Lesson 14
ECE 301

Transient Analysis:
These notes are only intended to supplement the material given in the text. In some cases, the material will be given in a different order than the text presentation. However, in the final analysis, the same material is covered. In any case, you should read the text in addition to these notes.

The general meaning of the word transient is well understood. This general definition applies also to circuits. We assume that the transient response of a circuit is what happens to various voltages or currents when the circuit changes from one state of excitation to another state of excitation. For example, consider the RC circuit shown in Figure 14.1.

\[ f(t) = 12V \]

Figure 14.1: An RC circuit.
Initially the voltage across the capacitor is zero (we assume this).

After the switch has been closed for a very long time, the voltage across the capacitor becomes 12 V. We illustrate this in Figure 14.2.

![Diagram showing transient and steady state of voltage across a capacitor.]

Figure 14.2: Illustrating transient region.

We know \( V_c(\infty) = 12 \text{ Volts} \) because \( V(t) \) through the capacitor is \( V(t) = 12 \text{ V} \) and we know \( V(\infty) = 0 \) because the capacitor acts as an open-circuit as \( t \to \infty \). This means \( I(\infty) = 0 \) and all of the 12 V must be across the capacitor.

Again we refer to the circuit of Figure 14.1 but with the condition that \( f(t) = 0 \) and \( V_e(\infty) = V_e \). The circuit for this case becomes as shown in Figure 14.3.
Figure 14.2: The source free RC circuit.

From this circuit we can write:
\[ i_\text{C} + i_\text{R} = 0 \]

We know:
\[ i_\text{C}(t) = C \frac{dV_\text{C}}{dt} \quad i_\text{R} = \frac{V_\text{C}}{R} \]

so:
\[ C \frac{dV_\text{C}}{dt} + \frac{V_\text{C}}{R} = 0 \quad \text{Eq. 14.1} \]

From the viewpoint of mathematics, this equation is called a first order, linear, time-invariant 
system. Since we have no forcing function, we call the 
solution to this equation the 

natural response \( \rightarrow \) complimentary solution

\( \rightarrow \) transient solution

\( \rightarrow \) transient response
There are several ways to solve this equation. We will use separation of variables here:

\[ \frac{dV_c(t)}{V_c(t)} = \frac{-dt}{RC} \]

Eq 14.3

Integrate Eq 14.3:

\[ \int_{t=0}^{t} \frac{dV_c(t)}{V_c(t)} = -\frac{1}{RC} \int_{0}^{t} dt \]

Eq 14.4

\[ \ln V_c(t) + k = -\frac{t}{RC} \]

\[ \ln(V_c(t)) = \frac{-t}{RC} + k_1 \]

\[ V_c(t) = e^{\frac{-t}{RC} + k_1} = e^{k_1} e^{\frac{-t}{RC}} = K e^{\frac{-t}{RC}} \]

We evaluate \( k \) by noting that \( V_c(0) = V_e \)

Therefore:

\[ V_c(t) = V_e \left. e^{\frac{-t}{RC}} \right|_{t=0}^{t} = K \]

Eq 14.5

\[ t = 0 \]

Thus:

\[ V_c(t) = V_e e^{\frac{-t}{RC}} \]

Eq 14.6

This is called the **free response** of the series RC circuit.
The product $RC$ has a special significance. Consider the sketch in Figure 14.4.

![Figure 14.4 Sketch](image)

**Figure 14.4** Unforced RC circuit response

Line A in the diagram represents a linear line of the form $y = mt + b$. We assume the slope of the line is defined by

$$m = \frac{\Delta y}{\Delta t} = \frac{-V_c e^{-\frac{t}{RC}}}{RC} = -\frac{V_c}{RC} \quad \text{Eq. 14.7}$$

Then

$$y = -\frac{V_c}{RC} + m$$

We evaluate $m$ by using the set $(x_0, y_0) = (0, V_c)$ so $m = V_c$. We have

$$y = -\frac{V_c}{RC} + V_c \quad \text{Eq. 14.8}$$
We define the period (time) at which this linear line intersects the t axis as \( T \). We define \( T \) as one time constant. We see from Eq. 14.8 that with \( y = 0 \),

\[
0 = -\frac{tV_c}{RC} + V_c
\]

or

\[
t = \frac{RC}{V_c}
\]

Eq. 14.9

Having defined \( T \) in this manner gives us a way to evaluate the state of the response in terms of \( T \).

Normally, one considers the circuit to be in steady state at

\[
t = 4T
\]

Eq. (14.10)

Actually,

\[
\frac{V_c}{4T} = 0.01832V_c
\]

Eq 14.11

And we may find that the response is within 7% of the final value.
The Forced RC Circuit

Now consider the RC circuit of Figure 14.5.

\[
V(t) = E u(t)
\]

Figure 14.5: The forced series RC circuit.

\( V(t) \) is a step input, starting at \( t = 0 \), of amplitude \( E \). The differential equation of the circuit in terms of \( V(t) \) is

\[
RC \frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = E u(t) \quad \text{Eq. 14.12}
\]

or

\[
\frac{\mathrm{d} V_C(t)}{\mathrm{d}t} + \frac{V_C(t)}{RC} = \frac{E}{RC} u(t) \quad \text{Eq. 14.13}
\]

We assume \( V_C(t) \) express as

\[
V_C(t) = V_{Cn}(t) + V_{ Cp}(t) \quad \text{Eq. 14.14}
\]

\( V_{Cn} = \text{natural (transient) response,} \)

\( V_{ Cp} = \text{particular (steady state) response.} \)
We first find \( V_{cp} \) by noting that the forcing function is a constant. We assume

\[ V_{cp} = K \]

and substitute this in Eq. 14.13 to find

\[ \frac{\partial}{\partial t} + \frac{K}{RC} = \frac{E}{RC} \quad \text{Eq. 14.15} \]

\[ K = E. \]

We use this in Eq. 14.14, later. First we find \( V_{en} \), which is the solution to

\[ \frac{\partial V_{en}}{\partial t} + \frac{V_{en}(t)}{RC} = 0 \quad \text{Eq. 14.16} \]

We solved this problem earlier using separation of variables. We solve it a different way below.

Assume

\[ V_{en}(t) = K Ne^{-t} \quad \text{Eq. 14.17} \]

There are some good reasons for assuming this form and they are explained in a first course in differential equations.
Substituting Eq. 14.17 into Eq. 14.16 gives

$$\frac{d}{dt} \left[ ke^{\frac{t}{RC}} \right] + \frac{ke^{\frac{t}{RC}}}{RC} = 0$$

or

$$ke^{\frac{t}{RC}} + \frac{ke^{\frac{t}{RC}}}{RC} = 0 \quad \text{Eq. 14.18}$$

giving:

$$\dot{s} + \frac{1}{RC} = 0 \quad \text{Eq. 14.19}$$

or

$$s = -\frac{1}{RC} \quad \text{Eq. 14.20}$$

Equation 14.19 is called the characteristic equation of the DE, and Eq. 14.20 gives the characteristic root, which is often called the eigenvalue.

We are led to

$$\text{ve}(t) = ke^{-\frac{t}{RC}} \quad \text{Eq. 14.21}$$

Using this in Eq. 14.14, along with the expression for $\text{ve}$, gives

$$\text{ve}(t) = E + ke^{-\frac{t}{RC}} \quad \text{Eq. 14.22}$$

It is at this point, not at Eq. 14.21, that we evaluate $k$ using initial conditions.
From the physics of the circuit we know that \( V_c(\infty) \) (steady state) will give
\[
V_c(\infty) = E \quad \text{Eq. 14.23}
\]
and we know that at \( t = 0 \)
\[
V_c(0) = \text{initial voltage on the capacitor} \quad \text{Eq. 14.24}
\]
We can solve these, from Eq. 14.27
\[
V_c(0) = E + K e^{-\frac{t}{RC}}
\]
\[
\frac{d}{dt} V_c(0) = E + K
\]
and
\[
K = V_c(0) - E
\]
then
\[
V_c(t) = E + (V_c(0) - E) e^{-\frac{t}{RC}} \quad \text{Eq. 14.23}
\]
We can also view the above, using
\[
V_c(\infty) = E \text{ as }
\]
\[
V_c(t) = V_c(\infty) + [V_c(0) - V_c(\infty)] e^{-\frac{t}{RC}} \quad \text{Eq. 14.24}
\]
We view this as the general expression for \( V_c(t) \) for any initial conditions.
If we had the differential equation,
\[ \frac{2x(t)}{R} + \frac{x(t)}{T} = \frac{f(t)}{T} \]
then
\[ x(t) = x(0) + \left[ x(\infty) - x(0) \right] e^{-\frac{t}{T}} \]

A special case of interest for Eq. 14.24 is when \( V_c(0) = 0 \). We then have
\[ V_c(t) = V_c(\infty) \left[ 1 - e^{-\frac{t}{RC}} \right] \quad \text{Eq. 14.25} \]

For \( V_c(\infty) = E \) (the input a constant value)
\[ V_c(t) = E \left( 1 - e^{-\frac{t}{RC}} \right) \quad \text{Eq. 14.26} \]

A sketch of this response is shown in Figure 14.6.

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**Figure 14.6**: Response of a series RC circuit with zero I.C. and a step input of \( V(t) = Eu(t) \).
Figure 14.6 shows that a line with an initial slope of

$$\frac{\Delta V_c(t)}{\Delta t} = \frac{d}{dt} \left[ E - E e^{-\frac{t}{RC}} \right]$$

gives

$$\frac{\Delta V_c(t)}{\Delta t} = \frac{E}{RC}$$

intersects the line, $V_c(t) = E$ at $t$. When this intersection point is projected down toward the time axis, it crosses $V_c(t)$ at

$$V_c(t) = V_c(T) = E \left(1 - e^{\frac{-t}{RC}}\right)$$

$t = T$

or

$$V_c(T) = E (1 - e^{-1}) = 0.632 E$$

Eq. 14.26

Eq. 14.27

This is significant. If we are given, or experimentally determined, the step response capacitor voltage of a series RC circuit, we can determine the time constant $T$ by dropping down from the final value of $V_c(T)$ to $0.632 \times$ final value, project horizontally until we strike the
response curve: the project 14.13
drawn to the time axis and we
can determine $R$ as what we read
on the time axis. Knowing $R$, we
know $V_C$. If we know $R$ we can
determine $C$.

A fundamental thing to remember about a
capacitor is that the voltage across
it cannot change instantaneously.
This means that when switching
occurs we remember to use:

$$V_C(0^+) = V_C(0^-)$$

Example 14.1

You are given the circuit shown
below. The switch has been closed for
a very long time. Determine $V_C(+) \geq 0$.

![Circuit Diagram]
Solution

The capacitor acts like an open circuit because it is fully charged if the switch has been closed for a very long time. Thus,

\[ I = \frac{40}{40} = 1 \text{ A} \]

So \[ V_e(0^-) = -20 + 30 = 10 \text{ V} \]

After the switch is opened we can view the circuit as

We can write the D.T. as

\[ 30C \frac{dV_e}{dt} + V_e(t) = -20 \]
\[
\frac{\partial V_c}{\partial t} + \frac{V_c}{3} = -\frac{20}{3}
\]

\[V_c(t) = -20\]

\[V_c(t) = K e^{-\frac{t}{3}}\]

\[V_c(0) = -20 + Ke^{-\frac{0}{3}}\]

\[K = 30\]

\[V_c(t) = -20 + 30e^{-\frac{t}{3}}\]

We could have used Eq. 14.24 directly:

\[V_c(t) = V_c(\infty) + [V_c(0) - V_c(\infty)] e^{-\frac{t}{3}}\]

and wrote

\[V_c(t) = -20 + [10 - (-20)] e^{-\frac{t}{3}}\]

\[V_c(t) = -20 + 30e^{-\frac{t}{3}}\] V

You should be able to sketch \(V_c(t)\) for \(t \geq 0\).
The Series L-R Circuit
We consider first the unforced case.

\[ R \frac{di}{dt} + L \frac{d^2i}{dt^2} = 0 \]

\[ \frac{\frac{di}{dt} + \frac{li}{R}}{\frac{L}{R}} = 0 \]

From application of KVL we have

\[ R \frac{di}{dt} + L \frac{d^2i}{dt^2} = 0 \]

Eq. 14.28

It is easy to show that

\[ i(t) = I_0 e^{-\frac{t}{\tau}} \]

We define \( \frac{1}{R} \) as \( \tau \), the circuit time constant and write

\[ i(t) = I_0 e^{-\frac{t}{\tau}} \]

Eq. 14.29

The thing to keep in mind with inductors is that the current cannot change instantaneously; in reality, this is the inductor looks like a short circuit.
Now consider the R-L circuit with a forcing function as shown in Figure 14.8.

\[ V = E(t) \]

**Figure 14.8: Forced R-L circuit.**

We have,

\[
\frac{d}{dt} E(t) + R I(t) = E(t)
\]

or

\[
\frac{d}{dt} I(t) + \frac{R}{L} I(t) = \frac{E(t)}{L}
\]  
Eq 14.30

We again have, similarly to the RC circuit,

\[ I(t) = I_N(t) + I_P(t) \]

\[ I_P(t) = k = \frac{E}{R} \]

\[ I_N(t) = k_N e^{-\frac{t}{\tau}} = k_N e^{-\frac{t}{\frac{L}{R}}} \]

\[ I(t) = \frac{E}{R} + k_N e^{-\frac{t}{\frac{L}{R}}} \]

\[ \gamma = \frac{L}{R} \]

We need to evaluate \( k_N \).
We assume \( I(0) = I_0 \). Then

\[
I(t) = \left[ \frac{E}{R} + K_N e^{-\frac{t}{T}} \right]_{t=0}
\]

\[
I(0) = I_0 = \left[ \frac{E}{R} + K_N e^0 \right]
\]

so

\[
K_N = I_0 - \frac{E}{R}
\]

Then

\[
I(t) = \frac{E}{R} + \left[ I_0 - \frac{E}{R} \right] e^{-\frac{t}{T}}
\]

Since

\[
I(\infty) = \frac{E}{R}, \quad I(0) = I_0
\]

we have

\[
I(t) = I(\infty) + \left[ I(0) - I(\infty) \right] e^{-\frac{t}{T}}
\]

Eq. 14.32

We see the similarity between current in the R-L circuit (Eq. 14.31) and voltage across the capacitor (Eq. 14.24). The question might arise as to how we can find the voltage across the inductor for the R-L circuit or the capacitor current for the R-C circuit. We now turn our attention to this question.
Example 14.2

The switch in the circuit below has been closed long enough for the current $i(t)$ to be constant. At $t=0$, the switch is open. Find $i(t) = 0$.

\[ i(t) \text{ for } t < 0 : \quad i = \frac{10}{6} = 2A = i(0) \]

So we know:

\[ i'(0) = i'(0) = 2A \]

For $t > 0$, the circuit becomes:

\[ i(t) = i(0) + \left( i(0) - i(0) \right) e^{-\frac{t}{\tau}} \]

\[ i(0) = 0, \quad i'(0) = 2A, \quad \tau = \frac{L}{R} = 2 \text{ sec} \]

\[ i(t) = 2e^{-0.5t} \]