \alpha = -\frac{1}{2} \ln \left( \frac{1}{5} \right) = 0.81 \text{ Np/m}. \]

**Problem 1.13** A certain electromagnetic wave traveling in sea water was observed to have an amplitude of 98.02 (V/m) at a depth of 10 m and an amplitude of 81.87 (V/m) at a depth of 100 m. What is the attenuation constant of sea water?

**Solution:** The amplitude has the form \( Ae^{-\alpha z} \). At \( z = 10 \) m,

\[ Ae^{-10\alpha} = 98.02 \]

and at \( z = 100 \) m,

\[ Ae^{-100\alpha} = 81.87 \]

The ratio gives

\[ \frac{e^{-10\alpha}}{e^{-100\alpha}} = \frac{98.02}{81.87} = 1.20 \]

or

\[ e^{-10\alpha} = 1.2e^{-100\alpha}. \]

Taking the natural log of both sides gives

\[ \ln(e^{-10\alpha}) = \ln(1.2e^{-100\alpha}), \]

\[-10\alpha = \ln(1.2) - 100\alpha, \]

\[ 90\alpha = \ln(1.2) = 0.18. \]

Hence,

\[ \alpha = \frac{0.18}{90} = 2 \times 10^{-3} \text{ (Np/m)}. \]
Section 1-6: Phasors

Problem 1.21 A voltage source given by \(v_s(t) = 25 \cos(2\pi \times 10^3 t - 30^\circ) \) (V) is connected to a series RC load as shown in Fig. 1-19. If \(R = 1 \text{ M\Omega} \) and \(C = 200 \text{ pF} \), obtain an expression for \(v_c(t) \), the voltage across the capacitor.

Solution: In the phasor domain, the circuit is a voltage divider, and

\[
\vec{V}_c = \frac{1/j\omega C}{R + 1/j\omega C} \vec{V}_s = \frac{1}{{R + 1/j\omega C}} \vec{V}_s.
\]

Now \(\vec{V}_s = 25e^{-j30^\circ} \) V with \(\omega = 2\pi \times 10^3 \) rad/s, so

\[
\vec{V}_c = \frac{25e^{-j30^\circ}}{1 + j((2\pi \times 10^3 \text{ rad/s}) \times (10^6 \text{ \Omega}) \times (200 \times 10^{-12} \text{ F})}) = \frac{25e^{-j30^\circ}}{1 + j2\pi/5} = 15.57e^{-j81.5^\circ} \text{ V}.
\]

Converting back to an instantaneous value,

\[
v_c(t) = \Re(e^{j\omega t}v_c) = \Re(15.57e^{j(\omega \pi/8^\circ)}) = 15.57 \cos(2\pi \times 10^3 t - 81.5^\circ) \text{ V},
\]

where \(t\) is expressed in seconds.

Problem 1.22 Find the phasors of the following time functions:

(a) \(v(t) = 3 \cos(\omega t - \pi/3) \) (V),
(b) \(v(t) = 12 \sin(\omega t + \pi/4) \) (V),
(c) \(i(t) = e^{-3t} \sin(\omega t + \pi/6) \) (A),
(d) \(i(t) = -2 \cos(\omega t + 3\pi/4) \) (A),
(e) \(i(t) = 4 \sin(\omega t + \pi/3) + 3 \cos(\omega t - \pi/6) \) (A).

Solution:

(a) \(\vec{V} = 3e^{-j\pi/3} \text{ V}\),
(b) \(\vec{v}(t) = 12 \sin(\omega t + \pi/4) = 12 \cos(\pi/2 - (\omega t + \pi/4)) = 12 \cos(\omega t - \pi/4) \text{ V}, \vec{V} = 12e^{-j\pi/4} \text{ V}\),
(c) \(i(t) = 2e^{-3x} \sin(\omega t + \pi/6) A = 2e^{-3x} \cos(\pi/2 - (\omega t + \pi/6)) A = 2e^{-3x} \cos(\omega t - \pi/3) A, \vec{i} = 2e^{-3x} e^{-j\pi/3} \text{ A}\).
Find the instantaneous time sinusoidal functions corresponding to the following phasors:

(a) \( \bar{V} = -5e^{j\pi/3} \) (V),
(b) \( \bar{V} = 6e^{-j\pi/4} \) (V),
(c) \( \bar{I} = (6 + j8) \) (A),
(d) \( \bar{I} = -3 + j2 \) (A),
(e) \( \bar{I} = j \) (A),
(f) \( \bar{I} = 2e^{j\pi/6} \) (A).

Solution:

(a) 
\[ \bar{V} = -5e^{j\pi/3} \quad V = 5e^{j(\pi/3-\pi)} = 5e^{-j2\pi/3} \]  
\[ v(t) = 5 \cos (\omega t - 2\pi/3) \]  
(b) 
\[ \bar{V} = 6e^{-j\pi/4} \quad V = 6e^{j(-\pi/4+\pi/2)} = 6e^{j\pi/4} \]  
\[ v(t) = 6 \cos (\omega t + \pi/4) \]  
(c) 
\[ \bar{I} = (6 + j8) \quad A = 10e^{j53.1^\circ} \]  
\[ i(t) = 10 \cos (\omega t + 53.1^\circ) \]  
(d) 
\[ \bar{I} = -3 + j2 = 3.61e^{j146.31^\circ} \]  
\[ i(t) = 9 \Re\{3.61e^{j146.31^\circ}\} = 3.61 \cos (\omega t + 146.31^\circ) \]
(e) \[ \vec{I} = j = e^{j\pi/2}, \]
\[ i(t) = \Re\{e^{j\pi/2}e^{j\omega t}\} = \cos(\omega t + \pi/2) = -\sin \omega t \text{ A}. \]

(f) \[ \vec{I} = 2e^{j\pi/6}, \]
\[ i(t) = \Re\{2e^{j\pi/6}e^{j\omega t}\} = 2\cos(\omega t + \pi/6) \text{ A}. \]

**Problem 1.24** A series RLC circuit is connected to a generator with a voltage \( v_s(t) = V_0 \cos(\omega t + \pi/3) \) (V).

(a) Write down the voltage loop equation in terms of the current \( i(t) \), \( R \), \( L \), \( C \), and \( v_s(t) \).

(b) Obtain the corresponding phasor-domain equation.

(c) Solve the equation to obtain an expression for the phasor current \( \vec{I} \).

![RLC circuit diagram](image)

Figure P1.24: RLC circuit.

**Solution:**

(a) \( v_s(t) = Ri + L \frac{di}{dt} + \frac{1}{C} \int i dt. \)

(b) In phasor domain: \( \vec{V_s} = R\vec{I} + j\omega LI + \frac{\vec{I}}{j\omega C} \).

(c) \[ \vec{I} = \frac{\vec{V_s}}{R + j(\omega L - 1/\omega C)} = \frac{\omega CV_0e^{j\pi/3}}{R + j(\omega^2 LC - 1)}. \]

**Problem 1.25** A wave traveling along a string is given by

\[ \gamma(x,t) = 2\sin(4\pi t + 10\pi x) \text{ (cm)}\]