Quantum mechanics is a new way of representing the world:

- An amplitude for every “event”
- $|\text{the amplitude of an event}|^2 = \text{the probability of the event}$

The amplitude is somewhat behind the scene. The probability is all that we can possibly know. This randomness is different from that in classic statistical mechanics.

Accept this as a basic assumption. The philosophical interpretation is up to you.

When the “event” is an object (electron) being at position $r$ at time $t$, the “amplitude” $\psi(r, t)$ is the wave function.

$\psi(r, t)$ can often be viewed as a superposition of plane waves $e^{i(k \cdot r - \omega t)}$ (Fourier transform).

For the special case of plane waves, $E = \hbar \omega$ (momentum somehow related to space periodicity or translational symmetry)
We now have a problem.

For an electron in uniform motion (not a quantum way to say things) $e^{i(k \cdot r - \omega t)}$

$$|e^{i(k \cdot r - \omega t)}|^2 = 1$$

A good representation of the particle (electron) is a wave packet (or short wave train).

Taken from http://universe-review.ca/R01-04-diffeq02.htm
If there are only two wavelengths
Remember moiré?

Constructive interference will occur in regular intervals.

Completely in phase

Completely out of phase
If there are lots of \textit{discrete} wavelengths
If there are a continuum of wavelengths

Constructive interference only at one point

There’s no common multiple of all these wavelengths

$\Delta x \Delta k \approx 1$

$\Delta x (\hbar \Delta k) \approx \hbar$

$\Delta x \Delta p \approx \hbar$
After this class, you will have the foundations for the next reading assignment:

Ch. 16, Ch. 19 of The Feynman Lectures on Physics


Everybody’s background is different. You may not be able to understand everything. Go back and forth between these. At the end, hopefully you get a vivid picture. You may find some chemistry books more helpful, coz they give you vivid graphics.

You are on the way to understand orbitals, hybridization, etc, and to answer the question Isaac raised in the last class.
Bra-ket (Dirac) Notation

Any state of a system is like a vector in the state space

\[ |\phi\rangle = \sum C_i |i\rangle \] basis states, must be a complete set.

\[ C_i = \langle i | \phi \rangle \]

\[ \langle i | j \rangle = \delta_{ij} = \begin{cases} 0, & |i\rangle \neq |j\rangle \\ 1, & |i\rangle = |j\rangle \end{cases} \]

Stationary states

If a state has a definite energy, it's a stationary state.

Its probabilities do not depend on time, although the corresponding amplitudes do, following

\[ e^{-i \frac{E_0}{\hbar} t} = e^{-iwt}, \quad E_0 = \hbar \omega \]

not that number i.

\[ \hat{r} = x \hat{x} + y \hat{y} + z \hat{z} \]

\[ x = \hat{x} \cdot \hat{r} \]

\[ y = \hat{y} \cdot \hat{r} \]

\[ z = \hat{z} \cdot \hat{r} \]

\[ \hat{x} \cdot \hat{x} = 1 \]

\[ x \cdot \hat{y} = 0 \]

You are free to choose the basis.
More on State Vectors

\[ |\Phi\rangle = \sum_i c_i |i\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \end{pmatrix} \]

\[ |\chi\rangle = \sum_i D_i |i\rangle = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_i \\ \vdots \end{pmatrix} \]

Projections or dot products

\[ \langle \chi | \Phi \rangle = (D_1^*, D_2^*, \ldots, D_i^*, \ldots) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_i \\ \vdots \end{pmatrix} = \sum_i D_i^* C_i \]

\[ \langle \Phi | \chi \rangle = (c_1^*, c_2^*, \ldots, c_i^*, \ldots) \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_i \\ \vdots \end{pmatrix} = \sum_i C_i^* D_i \]
\[ \langle \phi | \chi \rangle = \langle \chi | \phi \rangle^* \quad | \quad \vec{r}_1 \cdot \vec{r}_2 = \vec{r}_2 \cdot \vec{r}_1 \]

Origin of the resemblance (or difference):

Solutions of linear differential equations.

Again, the philosophical interpretation is up to you.

Operators

If you do something to the system (an operation, system going through an apparatus, you doing a measurement, etc.), you transform the state vector, which is nevertheless still in the same state space.

\[ A | \phi \rangle = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & \ddots & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_j A_{1j} c_j \\ \sum_j A_{2j} c_j \\ \vdots \end{pmatrix} \]

An operator is a matrix.
The "Waiting" Operator

\[ |\psi(t+\Delta t)\rangle = U(t, \Delta t) |\psi(t)\rangle \]

\[ = \sum_j U_{ij}(t, \Delta t) C_j(t) \quad \text{A state, not your usual wave function} \]

Let \( \Delta t \to 0 \),

\[ C_i(t+\Delta t) = C_i(t) + \sum_j K_{ij}(t) C_j \Delta t \]

In other words,

\[ U_{ij}(t, \Delta t) \xrightarrow{\Delta t \to 0} U_{ij}(t) = \delta_{ij} + K_{ij}(t) \Delta t \]

\[ \frac{C_i(t+\Delta t) - C_i(t)}{\Delta t} = \sum_j K_{ij}(t) C_j \]

\[ \frac{dC_i}{dt} \]

To conform to established conventions, let \( K_{ij}(t) = -\frac{i}{\hbar} H_{ij}(t) \)
\[ \frac{dC_i}{dt} = -\frac{i}{\hbar} \sum_j H_{ij} C_j(t) \]

A set of linear diff. eqs.

If we know \( H_{ij} \) of a system and the initial conditions, we know (in the quantum mechanics sense) the system.

The matrix \( H \) is called the Hamiltonian.

An important property of \( H \):

\[ H_{ij} = H_{ji}^* \]

This follows from \( \sum_i |C_i(t)|^2 = \text{Constant} \)
General 2-State Systems

\[ |\Psi\rangle = C_1 |1\rangle + C_2 |2\rangle \]

\[
\begin{align*}
\frac{i\hbar}{\partial t} C_1 &= H_{11} C_1 + H_{12} C_2 \\
\frac{i\hbar}{\partial t} C_2 &= H_{21} C_1 + H_{22} C_2
\end{align*}
\]

In general, \( H_{12} \neq 0 \), \( H_{21} \neq 0 \).

If we start with \( |1\rangle \), i.e., \( C_2 = 0 \).

\[ \frac{i\hbar}{\partial t} C_2 = H_{21} C_1 \neq 0. \]

We will end up with a combination of \( |1\rangle \) & \( |2\rangle \).

So, these are not stationary states.

The \( \text{H}_2^+ \) ion

Due to symmetry, we should have \( H_{11} = H_{22} \). Let \( H_{11} = H_{22} = E_0 \).

We know \( H_{12} = H_{21}^* \).

However, symmetry does not guarantee \( H_{12} = H_{21} \).

(Will talk about this later)

Let \( H_{12} = A \), \( H_{21} = A^* \).
Using matrices,

\[ i \hbar \frac{d}{dt} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} E_0 & A \\ A^* & E_0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \]

We just need to find another set of basis vector, say \( |a\rangle \times |b\rangle \), such that

\[ |\psi\rangle = c_a |a\rangle + c_b |b\rangle \]

\[ H_{ab} = H_{ba} = 0 \]

\[ i \hbar \frac{dC_a}{dt} = H_{aa} C_a \quad \text{stationary states} \]

\[ i \hbar \frac{dC_b}{dt} = H_{bb} C_b \quad \text{stationary states} \]

In linear algebra, this means we need to "diagonalize" the matrix \( H \), and find the eigenvalues \( H_{aa} \) and \( H_{bb} \), and
Let the eigenvalue be $\lambda$.

\[
\begin{vmatrix}
E_0 - \lambda & A \\
A^* & E_0 - \lambda
\end{vmatrix} = 0, \quad \Rightarrow (E_0 - \lambda)^2 = A^*A = |A|^2
\]

\[
\Rightarrow E_0 - \lambda = \pm |A|
\]

\[
\Rightarrow \lambda = E_0 \pm |A|
\]

These are the 2 eigenvalues.

Note: In general, $A$ is a complex number.

Now, let's find the eigenvectors.

First, the eigenvector corresponding to $E_0 - |A|$:

\[
\begin{pmatrix}
E_0 & A \\
A^* & E_0
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} = (E_0 - |A|)
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix}
\]

\[
E_0C_1 + AC_2 = (E_0 - |A|)C_1
\]

\[
AC_2 = -|A|C_1
\]

\[
C_1 = -\frac{A}{|A|} C_2
\]

\[
A^*C_1 + E_0C_2 = (E_0 - |A|)C_2
\]

\[
A^*C_1 = -|A|C_2
\]

\[
C_1 = -\frac{|A|}{A^*} C_2
\]
Are these two consistent with each other?

Sure, \( \frac{A}{|A|} = \frac{|A|}{A^*} \) since \( |A|^2 = AA^* \).

Then, the eigenvector corresponding to \( E_0 + |A| \):

\[
\begin{pmatrix}
E_0 & A \\
A^* & E_0
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2
\end{pmatrix} =
\begin{pmatrix}
(E_0 + |A|)C_1 \\
E_0 C_1 + A C_2 = (E_0 + |A|) C_1
\end{pmatrix}
\]

\[
A^* C_1 + E_0 C_2 = (E_0 + |A|) C_2
\]

\[
A C_2 = |A| C_1
\]

\[
C_1 = \frac{A}{|A|}
\]

\[
A \frac{A}{|A|} = \frac{|A|}{A^*}
\]

\[
\text{Let } A = |A| e^{i\Phi_a}, \quad \frac{A}{|A|} = e^{i\Phi_a}
\]
The two eigenvectors are \( \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\phi_a} \\ 1 \end{pmatrix} \) \& \( \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_a} \\ 1 \end{pmatrix} \) corresponding to eigenvalues \( E_0 - 1A1 \) \& \( E_0 + 1A1 \), respectively.

Now, let's look at \( |1\rangle \& |2\rangle \).

If \( |1\rangle \& |2\rangle \) make a complete orthogonal set, then \( |1\rangle \& e^{i\phi}|2\rangle \) with any \( \phi \) also make a complete orthogonal set. — Prove it.

\[ \langle 1|1\rangle = \langle 2|2\rangle = 1 \quad \langle 1|2\rangle = \langle 2|1\rangle = 0 \]

* There is always a phase uncertainty.
If $|1\rangle$ and $|2\rangle$ make a complete orthogonal set, so do $\frac{1}{\sqrt{2}} (|1\rangle + |2\rangle)$ and $\frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)$.

If $|1\rangle$ and $|2\rangle$ make a complete orthogonal set, so do $\frac{1}{\sqrt{2}} (|1\rangle + e^{i\phi} |2\rangle)$ and $\frac{1}{\sqrt{2}} (|1\rangle - e^{i\phi} |2\rangle)$.

Now, let's re-write the two eigenvectors as

$$\frac{1}{\sqrt{2}} (\begin{pmatrix} 1 \\ -e^{-i\phi_a} \end{pmatrix}) \quad \& \quad \frac{1}{\sqrt{2}} (\begin{pmatrix} 1 \\ e^{-i\phi_a} \end{pmatrix}), \quad \text{i.e.}$$

$$\frac{1}{\sqrt{2}} (|1\rangle - e^{-i\phi_a} |2\rangle) \quad \& \quad \frac{1}{\sqrt{2}} (|1\rangle + e^{-i\phi_a} |2\rangle), \quad \text{corresponding to} \quad E_0 - |A| \quad \& \quad E_0 + |A|, \quad \text{respectively.}$$
The choice of $|2\rangle$ has a phase uncertainty (w/ respect to $|1\rangle$). We can always choose $|2\rangle$ to be in phase w/ $|1\rangle$.

Then, due to symmetry, we have $H_{12} = H_{21}^*$. And, we already know $H_{12} = H_{21}^*$. 

$\therefore H_{12} = H_{21} = \mathbf{A}$ must be real.

$\therefore \phi_A = 0$ or $\phi_A = \pi$, i.e.

$A > 0$ or $A < 0$

If $A < 0$, then the two eigenvalues $E_0 - |A|$ & $E_0 + |A|$ correspond to

$\frac{1}{\sqrt{2}} (|1\rangle + |2\rangle)$ & $\frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)$, respectively.

This makes physical sense (which is why $A < 0$)!
Another Important 2-State System: Spin 1/2

Energy of magnet in a magnetic field

\[ U = -\mu \cdot B = -\mu B \cos \theta \]

Force on the magnet

\[ F = -\frac{\partial U}{\partial z} = \mu \frac{\partial B}{\partial z} \cos \theta \]

Stern-Gerlach Experiment

If the “quantum magnets” enter a second Stern-Gerlach apparatus after exiting the first one, you will still have two values for the projection of the magnetic moment along the direction of magnetic field gradient in the second Stern-Gerlach.

The ratio between the two possibilities depends on the rotation angle between the two apparatuses.

For 90° rotations, say the magnetic gradient of the first one is in z direction, you will have equal probabilities of observing the magnetic moment in +x and −x directions, and also equal probabilities in +y and −y directions.

Interpret this?

Eigen values, eigenstates, representations...
Now, let's talk about electrons.

\[ \mu_s = - g_s \mu_B \frac{\vec{S}}{\hbar} \]  
spin angular momentum

spin magnetic moment  
Bohr magneton

\[ \mu_B = \frac{e\hbar}{2m_e} \quad \text{in SI units} \]

\[ g_s = 2.0023 \ldots \approx 2 \quad \text{the spin g-factor} \]

In any given z direction:

\[ \mu_{sz} = - g_s \mu_B S_z = - g_s \mu_B m_s \frac{\hbar}{2\hbar} = - g_s \mu_B m_s \frac{\hbar}{2\hbar} \]

\[ = - m_s \mu_B \]
For convenience, define the dimensionless operator $\hat{s}$:

$$\hat{s} = \frac{\hbar}{2} \hat{\sigma}.$$  

$$\hat{\sigma} = \hat{x} \sigma_x + \hat{y} \sigma_y + \hat{z} \sigma_z.$$  

Take the $z$ direction as a "special" one, define "up" & "down":

$$|\uparrow\rangle = |x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Recall the phase uncertainty:

$$\begin{pmatrix} e^{i\theta} \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ e^{i\theta} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ e^{i\theta} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} e^{i\theta} \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ e^{-i\theta} \end{pmatrix}.$$  

Apparently,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Diagonalized in its own representation, with eigenvalues on the diagonals.
\begin{align*}
\begin{vmatrix}
1 - \lambda & 0 \\
0 & -1 - \lambda
\end{vmatrix} &= 0 \quad \Rightarrow \quad \lambda^2 - 1 = 0 \quad \lambda = \pm 1 \\
\text{(Here } \lambda \text{ is real, coz it's a physical quantity.)}

\text{Now find the eigenvectors (eigenstates)}
\begin{align*}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix} &=
\begin{pmatrix}
a \\
b
\end{pmatrix} \\
\begin{pmatrix}
a = a \\
-b = b \Rightarrow b = 0
\end{pmatrix}
\begin{align*}
|a|^2 &= 1 \\
\text{The eigenvector corresponding to } 
\sigma_3 &= +1 \text{ (or } m_3 = +1) \text{ is } \\
\begin{pmatrix}
e^{i\theta_1} \\
0
\end{pmatrix} \\
\text{Quantum number}
\end{align*}
\begin{align*}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix} &=
\begin{pmatrix}
-a \\
-b
\end{pmatrix} \\
\begin{align*}
\alpha &= -\alpha \Rightarrow a = 0 \\
-b &= -b
\end{align*}
\begin{align*}
|b|^2 &= 1 \\
\text{The eigenstate corresponding to } 
\sigma_3 &= -1 \text{ is } \begin{pmatrix} 0 \\ e^{i\theta_2} \end{pmatrix}
\end{align*}
\text{For convenience}
\begin{align*}
|\uparrow\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}
\end{align*}
If you get the math right (i.e., the outcome of the theory agrees with the outcomes of experiments), you have:

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

For \( \sigma_x \), the eigenvalues are \( \pm 1 \), and the eigenstates are

\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \text{and} \]

\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

For \( \sigma_y \), the eigenvalues are \( \pm i \), and the eigenstates are

\[ \frac{1}{\sqrt{2}} (|\uparrow\rangle + i |\downarrow\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \text{and} \]

\[ \frac{1}{\sqrt{2}} (i |\uparrow\rangle + |\downarrow\rangle) = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \]

Now, get some scribble paper...
Exercise 1. The eigenvalues & eigenvectors of $\sigma_x$

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 - 1 = 0 \quad \Rightarrow \lambda = \pm 1
\]

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}
\]

\[
\begin{align*}
\{ & b = a \\
& a = b
\end{align*}
\]

\[
|a|^2 + |b|^2 = 1
\]

\[
\therefore a = b = e^{\imath \theta}
\]

The eigenstate corresponding to the eigenvalue $\sigma_x = +1$ is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\imath \theta}
\]

You can set $\theta = 0$

\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}
\]

\[
\begin{align*}
\{ & b = -a \\
& a = -b
\end{align*}
\]

\[
|a|^2 + |b|^2 = 1
\]

\[
\therefore a = -b = e^{\imath \theta}
\]

The eigenstate for $\sigma_x = -1$ is

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{\imath \theta}
\]
Exercise 2. The eigenvalues & eigenvectors of $\sigma_y$

\[
\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1
\]

\[
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}
\]

\[
\begin{cases} -ib = a \\ ia = b \end{cases} \Rightarrow b = ia
\]

\[
|a|^2 + |b|^2 = 1
\]

\[
\therefore \text{The eigenvector for } \sigma_y = +1
\]

is \[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i\theta}
\]

\[
\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}
\]

\[
\begin{cases} -ib = -a \\ ia = -b \end{cases} \Rightarrow a = -ib
\]

\[
|a|^2 + |b|^2 = 1
\]

\[
\therefore \text{The eigenvector for } \sigma_y = -1
\]

is \[
\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{i\theta}
\]
Exercise 3. The eigenvalues & eigenvectors of $\sigma^2$

\[ S^2 = \hat{S} \cdot \hat{S} = \left( \frac{h}{2} \right)^2 \sigma \cdot \sigma = \left( \frac{h}{2} \right)^2 \sigma^2 \]

\[ \sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \]

\[
\begin{align*}
\sigma^2 &= (0 \ 1 \ 0)(0 \ 1 \ 0) + (0 \ -i \ 0)(0 \ -i \ 0) + (1 \ 0 \ 0)(0 \ 0 \ 1) \\
&= \begin{pmatrix}
1 & 0 \\
0 & 1 
\end{pmatrix} + \begin{pmatrix}
0 & -i \\
i & 0 
\end{pmatrix} + \begin{pmatrix}
1 & 0 \\
0 & 0 
\end{pmatrix}
\end{align*}
\]
Exercise 3. The eigenvalues & eigenvectors of $\sigma_3$

\[ S^2 = \tilde{S} \cdot \tilde{S} = \left( \frac{\hbar}{2} \right)^2 \tilde{S} \cdot \tilde{S} = \left( \frac{\hbar}{2} \right)^2 \sigma^2 \]

\[ \sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \]

\[ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \]

$\sigma_x^2$, $\sigma_y^2$, $\sigma_z^2$, & $\sigma^2$ are all diagonalized in the $\tilde{S}$ representation. Their eigenstates are all common eigenstates w/ those of $\tilde{S}_z$. Each have 2 degenerate eigenvalues

\[ s^2 = \left( \frac{\hbar}{2} \right)^2 \cdot 3 = \frac{3}{4} \hbar^2 \]
\[ s^2 = 3 \left( \frac{h}{2} \right)^2 = \frac{3}{4} \frac{h^2}{2} = \left( \frac{\sqrt{3}}{2} \frac{h}{2} \right)^2 \]

\[ s_3 = \pm \frac{h}{2}, \quad s_3^2 = \left( \frac{h}{2} \right)^2 \]

\[ s^2 = s_x^2 + s_y^2 + s_3^2 \]

\[ = (\sigma_x^2 + \sigma_y^2 + \sigma_3^2) \left( \frac{h}{2} \right)^2 \]

\[ \sigma_y^2 + \sigma_3^2 = 2 \left( \frac{h}{2} \right)^2 = \frac{h^2}{2} \]

Sure \[ s^2 = s_x^2 + s_y^2 + s_3^2 \]

\[ = \left( \frac{h}{2} \right)^2 + 2 \left( \frac{h}{2} \right)^2 \]

\[ = 3 \left( \frac{h}{2} \right)^2 \]