A Prelude

- Complex Numbers

- Euler's Identity

\[ e^{ix} = \cos x + j \sin x \]

\[ \text{Re}[e^{ix}] = \cos x \]

\[ \text{Im}[e^{ix}] = \sin x \]

- Rectangular Form

\[ a + j \, b \]

on a calculator

\[ (a, b) \]

Graphical plot

- Polar Form

\[ ML \theta \]

on a calculator

\[ (ML \theta) \]

Graphical plot
relationship between rectangular and polar

\[ a + jb = \sqrt{a^2 + b^2} \left( \cos \left( \frac{b}{a} \right) + j \sin \left( \frac{b}{a} \right) \right) \]

the exponential form

\[ e^{j\theta} = \frac{1}{e^{-j\theta}} \]

by definition

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

\[ |e^{j\theta}| = |\cos \theta + j \sin \theta| \]

\[ = \sqrt{\cos^2 \theta + \sin^2 \theta} \]

\[ |e^{j\theta}| = 1 \]

of note:

\[ e^{j(wt + \psi)} = e^{jwt} e^{j\psi} \]

\[ \text{Re} [e^{j(wt + \psi)}] = \cos(wt + \psi) \]

of note

\[ e^{j90} = \cos 90 + j \sin 90 = j \]

\[ e^{j90} = 1/\overline{180} = j \]
* Of Note
\[ j^2 = j \cdot j = 1150 \cdot 1150 = 1 \, \angle 180^\circ = -1 \]
\[ j^2 = -1 \]
Also,
\[ j^3 = j^2 \cdot j = -j = 1 \, \angle -90^\circ \]
\[ j^4 = j^2 \cdot j^2 = (-1)(-1) = 1 \]

- Calculator Exercises
Note: You need a TI 86, 89 Vintage

Perform
\[ (20 + j 30) \times (6 + j 25) \]
Express in rectangular and polar form
On the calculator (set to rectangular)
\[ (20, 30) \times (6, 25) \] enter
Ans: \[ 630 \pm j 680 \] rectangular
\[ 526.98 \, \angle 132.8^\circ \] polar

Perform
\[ \frac{320 \pm j 65}{6 + j 8} \]
On the calculator
\[ (320 \pm j 65) \div (6, 8) \] enter
Ans: \[ 31.3 \pm j 6.58 \] rectangular
\[ 32 \, \angle 11.87^\circ \] polar
- Converting cosine, sine functions

Suppose you have given want
\[ \cos (30^\circ) \quad \sin (?) \]
\[ \sin (-60^\circ) \quad \cos (?) \]
\[ \sin (\omega + 60^\circ) \quad \cos (\omega + ?)? \]

If you go back to basic trig relationships you can perform the above operations.

However, there is a simple scheme for doing this, as presented below.

Express the quadrant axis as below

\[ \begin{array}{c|c|c}
\hline
- \cos x & 0 & \cos x \\
\hline
- \sin x & \cdot & \sin x \\
\hline
\end{array} \]

Positive angle is counter clockwise
Given
\[ \cos(-30^\circ) \rightarrow \sin(?) \]

\[ \cos(-30^\circ) = \sin(60^\circ) \]

\[ \text{Given} \]
\[ -A \sin(wt - 20^\circ) \rightarrow \]

\[ -A \sin(wt - 20^\circ) = A \cos(wt + 90^\circ) \]

We need to know how to do the above when dealing with AC voltage and current sources expressed in the time domain.
**Physics and Background:**

These notes give a very abbreviated background on the theory (analysis) of circuits with sinusoidal sources.

In AC circuits we are nearly always interested in the steady state solution.

**Illustrating**

**Given**

\[
\begin{array}{c}
\text{\textbf{Circuit Diagram}} \\
\text{\textbf{Equations}}
\end{array}
\]

\[
R_i(t) + L \frac{\partial i}{\partial t} = V_m \cos(\omega t)
\]

\[
\frac{\partial i}{\partial t} + \frac{R}{L} i(t) = \frac{V_m}{L} \cos(\omega t) \quad (A)
\]

Change the source to \( \frac{V_m}{L} e^{j\omega t} \)

Find \( i(t) \). Take \( \text{Re} \{i(t)\} \) to get the answer to \( (A) \).

\[
\frac{\partial i}{\partial t} + \frac{R}{L} i(t) = \frac{V_m}{L} e^{j\omega t}
\]
\[ i_{ss} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \theta) \]

\[ i_{ss} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-\frac{R}{2L}t} \]

\[ i_{ss} = \frac{V_m}{R + j\omega L} e^{-\frac{R}{2L}t} \]

We want the usual peak of this

\[ K = \frac{L}{R + j\omega L} = \frac{L}{R + j\omega L} \]

\[ K_{out} = K e^{i\omega t} \]

Assume \[ i_{ss} = K e^{i\omega t} \]

\[ i_{ss} = \frac{V_m}{R + j\omega L} e^{-\frac{R}{2L}t} \]
In AC circuits we are mainly interested in the steady state. It turns out there is not a need to express as a differential equation. We express things in phasor form and basically use all the theorems and techniques we learned for DC circuits.

So, what is a phasor?

Let 
\[ v(t) = V_m \cos (\omega t + \phi) \]

then 
\[ V/\dot{E} = \Re \left[ V_m e^{j(\omega t + \phi)} \right] \]
\[ = \Re \left[ V_m \cdot e^{j\phi} \cdot e^{j\omega t} \right] \]
\[ V/\dot{E} = \Re \left[ V e^{j\omega t} \right] \]
In AC circuits we are mainly interested in the steady state. It turns out there is not a need to express as a differential equation. We express things in *phasor* form and basically use all the theorems and techniques we learned for DC circuits.

So what is a phasor?

Let \( v(t) = V_m \cos(\omega t + \phi) \)

then

\[
\frac{v(t)}{t} = \Re \left[ V_m e^{j(\omega t + \phi)} \right]
\]

\[
= \Re \left[ V_m - e^{j\phi} \cdot e^{j\omega t} \right]
\]

\[
v(t) = \Re \left[ V e^{j\omega t} \right]
\]
We define
\[ \mathbf{V} \] as a phasor
\[ \mathbf{V} = V_m e^{j\theta} = V_m |\theta| \]
\[ \mathbf{V} \] is not a vector. It does not obey the rules of vector analysis (product, cross product).
\[ \mathbf{V} \] is a number at an angle.

The sine function \([V e^{j\omega t}]\) is a rotating line.

Illustrating a Sine
Going from Phasor to Time Domain

\[
\begin{align*}
\text{Phasor} & \\ V_m e^{j\theta} & \iff V_m (\cos(\omega t + \theta)) \\
\end{align*}
\]

A Phasor Diagram

\[
\begin{align*}
V & = V_m e^{j30^\circ}, \quad I = I_m e^{j-50^\circ}
\end{align*}
\]

Notes: Voltage & current must have a different scale.

We will use the above in AC circuit analysis.

We now turn our attention to formulating the AC circuit as a phasor problem.
Phasor Relations of Circuit Elements

The Resistor

Assume

\[ v(t) = i_1(t) R \]

\[ i(t) = \text{Im} \cos (\omega t + \theta) \]

\[ i^2 = \text{Im} |L|^2 R \]

\[ v = \text{Re} [\text{Im} R e^{j\theta} e^{j\omega t}] \]

\[ v = \text{Re} [V e^{j\omega t}] \]

\[ V = \text{Im} |L|^2 R = \frac{v^2}{R} \]

AC Ohm's Law for a Resistor

\[ V = R \hat{i} \]

\[ \frac{R}{i(t)} \rightarrow \frac{1}{1 + \frac{1}{R}} \]

\[ \frac{v(t)}{i(t)} = j \omega R \]

The Inductor

Assume

\[ i(t) = \text{Im} \cos (\omega t + \theta) \]

\[ i^2 = \text{Im} |L|^2 \]

\[ v(t) = -\omega L \text{Im} \sin (\omega t + \theta) \]
\[ V(t) = -wL \Im \sin(wt + \Theta_L) \]
\[ = wL \Im \cos(wt + 90^\circ + \Theta_L) \]
\[ = \Re \left[ wL \Im e^{i(wt + 90^\circ + \Theta_L)} \right] \]
\[ = \Re \left[ wL \Im e^{i\Theta_L} e^{j90^\circ} e^{j\omega t} \right] \]
\[ = \Re \left[ V e^{j\omega t} \right] \]
\[ V = wL \Im L \Theta_L \cdot j \]
\[ V = jwL \Im L \Theta_L \]
\[ V = jwL \mathbf{i} \]

*AC Ohm’s law for an inductor*

\[ \frac{i}{\mathbf{R}} \rightarrow jwL \]
\[ + \quad \frac{\mathbf{V}}{\mathbf{R}} \]
\[ V = jwL \mathbf{i} \]

**The Capacitor**

Let \( V_c(t) = V_m \cos(wt + \Theta_c) \)
\[ \dot{V}_c = V_m \cdot L \Theta_c \]

Now recall
\[ i(t) = C \frac{dV_c}{dt} \]
\[ i(t) = - \omega C V_m \sin(\omega t + \theta_c) \]
\[ = \omega C V_m \cos(\omega t + 90^\circ + \theta_c) \]
\[ = \mathcal{R}e \left[ \omega C V_m e^{j90^\circ} e^{j\theta_c + j\omega t} \right] \]
\[ = j\omega C V_m L \theta_c = j\omega C V \]

\[ V = \left[ \frac{1}{j\omega C} \right] \times I \]

\[ \mathcal{I} = \frac{1}{j\omega C} \]

\[ + \frac{\hat{V}}{\hat{V}} \]

**Impedance and Admittance**

We have developed the following:

\[ \frac{\hat{V}}{\hat{I}} = R \]
\[ \frac{\hat{V}}{\hat{I}} = j\omega L \]
\[ \frac{\hat{V}}{\hat{I}} = \frac{1}{j\omega C} \]

In a more general sense
We say
\[ \frac{\bar{V}}{\bar{I}} = \bar{Z} \]

where \( \bar{Z} \) is impedance (units of ohms)
\[ \bar{Z} = R + jX \]

 resistance (ohms)  reactance (ohms)

\[ \bar{V} = (10 + j5) \bar{I} \]
\[ \bar{Z} = (10 + j5) \text{ ohms} \]

\[ \bar{V} = (5 + j15 - j10) \bar{I} = (5 + j5) \bar{I} \]
\[ \bar{Z} = (5 + j5) \text{ ohms} \]
We define admittance as

\[ Y = \frac{1}{Z} = G + jB \]  \hspace{1cm} (5)

- Conductance \hspace{1cm} (S)
- Susceptance \hspace{1cm} (S)

\[ Y \rightarrow \text{Admittance (siemens, S)} \]

\[ G \rightarrow \text{Conductance (siemens, S)} \]

\[ B \rightarrow \text{Susceptance (siemens, S)} \]

Example 1

Find the impedance seen looking into terminals a-b

This becomes

\[ \begin{align*}
\text{a} & \rightarrow 5 \Omega \rightarrow 15 \Omega \\
\text{b} & \rightarrow 10 \Omega \rightarrow 15 + j20 \Omega
\end{align*} \]
\[ z = 5 + 10 \frac{1}{1+15+j20} = 5 + \frac{10 \times (15+j20)}{10+15+j20} = 5 + 7.56 + j1.95 = 12.56 + j1.95 \]

\[ |z| = 12.74 \angle 8.8^\circ \]

From this point it is an easy step to move to solution of A.C. circuits. We do this in the next lesson. Think about the following.

Find the phasor current \( \dot{I} \).