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ECE 301 — Lesson 10

RC & RL Circuits — 9/28/06

In considering the RC and RL circuit we assume a single structure at first and add more complexity later. There are two divisions we make.

A. The RC & RL circuit with no external forcing function. (Initial condition only)

B. The RC & RL circuit with a forcing function. (Can also include an initial condition).

A. Consider the circuit below.

Be Kind

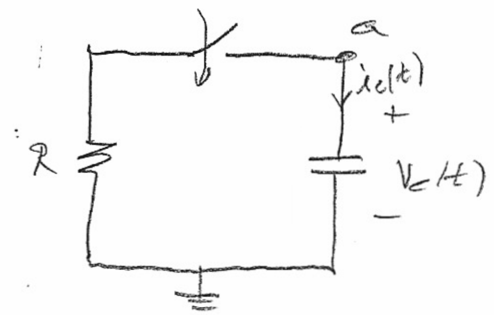


Figure 10: RC circuit without a forcing function.

We assume $V_c(0^-) = V_0$. We know $V_c(0^+) = V_c(0^-)$.
Writing a nodal equation at "a" gives

$$\frac{V_c(t)}{R} + C\frac{dV_c}{dt} = 0 \qquad (10.1)$$

or

$$\frac{dV_c}{dt} + \frac{V_c(t)}{RC} = 0 \qquad (10.2)$$

We can also put the equation in the form

$$RC\frac{dV_c(t)}{dt} + V_c(t) = 0 \qquad (10.3)$$

The solution to (10.1) is of the form

$$V_c(t) = Ae^{st} \qquad (10.4)$$

where A & s are unknowns. Placing this assumed solution in (10.2) gives

$$\frac{d[Ae^{st}]}{dt} + \frac{Ae^{st}}{RC} = 0$$

or

$$sAe^{st} + \frac{Ae^{st}}{RC} = 0$$

or

$$s + \frac{1}{RC} = 0 \qquad (10.5)$$

giving

$$s = -\frac{1}{RC} \qquad (10.6)$$

Placing (10.6) into (10.4) gives

$$V_c(t) = Ae^{-\frac{t}{RC}} \qquad (10.7)$$

We now use the initial condition

$$V_c(0^+) = V_o$$

giving

$$V_o = Ae^{-\frac{t}{RC}}\Big|_{t=0} = A$$

And

$$V_c(t) = V_o e^{-\frac{t}{RC}} \qquad (10.8)$$

This is called the natural response of the circuit. A sketch of $V_c(t)$ is shown in Figure 10.2

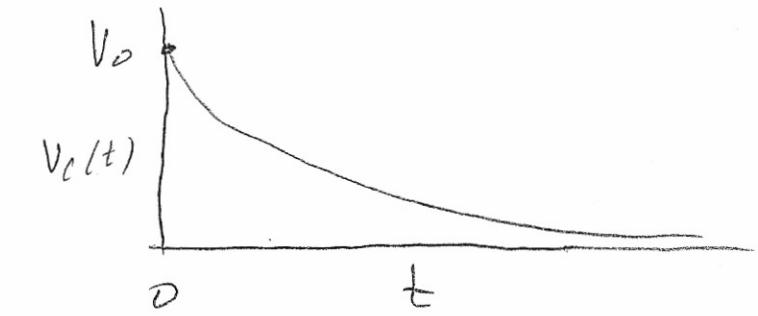


Figure 10.2: Natural Response of RC circuit.

RC is defined as the time constant

$$\tau = \text{time constant} = RC$$

If you find the equation of a line tangent to $V_c(t)$ at $t=0$, the line intersects the t axis at $t=\tau$.

At $t=\tau$

$$V_c(t) = V_o e^{-\frac{t}{\tau}} \Big|_{t=\tau} = V_o e^{-1} = .368\, V_o \qquad (10.9)$$

So;

Figure 10:3: Show the time constant, graphically.

A side note;

If we take the LaPlace Transform of (10.2) we have

$$\mathcal{L}\left[\frac{dV_c(t)}{dt}\right] + \mathcal{L}\left[\frac{V_c(t)}{RC}\right] = 0 \qquad (10.10)$$

$$sV_c(s) - V_c(0^+) + \frac{V_c(s)}{RC} = 0$$

$$\left(s + \frac{1}{RC}\right)V_c(s) = V_o$$

$$V_c(s) = \frac{V_o}{s + \frac{1}{RC}} \qquad (10.11)$$

Now

$$\mathcal{L}^{-1}\left[V_c(s)\right] = V_c(t) = \mathcal{L}\left[\frac{V_o}{s + \frac{1}{RC}}\right]$$

so

$$V_c(t) = V_o e^{-\frac{t}{RC}} \qquad (10.12)$$

If we want $i_c(t)$, we know

$$i_c(t) = C\,\frac{dV_c}{dt}.$$

<u>Now what about the inductor?</u>

Consider Figure 10.4

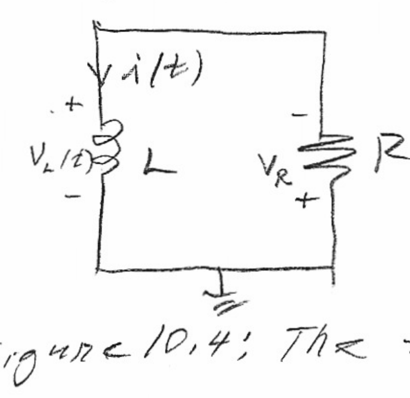


Figure 10.4: The force free RL ckt.

We assume, that by some way or the other, that $i(0^-) = I_0$. Current can't change instantaneously through the inductor so

$$i(0^+) = i(0^-) = I_0$$

From Figure 4.4;

$$V_L + V_R = 0$$

$$L\frac{di}{dt} + Ri = 0$$

$$\frac{di}{dt} + \frac{R}{L} i = 0 \qquad (10.12)$$

The solution is

$$i(t) = I_0 e^{-\frac{Rt}{L}} \qquad (10.13)$$

We define

$$\tau = \text{time constant} = \frac{L}{R}$$

$$i(t) = I_0 e^{-\frac{t}{\tau}} \qquad (10.14)$$

The response is as shown in Figure 10.5

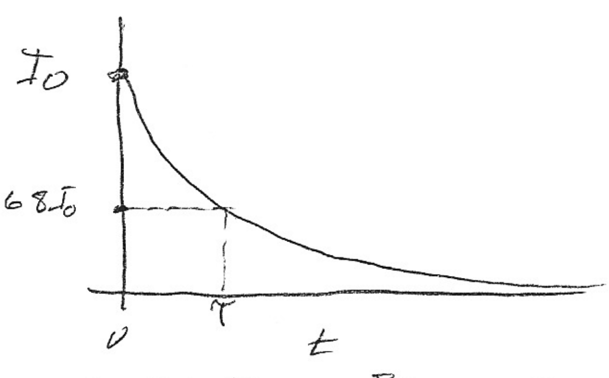


Figure 10.5: Free Response of RL circuit

Lets see how we use this at a little different level.

Example 10.1

For the circuit in Figure 10.6, find $V_o(t)$. Also find the time required for $V_o(t)$ to decay to $\frac{V_o(0^+)}{3}$.

switch has been closed for a very long time.

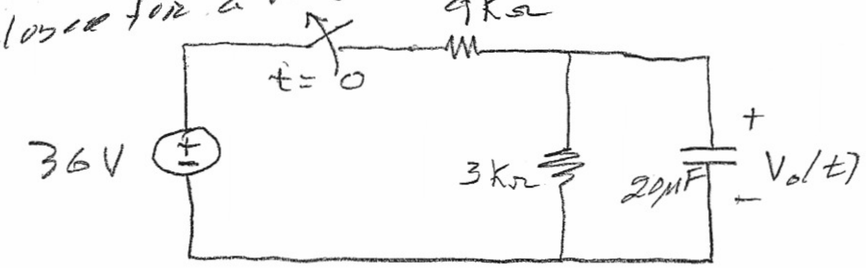


Figure 10.6: Circuit for Example 10.1

Solution:

Since the switch has been closed for a very long time, the capacitor looks like an open circuit for $t = 0^-$. The voltage across the capacitor is given by

$$V_o(0^-) = \frac{36 \times 3K}{3K + 9K} = 9\,V = V_o$$

We know from previous work

$$V_o(t) = V_o e^{-\frac{t}{\tau}}$$

$$\tau = RC = 20\mu F \times 3K = 60 \times 10^{-3}$$

$$V_o(t) = 9 e^{-16.67t} \quad u(t)$$

To find the time to reach $\frac{V_o(0^+)}{3}$.

Now $V_o(0^+) = V_o = 10$

So

$$V_c(t) = 9e^{-16.67t} = \frac{10}{3}$$

$$27e^{-16.67t} = 10$$

$$-16.67t = \ln\left(\frac{10}{27}\right) = -.9933$$

$$\boxed{t = 59.6 \text{ m sec}}$$

Example 10.2

Find $i_o(t)$ for $t > 0$ for the circuit of Figure 10.7

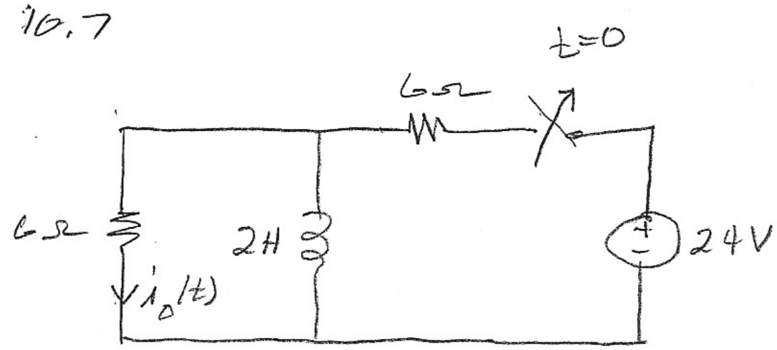
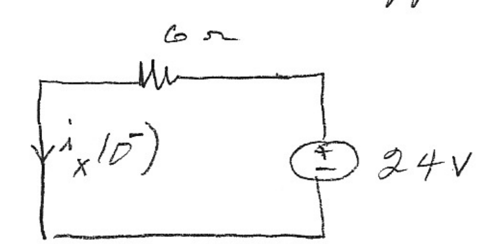


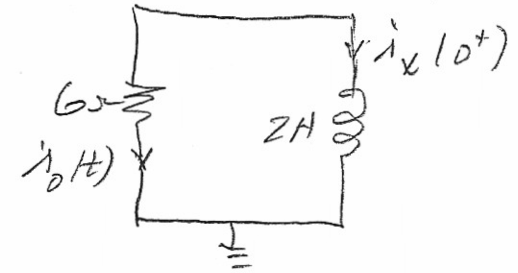
Figure 10.7: Circuit for example 10.2

For $t < 0$, the circuit appears as



$i_x(0^-) = \frac{24}{6} = 4\ A$

For $t \geq 0$, the circuit is



$i_x(0^+) = i_x(0^-) = 4A$

We know that the solution for $i_o(t)$ is

$$i_o(t) = I_o e^{-\frac{Rt}{L}} = I_o e^{-\frac{t}{\tau}}$$

Test 2 Oct 10

$$I_o = -i_x(0^+) = -4A$$

$\therefore$

$$\boxed{i_o(t) = -4e^{-3t}\ u(t)}$$

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Example 10.3

For the circuit of Figure 10.8, the switch has been closed for a very long time. It is opened at $t=0$. Find $i_L(t)$, $0 \leq t$.

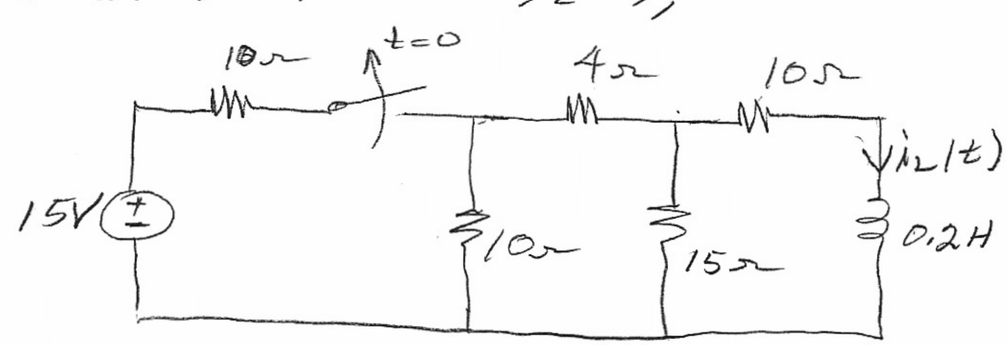
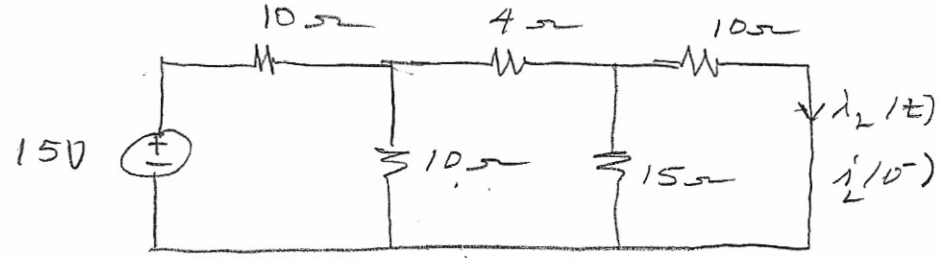


Figure 10.8: Circuit for Example 10.3

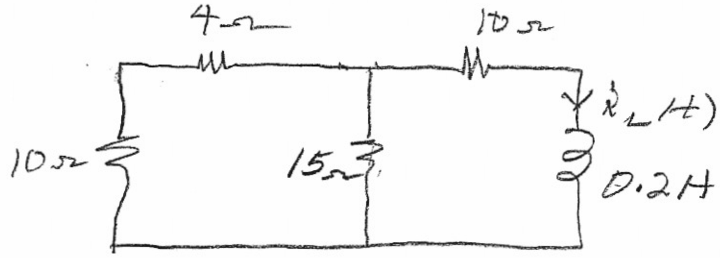
10.9

At $t = 0^-$ the circuit appears as



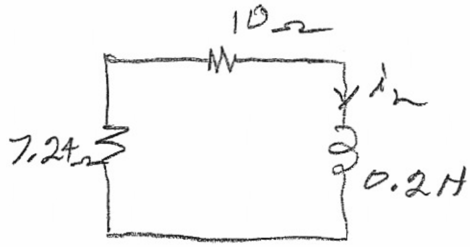
We can find that $\boxed{i_L(0^-) = 0.3\,A = i_L(0^+)}$.

Now for $t = 0^+$ the circuit is as follows.



or

15‖14



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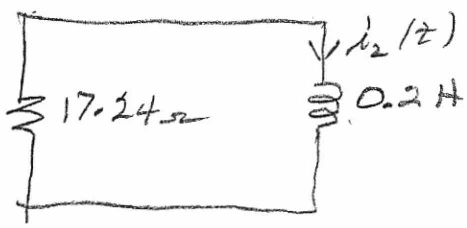


We know
$$i_L(t) = I_0 e^{-\frac{Rt}{L}} \qquad I_0 = i_L(0^+)$$

so
$$\boxed{i_L(t) = 0.3 e^{-86.2t}}$$

The Forced RC Circuit:

We consider the circuit shown in Figure 10.9.

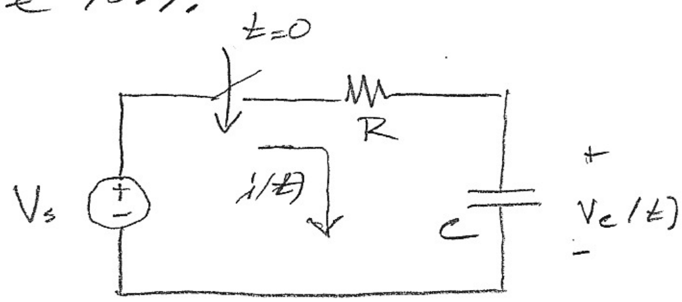


Figure 10.9: The forced RC Circuit.

Given:

$$V_c(0^-) = V_0$$

then we know $V_c(0^+) = V_c(0^-) = V_0$

and

$$i(t) = C\frac{dV_c}{dt}$$

We can write KVL for Fig 10.9 as, $t>0$

$$-V_s + Ri(t) + V_c(t) = 0$$

or

$$RC\frac{dV_c}{dt} + V_c(t) = V_s$$

or

$$\frac{dV_c}{dt} + \frac{V_c(t)}{RC} = \frac{V_s}{RC} \qquad (10.15)$$

We assume

$$V_c(t) = V_N(t) + V_{ss}(t)$$

where $V_N(t)$ = natural response

$V_s(t)$ = steady state response

The natural response comes from

$$\frac{dV_N}{dt} + \frac{V_N(t)}{RC} = 0 \qquad (10.16)$$

which has a solution of

$$V_N(t) = A e^{-\frac{t}{RC}} \qquad (10.17)$$

The forced response is

$$V_{SS}(t) = K$$ (A constant K since the right side of 10.15 is a constant)

or

$$\frac{dV_{SS}}{dt} + \frac{V_{SS}}{RC} = \frac{V_S}{RC} \quad \Big|_{V_F = K}$$

which gives

$$\frac{K}{RC} = \frac{V_S}{RC}$$

$$K = V_S$$

Thus, combining $V_N + V_{SS}$

$$V_C(t) = A e^{-\frac{t}{RC}} + V_S \qquad (10.18)$$

We need to evaluate A:
From the initial condition,

$$V_C(0^+) = V_0$$

From (10.18) we have

$$V_C(0^+) = V_0 = \left[A e^{-\frac{t}{RC}} + V_S\right]\Big|_{t=0^+}$$

$$V_C(0^+) = V_0 = A + V_S$$

$$A = V_0 - V_S = V_c(0^+) - V_S$$

We note from (10.18) that

$$V_c(\infty) = V_S$$

$$A = V_c(0) - V_c(\infty)$$

So we can write (10.18) as

$$\boxed{V_c(t) = V_c(\infty) + \left(V_c(0^+) - V_c(\infty)\right)e^{-\frac{t}{\tau}}} \quad (10.19)$$

We can also write this as

$$V_c(t) = \underbrace{V_0 e^{-\frac{t}{\tau}}}_{V_{NATURAL}} + \underbrace{\left(V_S - V_S e^{-\frac{t}{\tau}}\right)}_{V_{FORCED}} \quad (10.20)$$

We can re-arrange as

$$v_c(t) = \underbrace{V_S}_{V_{SS}} + \underbrace{(V_0 - V_S)e^{-\frac{t}{\tau}}}_{V_{TRANSIENT}} \quad (10.21)$$

We can save ourselves a lot of work if we use the form of (10.19). We only need to $V_c(\infty)$, $V_c(0^+)$, and τ and we write the solution.

This format works for RC and RL circuits (first order).

Example 10.4

You are given the following circuit (circuit of 5.37 in the text).

(a) Use the step-by-step method (10.19) to find $V_c(t)$,

(b) Use nodal analysis, find the DE, solve for $V_c(t)$.

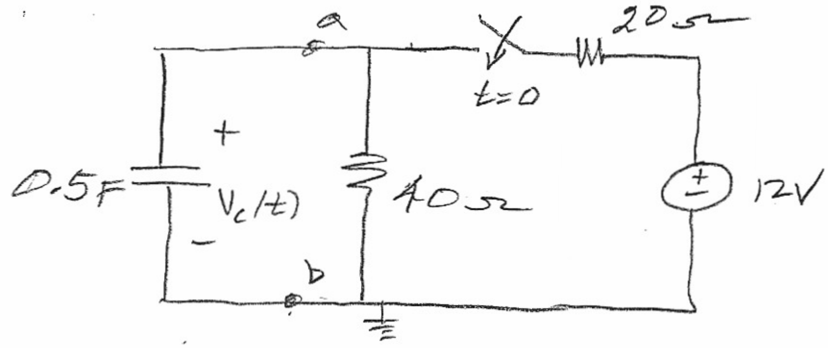


Figure 10.10:

(a) $V_c(\infty) = V_c(ss) = \dfrac{12 \times 40}{20+40} = 8\,V$

$V_c(0^+) = V_c(0^-) = 0$

To find R_{eq} look into a-b as if finding R_{TH}

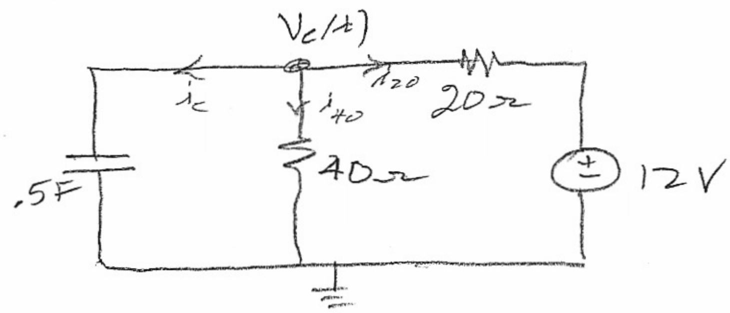
$R_{eq} = 20 \| 40 = \dfrac{20 \times 40}{20+40} = 13.33\,\Omega$

so

$$V_c(t) = V_c(\infty) + [V_c(0^+) - V_c(\infty)]\,e^{-\frac{t}{R_{eq}C}}$$

$$V_c(t) = (8 - 8e^{-0.15t})\,u(t)$$

(b) By nodal analysis



$$C\frac{dV_c}{dt} + \frac{V_c}{40} + \frac{V_c - 12}{20} = 0$$

$C = 0.5$

$$0.5\frac{dV_c}{dt} + .025V_c + .05V_c = 0.6$$

$$\boxed{\frac{dV_c}{dt} + 0.15V_c = 1.2}$$

$$V_N = Ae^{-0.15t}$$

$$V_{ss} = \frac{1.2}{.15} = 8$$

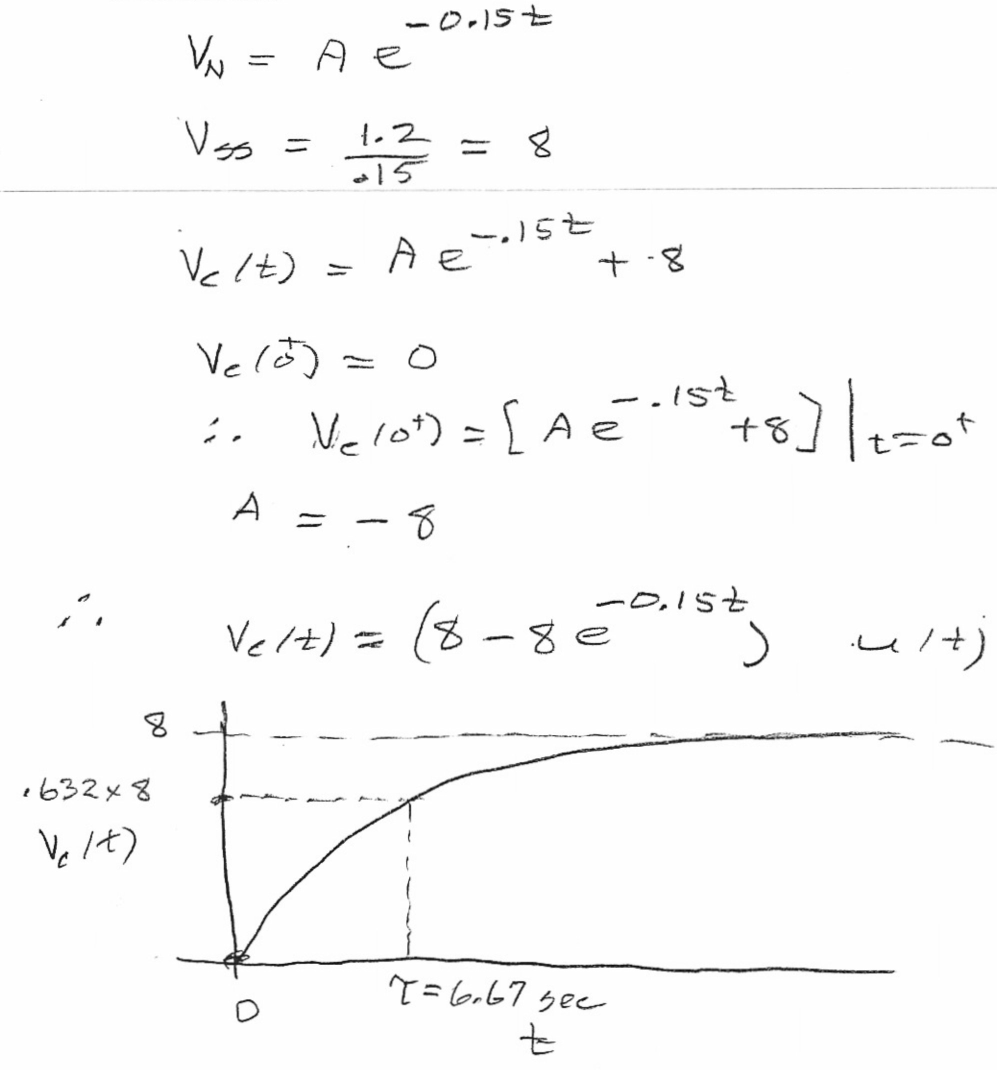
$$V_c(t) = Ae^{-.15t} + 8$$

$$V_c(0^+) = 0$$

$$\therefore \quad V_c(0^+) = \left[Ae^{-.15t} + 8\right]\Big|_{t=0^+}$$

$$A = -8$$

$$\therefore \quad V_c(t) = \left(8 - 8e^{-0.15t}\right)u(t)$$



The RL Forced Circuit:

We are given the following circuit of Figure 10.11.

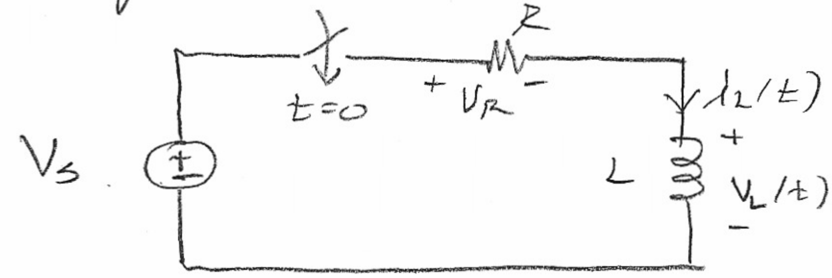


Figure 10.11: The Forced RL Circuit.

Known:

$$i_L(0^-) = I_0$$

$$\therefore\ i_L(0^+) = I_0$$

$$V_L(t) = L\frac{di_L}{dt}$$

Solution:

K.V.L

$$V_R(t) + V_L(t) = V_s$$

$$L\frac{di_L}{dt} + R i_L(t) = V_s$$

$$\frac{di_L}{dt} + \frac{R}{L} i_L(t) = \frac{V_s}{L}$$

$$\frac{di_L}{dt} + \frac{i_L(t)}{\tau} = \frac{V_s}{L} \qquad (10.23)$$

$$\tau = \text{time constant} = \frac{L}{R}$$

Solve using the same reasoning as applied for the forced RC circuit.

$$i_L(t) = i_N(t) + i_{ss}(t) \qquad (10.23)$$

$$i_N(t) = A e^{-\frac{t}{\tau}}$$

$$i_{ss} = K \qquad \text{(substitute in (10.24))}$$

$$\therefore \frac{K}{\tau} = \frac{V_s}{L}$$

$$K = \frac{\tau V_s}{L} = \frac{L}{R}\frac{V_s}{L} = \frac{V_s}{R}$$

so

$$i_L(t) = A e^{-\frac{t}{\tau}} + \frac{V_s}{R}$$

At $t = 0^+$, $i_L(0^+) = I_0 = i_L(0^+)$

$$i_L(0^+) = \left[A e^{-\frac{t}{\tau}} + \frac{V_s}{R}\right]\Big|_{t=0^+}$$

$$i_L(0^+) = A + \frac{V_s}{R}$$

$$A = i_L(0^+) - \frac{V_s}{R}$$

Now $\frac{V_s}{R} = i_L(ss) = i_L(\infty)$

$$A = i_L(0^+) - i_L(\infty)$$

$$i_L(t) = i_L(\infty) + \left(i_L(0^+) - i_L(\infty)\right)e^{-\frac{t}{\tau}} \qquad (10.25)$$

We see this is the same form as (10.19). We can therefore find $i_L(\infty) = i_L(ss)$, $i_L(0^+)$, τ and write down the solution.

Example 10.5

You are given the following circuit. Use step-by-step to find $V_o(t)$. (FROM IRWIN 7.42)

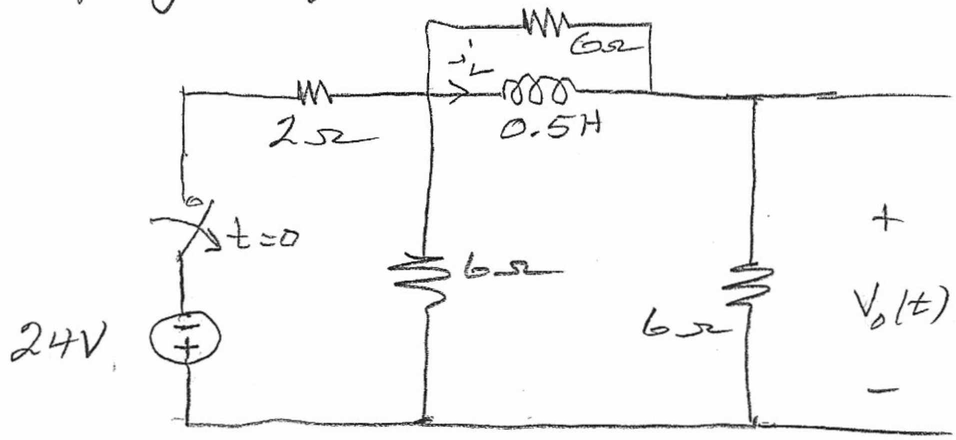


Figure 10.12: Circuit for Example 10.5.

The expression to use is

$$V_o(t) = V_o(\infty) + [V_o(0^+) - V_o(\infty)]e^{-\frac{t}{\tau}}$$

We must determine $V_o(\infty)$, $V_o(0)$, τ

Solution: for $t<0$

Looking at the circuit, $V_o(\infty) = 0$

For $t<0$ the circuit is as follows.

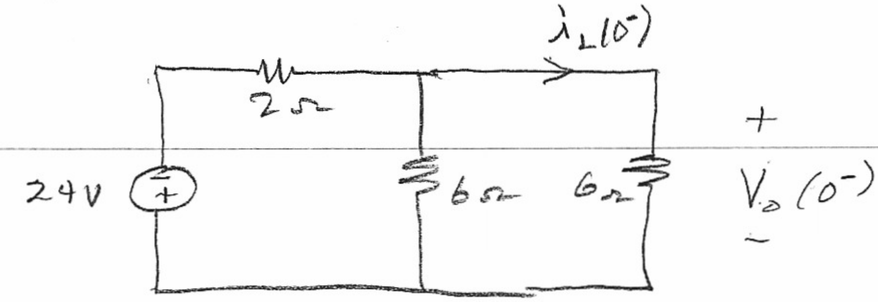


Figure 10.13

10.18

We see that

$$V_o(0^-) = \frac{-24 \times 3}{3+2} = -14.4\ V$$

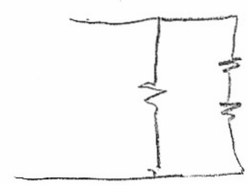
The problem is, we don't know if this is $V_o(0^+)$. The only thing we know is that $i_L(0^+) = i_L(0^-)$

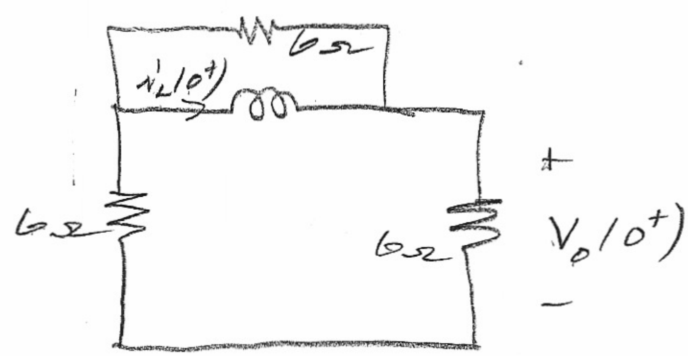
We see from Figure 10.13 that

$$i_L(0^-) = \frac{V_o(0^-)}{6} = 2.4\ A$$

$\therefore\ i_L(0^+) = 2.4A$

Now for $t = 0^+$





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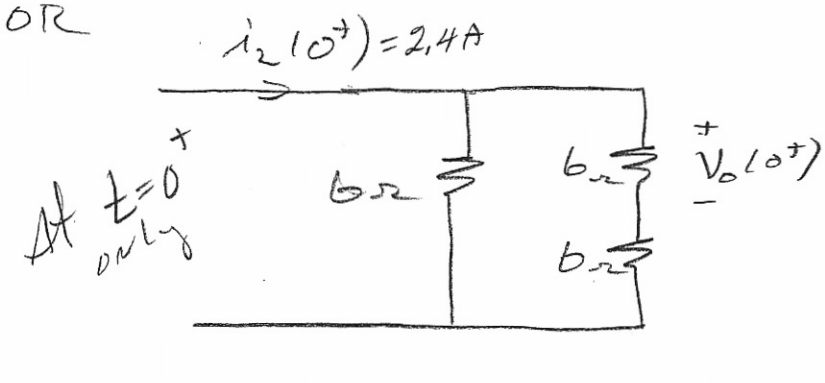

$$V_o(0^+) = \left(\frac{2.4 \times 6}{1.8}\right) \times 6 = 4.8V$$

10.19

It is easy to see that

$$R_{eq} = 6 \| 12 = \frac{6 \times 12}{6 + 12}$$

$$R_{eq} = 4\,\Omega$$

$$V_0(t) = V_0(0^+)e^{-\frac{R_{eq}t}{L}}$$

$$\boxed{V_0(t) = 4.8\,e^{-8t}}$$

pretty tough problem.