Nodal & Mesh Analysis

In Nodal analysis we encounter 4 cases to analyze.

- Ckt with all current sources (ind. or gen.) plus resistors
- Ckt with mixture of current sources and voltage sources (both ind. or gen.) plus resistors
- Ckt with independent voltage and current sources but with single node plus resistance
- Ckt with mixture of independent and dependent voltage and current sources plus resistance.

An example for each case will be presented below.

**Example 5.1**

Nodal analysis: All independent current sources plus resistors.

Consider the following ckt.
Regarding reference nodes.
We can redraw the circuit of Example 5.1.

We could just as well use $V_1$ as ground and relable the remaining nodes.
In this case we have 3 node voltages as indicated. We wish to solve for $V_1$, $V_2$, $V_3$ and $I$

At $V_1$
\[ \frac{V_1 - V_2}{10} + \frac{V_1 - V_3}{40} = 2 \]
or
\[ 4V_1 - 4V_2 + V_1 - V_3 = 80 \]
giving
\[ 5V_1 - 4V_2 - V_3 = 80 \] (5.1)

At $V_2$
\[ \frac{V_2 - V_1}{10} + \frac{V_2}{5} + \frac{V_2 - V_3}{20} = 0 \]
\[ 2V_2 - 2V_1 + 4V_2 + V_2 - V_3 = 0 \]
\[ -2V_1 + 7V_2 - V_3 = 0 \] (5.2)

At $V_3$
\[ \frac{V_3 - V_1}{40} + \frac{V_3 - V_2}{20} = -5 \]
or
\[ V_3 - V_1 + 2V_3 - 2V_2 = -200 \]
\[ -V_1 - 2V_2 + 3V_3 = -200 \] (5.3)
Express (5.1), (5.2) and (5.3) in matrix form:

\[
\begin{bmatrix}
5 & -4 & -1 \\
-2 & 7 & -1 \\
-1 & -2 & 3
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
=
\begin{bmatrix}
80 \\
0 \\
-200
\end{bmatrix}
\]

Solve to find:

\[V_1 = -12.14V\quad V_2 = -15V\quad V_3 = -80.71V\]
\[I = \frac{V_2}{5} = -3A\]

Example 5.2

Nodal analysis with super node present.

Given the following ckt:

\[
\begin{align*}
V_1 & \quad 20V \\
5 & \quad V_2 \\
10 & \quad 4 \quad V_3
\end{align*}
\]

Find \(V_1, V_2, V_3\) and \(I\).

Method 1

At \(V_1\):

\[
\frac{V_1}{20} + \frac{V_1 - V_3}{5} + \frac{V_1 + 20 - V_2}{10} = 0 \quad (5.4)
\]
clear (5.4) gives

\[ V_1 + 4V_1 - 4V_2 + 2V_1 + 40 - 2V_2 = 0 \]

or

\[ 7V_1 - 2V_2 - 4V_3 = -40 \] (5.5)

When we have a voltage source between two nodes (nothing else) we can form a super node and write KCL for branches leaving the super node. Thus:

At the super node

\[ \frac{V_2 - 20 - V_1}{10} - 2 + \frac{V_3 - V_1 + V_3}{5} + \frac{V_3}{40} = 0 \]

or

\[ 4V_2 - 80 - 4V_1 - 80 + 8V_3 - 8V_1 + V_3 = 0 \]

\[ -12V_1 + 4V_2 + 9V_3 = 160 \] (5.6)

Constraint Equation

The following equation involving the super node must hold:

\[ V_2 - 10 - V_3 = 0 \]

or

\[ 0V_1 + V_2 - V_3 = 10 \] (5.7)

We put (5.5), (5.6) and (5.7) in matrix form and solve.
\[
\begin{bmatrix}
7 & -2 & -4 \\
-12 & 4 & 9 \\
10 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} =
\begin{bmatrix}
-40 \\
160 \\
10
\end{bmatrix}
\]

\[v_1 = 24.21 \text{ V}, \quad v_2 = 41.58 \text{ V}, \quad v_3 = 31.58 \text{ V} \]

To find \( I \):

At node \( v_2 \) we have, by KCL,

\[
\frac{v_2 - 20 - v_1}{10} - 2 + I = 0
\]

\[
\frac{41.58 - 20 - 24.21}{10} - 2 + I = 0
\]

Or

\[I = -2.26 + 2 = 2.26 \text{ A} \]

**Method 2**

Write the following equations, directly:

At \( v_1 \)

(Same as before, gives) (5.5)

\[7v_1 - 2v_2 - 4v_3 + 0I = -40 \]

At \( v_2 \)

\[
\frac{v_2 - 20 - v_1}{10} - 2 + I = 0
\]

\[
v_2 - 30 - v_1 - 20 + 10I = 0
\]

\[-v_1 + v_2 + 0v_3 + 10I = 40 \] (5.9)
A \leq V_3 \quad \text{(method 2)}

\begin{align*}
\frac{V_3 - V_1}{5} + \frac{V_3}{40} - I &= 0 \\
8V_3 - 8V_1 + V_3 - 40I &= 0 \\
-8V_1 + 0V_2 + 9V_3 - 40I &= 0 \quad (5.10)
\end{align*}

\text{Constraint Eq.}

\begin{align*}
V_2 - 10 - V_3 &= 0 \\
0V_1 + V_2 - V_3 + 10I &= 10 \quad (5.11)
\end{align*}

This gives

\begin{bmatrix}
7 & -2 & -4 & 0 \\
-1 & 1 & 0 & 10 \\
-8 & 0 & 9 & -40 \\
0 & 1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
I
\end{bmatrix}
= \begin{bmatrix}
-40 \\
40 \\
0 \\
10
\end{bmatrix}

Solving:

\begin{align*}
V_1 &= 24.21 \text{ V} \\
V_2 &= 41.56 \text{ V} \\
V_3 &= 31.57 \text{ V}
\end{align*}

I = 2.263 \text{ A}

Check
Example 5.3

Modeled: With independent voltage and current source mix plus resistors.

Given the following circuit:

\[
\begin{align*}
40 \Omega \\
\downarrow \\
V_1 \\
\uparrow \\
5 \Omega \\
\downarrow \\
10V \\
\uparrow \\
10 \Omega \\
\downarrow \\
0V
\end{align*}
\]

Use nodal analysis to find \( V_1, V_2 \), and \( I \).

At \( V_1 \)

\[
\frac{V_1}{10} + \frac{V_1 - V_2}{40} + \frac{V_1 + 10 - V_2}{5} = 0
\]

\[
4V_1 + V_1 - V_2 + 8V_1 + 80 - 8V_2 = 0
\]

\[
13V_1 - 9V_2 = -80 \quad (5.12)
\]
At $V_2$ \[ \Rightarrow (\text{echoing} = 0) \]

\[
\frac{V_2 - V_1}{40} + \frac{V_2 - 10 - V_1}{5} + \frac{V_2}{10} + 4 = 0
\]

\[
V_2 - V_1 + 8V_2 - 80 - 8V_1 + 4V_2 + 160 = 0
\]

\[
-9V_1 + 13V_2 = -80 \quad (5.13)
\]

In matrix form

\[
\begin{bmatrix}
13 & -9 \\
-9 & 13
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
-80 \\
-80
\end{bmatrix}
\]

\[V_1 = -20 \text{V} \quad V_2 = -120 \text{V} \]

\[I = \frac{V_1 - V_2}{40} = \frac{-20 + 20}{40} = 0\]

**Example 5.4**

Model with a dependent source

Given the following circuit
Find $V_0$ and $T_0$:

At $V_1$

\[ \frac{V_1}{2} + \frac{V_1 - V_0}{1} + \frac{V_1 - 16}{3} = 0 \]

\[ -3V_1 + 6V_1 - 6V_0 + 2V_1 - 32 = 0 \]

\[ 11V_1 - 6V_0 = 32 \quad (6.14) \]

At $V_0$

\[ \frac{V_0 - V_1}{1} + \frac{V_0 - 16}{2} - 2T_0 = 0 \quad (5.15) \]

\[ d\phi \quad T_0 = \frac{V_1}{2} \quad \text{Substitute this into} \quad (5.15) \]

\[ \frac{V_0 - V_1}{1} + \frac{V_0 - 16}{2} = \frac{1}{2} \left( \frac{V_1}{2} \right) = 0 \]

\[ 2V_0 - 2V_1 + V_0 - 16 - 2V_1 = 0 \]

\[ -4V_1 + 3V_0 = 16 \quad (5.16) \]
we now look at mesh variables.

\[ V_o = 21.33 \, V \]

\[ V_i = 33.78 \, V \]

\[
\begin{bmatrix}
16 \\
32 \\
22
\end{bmatrix} = 
\begin{bmatrix}
y_{10} \\
y_{1} \\
y_{11} \\
1
\end{bmatrix} \begin{bmatrix}
6 \\
3 \\
4
\end{bmatrix}
\]
We illustrate by example

**Example 5.5**

Mesh with all voltage source (independent) plus resistors.

\[ 20I_1 + 10I_1 - 5 + 40(I_1 - I_2) + 10 = 0 \]

\[ 70I_1 - 40I_2 = -5 \]
Example 5.6

Mesh analysis with a mix of independent voltage and current sources, and resistors. This example will be the same as Example 5.2 worked by nodal analysis.
Method I

The procedure is to assign mesh currents as shown but "omit" the current source but note the constraint equation. In this case the constraint equation is

$$I_2 - I_1 = 2A$$

or

$$-I_1 + I_2 + 0I_3 = 2 \quad (5.15)$$

The "modified" circuit is

We still retain the identity of $I_1, I_2,$ and $I_3$. In the above case we have 2 closed paths. We write KVL around these paths. This gives 2 equations.
We also have the constraint 14 equation (5.15). This gives 3 equations and 3 unknowns.

For the diagram of the ckt on page 13, start at A & go cw using \( \sum \text{drops} = 0 \):

\[ 20I_1 + 10(I_1-I_3) - 20 + 10 + 40I_2 = 0 \]

or

\[ 30I_1 + 40I_2 - 10I_3 = 10 \] (5.16)

Around the next mesh, starting at B, going cw, using \( \sum \text{drops} = 0 \) gives

\[ 5I_3 - 10 + 20 - 10(I_1-I_3) = 0 \]

or

\[ -10I_1 + 0 I_2 + 15I_3 = -10 \] (5.17)

The constraint equation (5.15)

\[ -I_1 + I_2 + 0 I_3 = 2 \]

We put these in matrix form.
\[
\begin{bmatrix}
30 & 40 & -10 \\
-10 & 0 & 15 \\
-1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
10 \\
-10 \\
2 \\
\end{bmatrix}
\]

\[I_1 = -1.21\text{ A},\quad I_2 = 0.769\text{ A},\quad I_3 = -1.47\text{ A}\]

We see that
\[I = I_2 - I_3 = 2.26\text{ A}\]

As we had in example 5.2.

**Method 2**

We work the following circuit.

We assign a voltage \(V_x\) across the 2 A current source.
We have an equation for each mesh, giving 3 equations. We still have the constraint equation. This gives 4 equations from which we can solve $I_1$, $I_2$, $I_3$ and $V_x$.

From the set on the previous page we have:

**Mesh 1**

\[
20I_1 + 10(I_1 - I_3) - 20 + V_x = 0
\]

\[
30I_1 + 0I_2 - 10I_3 + V_x = 20
\] (5.18)

**Mesh 2**

\[
-V_x + 10 + 40I_2 = 0
\]

\[
0I_1 + 40I_2 + 0I_3 - V_x = -10
\] (5.19)

**Mesh 3**

\[
5I_3 - 10 + 20 - 10(I_1 - I_3) = 0
\]

\[
-10I_1 + 0I_2 + 15I_3 + 10V_x = -10
\] (5.20)

**Constraint**

\[
-I_1 + I_2 + 0I_3 + 0V_x = 2
\] (5.21)
In matrix form

\[
\begin{bmatrix}
30 & 0 & -10 & 1 \\
0 & 40 & 0 & -1 \\
-10 & 0 & 15 & 0 \\
-1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
V_x
\end{bmatrix}
=
\begin{bmatrix}
20 \\
-10 \\
-10 \\
2
\end{bmatrix}
\]

\[I_1 = -1.21 \text{ A}, \quad I_2 = 0.789 \text{ A}, \quad I_3 = -1.47 \text{ A}\]

\[V_x = 41.58 \text{ V}\]

This checks with the solutions of Ex 5.2.

We can apply the same technique to previous Ex 5.4 but just work with the dependent source current as previous.