wlg ECE 301 Lecture 9

9/26/06

About Capacitors and Inductors

Capacitors

A capacitor can be defined as two conductors separated by a dielectric. This is shown in Figure 8.1

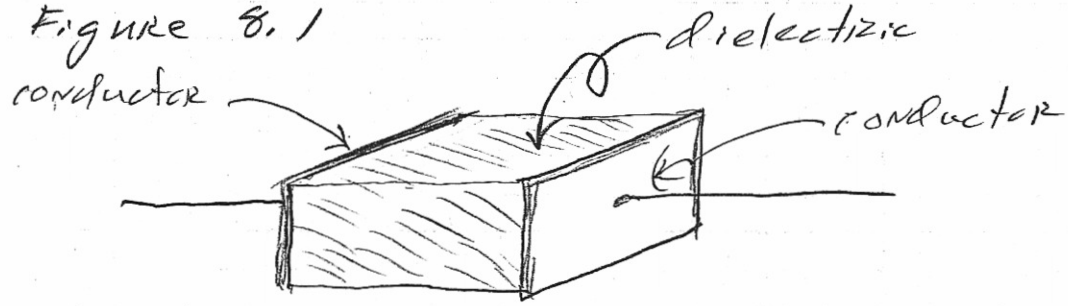


Figure 9.1: Defining a capacitor.

A fundamental expression for capacitance is

$$C = \frac{\epsilon A}{l} \qquad (9.1)$$

where

ϵ is the permittivity of the dielectric

A is the area of the conducting plates

l is the separation of the plates

We recall

$$q = CV \qquad (9.2)$$

Taking the derivative of both sides

$$\frac{dq}{dt} = C\frac{dv}{dt}$$

OR

$$i = \frac{dq}{dt} = C\frac{dv}{dt}$$

$$\boxed{i = C\frac{dv}{dt}} \qquad (9.3)$$

Equation (9.3) is considered to be a fundamental equation for a capacitor.

From (9.3) we have

$$\int_{t_0}^{t} dv = \frac{1}{C}\int_{t_0}^{t} i\,dt$$

or

$$\boxed{v(t) = \frac{1}{C}\int_{t_0}^{t} i(t)\,dt + v(t_0)} \qquad (9.4)$$

Equation (9.4) is also a fundamental equation for a capacitor.

In general

$$p(t) = v(t)\,i(t)$$

Using Equation (9.3) we have

$$p(t) = v(t)\,C\frac{di}{dt} \qquad (9.5)$$

Recalling,

$$w(t) = \int_{-\infty}^{t} p(t)\,dt \qquad (9.6)$$

substituting (9.5) into (9.6) gives

$$W = C\int_{-\infty}^{t} v(t)\frac{dv}{dt}\,dt = C\int_{-\infty}^{t} v(t)\,dv$$

$$\boxed{W = \frac{1}{2}CV^2} \qquad (9.7)$$

where we assume the capacitor is uncharged at $t = -\infty$.

Equation (9.7) tells us the energy stored in the dielectric field of the capacitor.

<u>Capacitors do not absorb power as do resistors</u>

Some properties of capacitors are considered below

(1) A capacitor looks like an open circuit to dc, in steady state. Consider the circuit of Figure 9.2

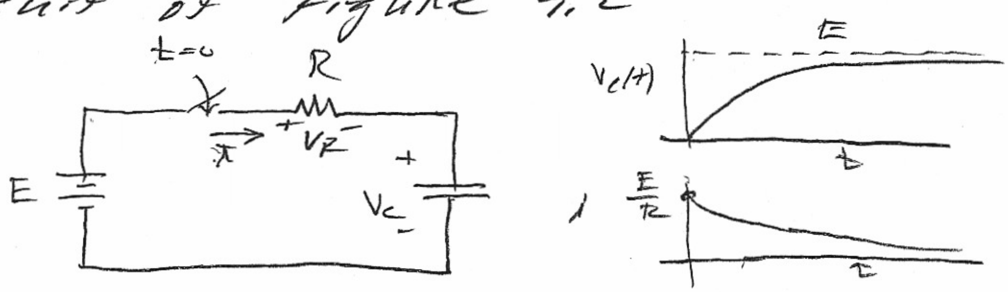


Figure 9.2: An RC circuit.

Once the switch is closed, the energy associated with the capacitor ($w = \frac{1}{2}CV^2$) is zero. Since energy cannot change instantaneously, V (voltage across the capacitor) cannot change instantaneously. This means that

$$V_c(0^-) = V_c(0^+)$$

Considering the circuit of figure 9.2, this means that at $t = 0^+$

$$i(0^+) = \frac{E}{R}$$

and

$$V_R = E$$

or

$$V_c(0^+) = 0$$

since all the voltage is across R we say the capacitor looks like a short circuit at $t = 0^+$.

As time goes on, the capacitor becomes fully charged, $\frac{dq}{dt}$ goes to zero, i goes to zero. Thus $V_R = 0$ and $V_c = E$. So we say the capacitor looks like an open circuit to dc.

Summarizing:

* (a) the capacitor looks like a short circuit at $t = 0^+$.

* (b) the capacitor looks like an open circuit in steady state.

We will use these properties when we study transients of RC circuits.

Configuration of Capacitors:

Consider 3 capacitor in parallel as shown in Figure 9.3

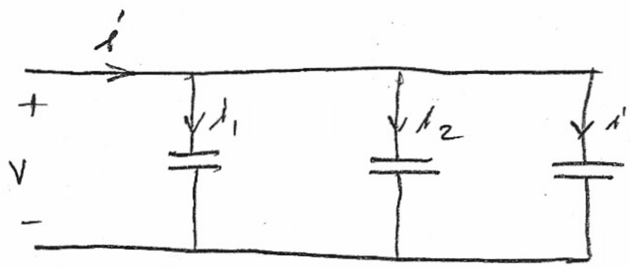


Figure 9.3: Three capacitors in parallel.

We know $i = i_1 + i_2 + i_3$ and that

$$i = C_{eq} \frac{dV}{dt} \qquad (9.8)$$

Furthermore,

$$i_1 = C_1 \frac{dV}{dt} ; \quad i_2 = C_2 \frac{dV}{dt}, \quad i_3 = C_3 \frac{dV}{dt} \qquad (9.9)$$

Using Equations (9.8) and (9.9) we have

$$C_{eq} \frac{dV}{dt} = C_1 \frac{dV}{dt} + C_2 \frac{dV}{dt} + C_3 \frac{dV}{dt}$$

OR

$$C_{eq} = C_1 + C_2 + C_3 \qquad (9.10)$$

In words, capacitors in parallel, add. Said another way: they are treated as resistors in series.

Now consider 3 capacitors in series as shown in Figure 9.4.

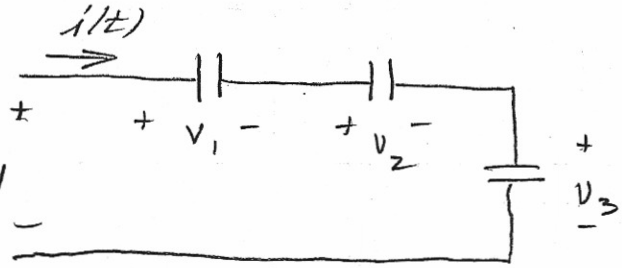


Figure 9.4: Three capacitors in series.

We have,

$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

$$\frac{1}{C_{eq}}\int_0^t i\,dt = \frac{1}{C_1}\int_0^t i\,dt + \frac{1}{C_2}\int_0^t i\,dt + \frac{1}{C}\int_0^t i\,dt$$

or

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \qquad (9.11)$$

In words; capacitors in series are treated as resistors in parallel insofar as determining C_{eq} is concerned.

Let us see how some of the above can be applied to a circuit containing two capacitors in series as shown in Figure 9.5. We want to find V_1 & V_2 in terms of V and C_1, C_2; similarly to two resistors in series.

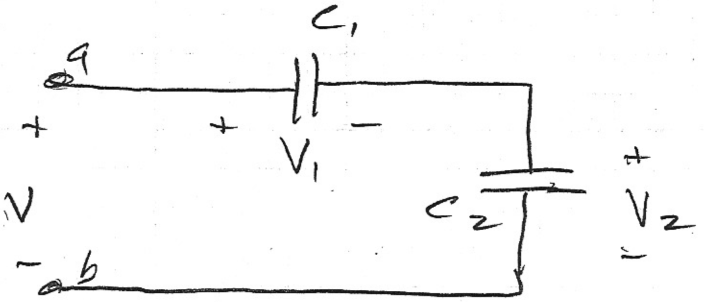


Figure 9.5: Finding voltage across capacitor in series.

We assume that V volts is applied to the circuit in Figure 9.5 and that the circuit is in steady state. We want to find V_1 and V_2.

At the terminals a-b

$$q = C_{eq} V$$

When capacitors are in series, they each have the same q. We conclude from this that, above, C_1 has charge q, C_2 has charge q. Now

$$q = C_1 V_1 = C_{eq} V \qquad (9.12)$$

$$q = C_2 V_2 = C_{eq} V \qquad (9.13)$$

Using (9.12), and (9.13)

$$V_1 = \frac{C_{eq} V}{C_1} \; ; \; V_2 = \frac{C_{eq} V}{C_2}$$

This gives the interesting results,

$$V_1 = \frac{C_2 V}{C_1 + C_2} \quad ; \quad V_2 = \frac{C_1 V}{C_1 + C_2} \qquad (9.14)$$

Almost the same as resistors in series and voltage division.

Distribution of charge

Consider the circuit of Figure 9.6.

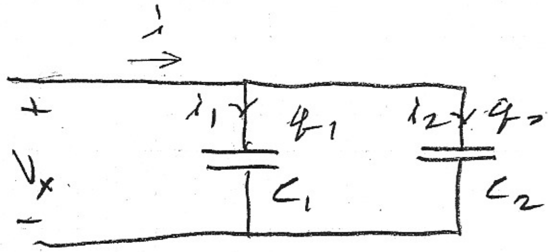


Figure 9.6: Distribution of charge for two resistors in parallel.

We know,

$$q_1 = C_1 V_x$$
$$q_2 = C_2 V_x$$
$$q_1 + q_2 = (C_1 + C_2) V_x \qquad (9.14)$$

but

$$q = C_{eq} V_x \qquad (9.15)$$

Using (9.14) and (9.15) gives

$$\boxed{q = q_1 + q_2} \qquad (9.16)$$

We can also write

$$q_1 = C_1 V_x$$

$$q = C_q V_x$$

so

$$q_1 = q \frac{C_1}{C_q}$$

or

$$\boxed{q_1 = q \frac{C_1}{C_1 + C_2}} \qquad (9.17)$$

and it follows that

$$\boxed{q_2 = q \frac{C_2}{C_1 + C_2}} \qquad (9.18)$$

Now look at current distribution in Figure 9.6.

$$i_1 = C_1 \frac{dV_x}{dt}$$

$$i = C_q \frac{dV_x}{dt}$$

Then

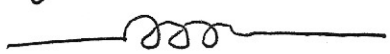
$$i_1 = \frac{i C_1}{C_q} = \frac{i C_1}{C_1 + C_2}$$

Therefore;

$$i_1 = \frac{i C_1}{C_1 + C_2} \quad ; \quad i_2 = \frac{i C_1}{C_1 + C_2}$$

Inductors

In the most simpliest terms, an inductor is a coil of wire. An inductor is indicated by



An approximation for an inductor is

$$L = \mu \frac{N^2 A}{l} \qquad (9.18)$$

where

μ is the permeability
N is the number of windings
A is the cross sectional area of the coil
l is the length of the coil.

Using (9.18) in constructing a coil gives reasonable accuracy of L.

There is no perfect L. Any coil of wire will have resistance. Often we can neglect R. Sometimes we cannot.

Consider Figure 9.7

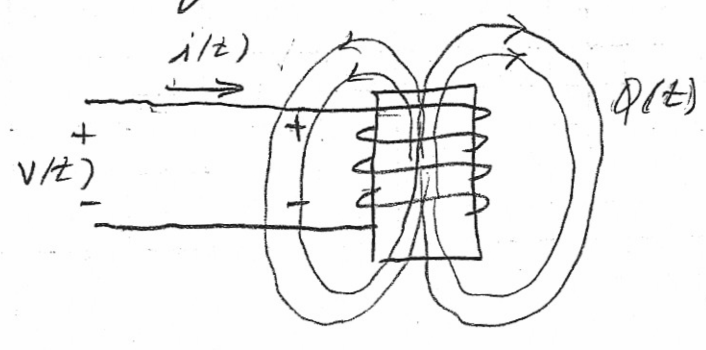


Figure 9.7: Basic Coil.

From Faraday's Law, we write

$$v(t) = N \frac{d\Phi}{dt}$$

but the flux, Φ, is a function of $i(t)$. So we write

$$v(t) = N \frac{d\Phi}{di} \times \frac{di}{dt}$$

We define $$L = N \frac{d\Phi}{di}$$ giving

$$\boxed{v_L(t) = L \frac{di}{dt}} \qquad (9.19)$$

From this equation we can show that

$$i(t) = \frac{1}{L} \int_{t_0}^{t} v_L(t)\,dt + i_L(t_0) \qquad (9.20)$$

We look at energy of the coil:

First

$$P_L(t) = v(t)\,i(t)$$

and

$$W = \int_{-\infty}^{t} p(t)\,dt = \int_{-\infty}^{t} i(t) \times L\frac{di}{dt}\,dt$$

or

$$W = L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2$$

(Assuming $i(-\infty) = 0$).

$$\boxed{W = \frac{1}{2}Li^2} \qquad (9.21)$$

This is a fundamental equation for an inductor. The energy is stored in the magnetic field around the inductor. Like the capacitor, an inductor absorbs zero average power.

Some fundamental properties of the inductor include

- The current through an inductor cannot change instantaneously.
  Reason: $W = \frac{1}{2}Li^2$.
  W cannot change instantaneously.

- Consider the following

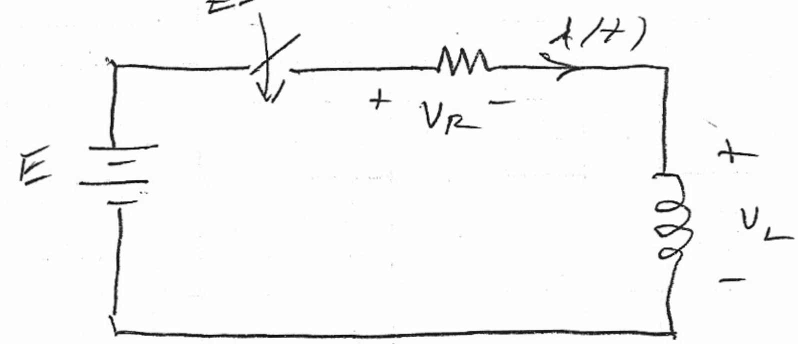


Figure 9.8: Basic RL circuit.

Just after the switch closes, $t = 0^+$, the current $i(0^+) = 0$. Therefore $V_R(0^+) = 0$. We can conclude that $V_L(0^+) = E$.

* So the coil looks like an open-circuit at $t = 0^+$

- In steady state, $t \to \infty$, the current in the circuit of Figure 9.8 ceases to change. Thus $v_L(t) = L\frac{di}{dt} = 0$. This means all the voltage is across R. The inductor looks like

* a short circuit in steady state.

It is easy to show that equivalent inductance is determine by the same procedures we use to find equivalent resistance.

<u>Voltage across series inductors</u>

Consider the circuit of Figure 9.9

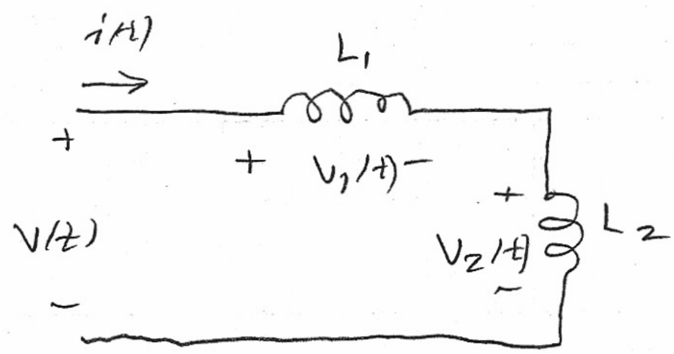


Figure 9.9; Voltage across two series inductors.

We can write,

$$v_1(t) = L_1 \frac{di}{dt} \quad ; \quad v_2 = L_2 \frac{di}{dt}$$

$$v = L_{eq} \frac{di}{dt}$$

so

$$\boxed{v_1(t) = \frac{v(t)\,L_1}{L_1 + L_2} \quad ; \quad v_2(t) = \frac{v(t)\,L_2}{L_1 + L_2}} \quad (9.22)$$

Very similar to voltage division for resistors.

Consider the two coils in parallel as in Figure 9.10

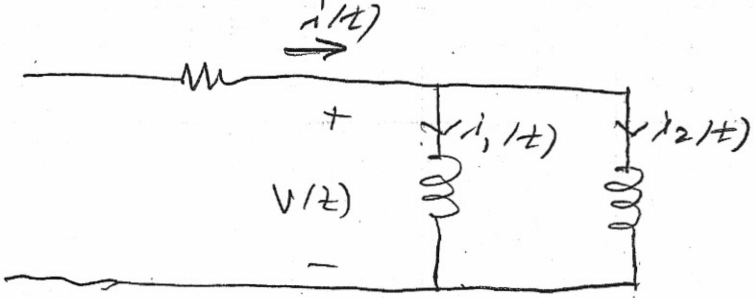


Figure 9.10: Coils in parallel.

We know that in steady state $i_1(t)$, $i_2(t) \rightarrow i_1(\infty)$; $i_2(\infty) = 0$. But we are interested in the currents $0 < t < \infty$. We know the following: when $i_1(0) = i_2(0) = 0$ (Assumption)

$$i_1(t) = \frac{1}{L_1}\int_0^t V(t)\,dt \; ; \; i_2(t) = \frac{1}{L_2}\int_0^t V(t)\,dt$$

$$i(t) = \frac{1}{L_{eff}}\int_0^t V(t)\,dt$$

Using the above we have

$$\boxed{\begin{aligned} i_1(t) &= \frac{i(t)\,L_2}{L_1+L_2} \\ i_2(t) &= \frac{i(t)\,L_1}{L_1+L_2} \end{aligned}} \qquad (9.23)$$

Very similar to resistors.