wlg ECE 301 Lesson 11

The RLC Circuit

The Series RLC Circuit

Consider the circuit of Figure 11.1

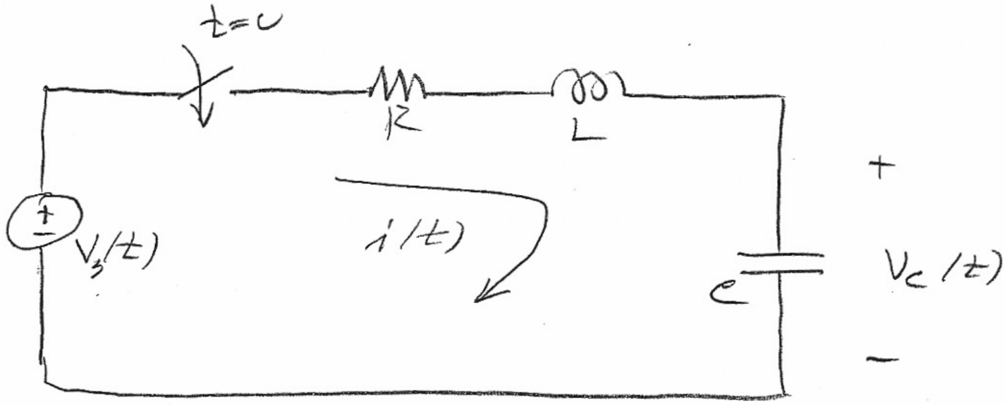


Figure 11.1 Basic series RLC circuit.

Generally, the current $i(0^+)$ can be some value, I_0, an initial condition. Also, $v_c(0^+)$, can be some initial value V_{C0}. These initial values come into play when we solve the differential equation of the circuit. More about this later.

The initial plan is to develop the D.E. for solving for $V_c(t)$.

We proceed by writing KVL around the loop.

We have

$$Ri(t) + L\frac{di}{dt} + V_c(t) = V_s(t) \qquad (11.1)$$

We use

$$i(t) = C\frac{dV_c}{dt} \qquad (11.2)$$

substituting into (11.1) gives

$$LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c(t) = V_s(t) \qquad (11.3)$$

or

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c(t)}{LC} = \frac{V_s(t)}{LC} \qquad (11.4)$$

We will only use $V_s(t) = V_s$, a constant voltage in ELE 301. Equation (11.4) becomes

$$\frac{d^2V_c(t)}{dt^2} + \frac{R}{L}\frac{dV_c(t)}{dt} + \frac{V_c(t)}{LC} = \frac{V_s}{LC} \qquad (11.5)$$

The solution will be compose of two parts

$$V_c(t) = V_{ct}(t) + V_{css} \qquad (11.6)$$

To find $V_{c_t}(t)$ we solve the homogeneous d.e. which is also called the complimentary solution. Thus

$$\frac{d^2 V_{c_t}}{dt^2} + \frac{R}{L}\frac{dV_{c_t}}{dt} + \frac{1}{LC}V_{c_t}(t) = 0$$

We assume

$$V_{c_t}(t) = k e^{st}$$

This leads to

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad (11.7)$$

We call (11.7) the characteristic equation. It is customary to express (11.7) in a parameteric form of

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \qquad (11.8)$$

in which case,

$$2\xi\omega_n = \frac{R}{L} \qquad (11.9)$$

$$\omega_n^2 = \frac{1}{LC} \qquad (11.10)$$

We use the quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to express the roots of (11.8).

Thus,

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} \qquad (11.11)$$

The nature of the roots of (11.11) fall into 3 cases, depending on the value(s) of ξ, relatively to 1.

ξ is called the damping ratio

ω_n is called the undamped natural resonant frequency

Case 1: $\xi > 1$

This is called the overdamped case. s_1 and s_2 are real and unequal.

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$

$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

We have

$$V_{ct}(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad (11.7)$$

It is easy to establish that

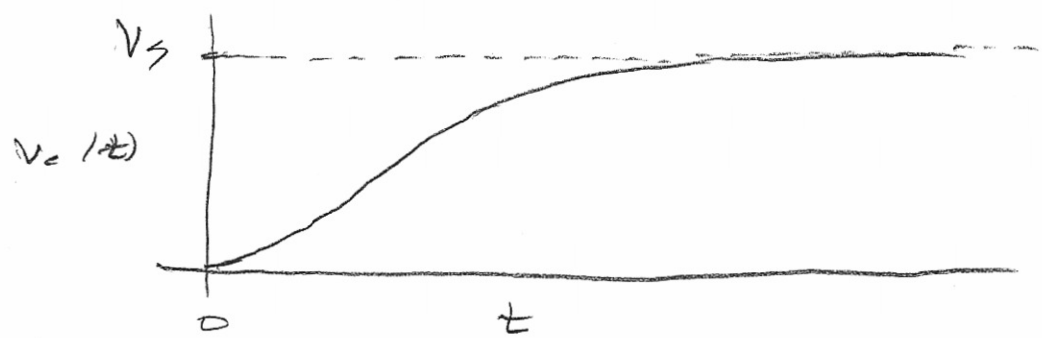
$$v_{ss} = V_s \qquad (11.8)$$

combining (11.7) and (11.8) gives

$$V_c(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} + V_s \qquad (11.9)$$

We solve (11.9) using initial conditions $V_c(0^+)$ and $\dot{V}_c(0^+)$

In any case, the voltage across $V_c(t)$ will have a form as shown in Figure 11.2



This looks almost like the response of a simple RC circuit. One major difference is that $\frac{dV_c(0)}{dt} = 0$ here, but not for the RC circuit.

CASE 2 $\xi = 1$

This is called the critically damped case. The roots will be repeated. The characteristic equation becomes

$$s_1 = s_2 = \left[-\xi \omega_n \pm \omega_n \sqrt{\xi^2 - 1} \right] \Big|_{\xi=1}$$

$$s_1 = s_2 = -\omega_n$$

We recall from differential eq's that when we have repeated roots we use a form of

$$V_{c_t}(t) = k_1 + k_2 t e^{-\omega_n t}$$

So the solution is,

$$V_c(t) = V_{c_t}(t) + V_{ss} = k_1 + k_2 t e^{-\omega_n t} + V_s \quad (11.10)$$

We again evaluate k_1 and k_2 from initial conditions. The response will have the form as shown in Figure 11.3

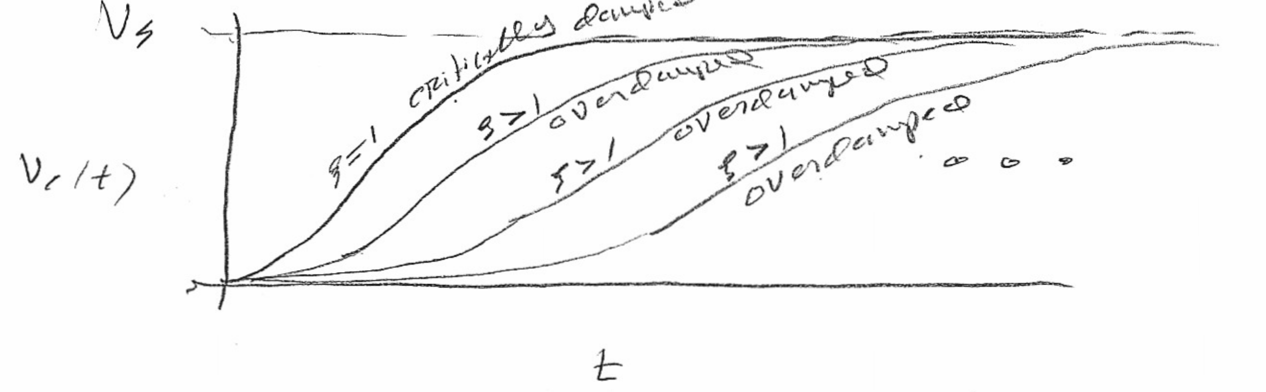


Figure 11.3: Curves showing critically damped and over

The critically damped response is the response curve, furthest to the left, that has no overshoot — meaning the response does not go over V_s.

The 3rd case is the most interesting, mathematically.

<u>CASE 3</u> $\xi < 1$

This is called underdamped.

The roots become,

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$
$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

Since $\xi < 1$, the radical is negative to handle this, we reverse the order of ξ^2 and 1 and use i $(\sqrt{-1})$ to give

$$s_1 = -\xi\omega_n + i\,\omega_n\sqrt{1-\xi^2}$$

$$s_2 = -\xi\omega_n - i\,\omega_n\sqrt{1-\xi^2}$$

Since we use i for current, we change this to j. So the roots appear as

$$s_1 = -\xi\omega_n + j\,\omega_n\sqrt{1-\xi^2} \qquad (11.11)$$

$$s_2 = -\xi\omega_n - j\,\omega_n\sqrt{1-\xi^2} \qquad (11.12)$$

We define

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad (11.13)$$

ω_d is the damped resonant frequency.

So the final form is

$$s_1 = -\xi\omega_n + j\,\omega_d$$

$$s_2 = -\xi\omega_n - j\,\omega_d$$

$$V_{c_t}(t) = k_1 e^{(-\xi\omega_n + j\omega_d)t} + k_2 e^{(-\xi\omega - j\omega_d)t}$$

$$V_{c_t}(t) = k_1 e^{-\xi\omega_n t}\left[e^{j\omega_d t}\right] + k_2 e^{-\xi\omega_n t}\left[e^{-j\omega_d t}\right]$$

$$= k_1 e^{-\xi\omega_n t}\left[\cos\omega_d t + j\sin\omega_d t\right] + k_2 e^{-\xi\omega_n t}\left[\cos\omega_d t - j\sin\omega_d t\right]$$

$$= e^{-\xi\omega_n t}(k_1 + k_2)\cos\omega_d t + e^{-\xi\omega_n t} j(k_1 - k_2)\sin\omega_d t$$

$$V_{c_t}(t) = e^{-\xi\omega_n t}\left[A\cos\omega_d t + B\sin\omega_d t\right] \qquad (11.14)$$

The total response will be

$$V_c(t) = V_s + e^{-\xi \omega_n t}\left[A\cos\omega_d t + B\sin\omega_d t\right] \qquad (11.15)$$

A typical response is given in Figure 11.4

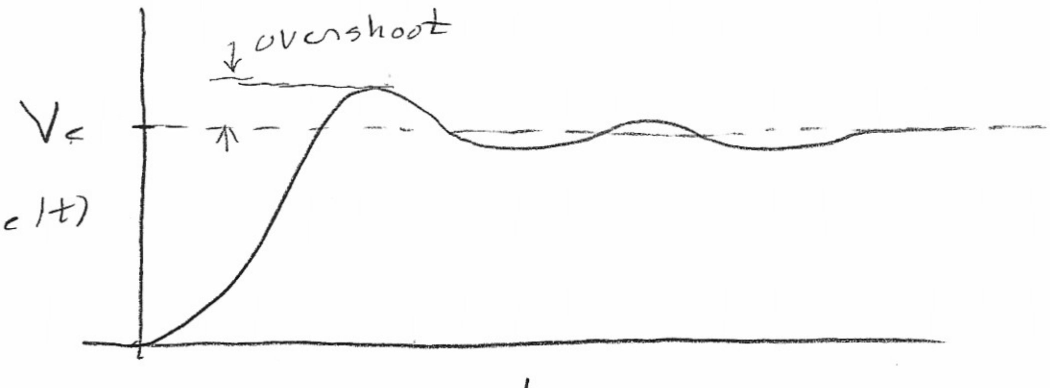


Figure 11.4: Typical 2nd order respon

If we normalize time (t) by letting $t \to \hat{t} = \omega_n t$ then (11.15) can be written as

$$V_c(\hat{t}) = V_s + e^{-\xi \hat{t}}\left[A\cos(\sqrt{1-\xi^2})\hat{t} + B\sin(\sqrt{1-\xi^2})\hat{t}\right]$$

forms a family of curves, based on ξ

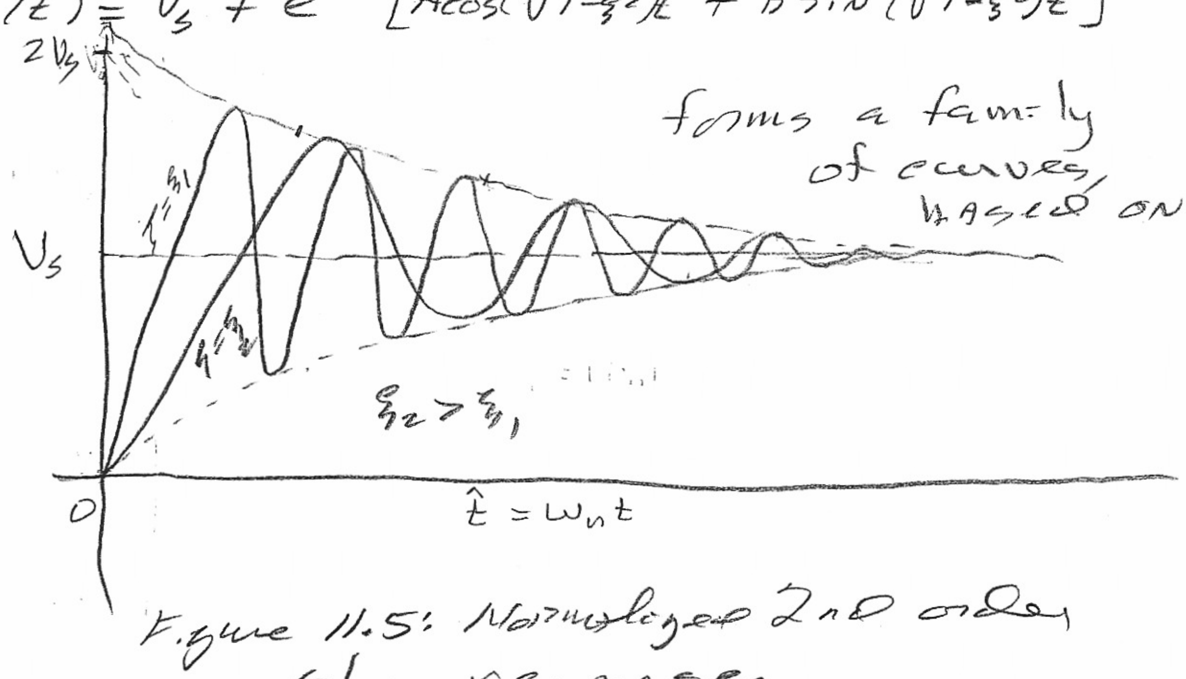


Figure 11.5: Normalized 2nd order step response.

Example 11.1

Given the following circuit. Assume $i(0^-) = 0$, $V_c(0^-) = 0$. Let $V_s = 20V$. Find $V_c(t)$.

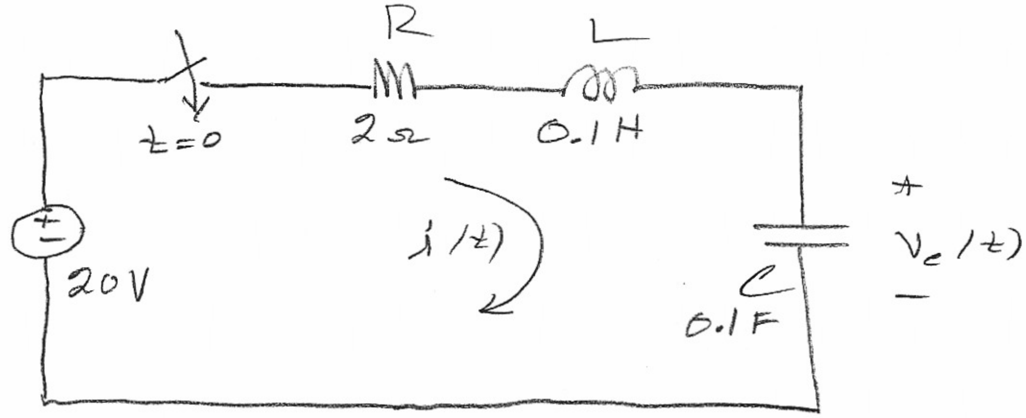


Figure 11.6: Circuit for Example 1.

From the previous development we know the D.E. is

$$Ri(t) + L\frac{di}{dt} + V_c(t) = V_s \qquad (11.16)$$

We know this leads to

$$\frac{d^2 V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c(t)}{LC} = \frac{V_s}{LC}$$

and the characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \qquad (11.17)$$

Putting numbers in for R, L, C gives

$$s^2 + \frac{2}{.1} + \frac{1}{(.1)(.1)} = 0 \qquad (11.18)$$

OR

$$s^2 + 20s + 100 = 0$$

which factors to

$$(s+10)(s+10) = 0$$

We have repeated roots,

$$V_c(t) = (k_1 + k_2 t)e^{-\xi \omega_n t} + V_s$$

We compare (11.18) to

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \qquad (11.19)$$

giving

$$2\xi\omega_n = 20$$

$$\omega_n^2 = 100$$

so $\omega_n = 10$, $\xi = 1$ (As expected)

$$V_c(t) = (k_1 + k_2 t)e^{-10t} + 20 \qquad (11.20)$$

We need $V_c(0)$ and $\dot{V}_c(0)$ to find k_1 and k_2.

11.12

We know (given) that

$$V_c(0^-) = \boxed{V_c(0^+) = 0}$$

We also know

$$i(t) = C\frac{dV_c}{dt} = C\dot{V}_c$$

We know (given) that $i(0^-)=0$.
current through the inductor cannot change instantaneously so $i(0^+)=0$
Then

$$\boxed{\dot{V}_c(0^+) = 0}$$

We go to Equation (11.20)

$$V_c(0^+) = 0 = \left[(k_1 + tk_2)e^{-10t} + 20\right]\Big|_{t=0}$$

$$0 = k_1 + 20$$

$$k_1 = -20$$

$$\frac{dV_c}{dt} = \Big[-10k_1 e^{-10t} + t k_2(-10)e^{-10t} + k_2 e^{-10t}\Big]_{t=0}$$

$$0 = -10k_1 + k_2$$

$$k_2 = 10k_1 = -200$$

i_o

$$V_c(t) = [-20 - 200t]e^{-10t} + 20.$$

$$-(11-21)$$

We can use MATLAB to find $V_c(t)$ with the Symbolic Tool kit.

Our d.e. is

$$\frac{d^2 V_c}{dt^2} + 20\frac{dV_c}{dt} + 100 V_c(t) = 2000$$

For the above equation we write

```
>>
>> dsolve('D2v + 20*Dv + 100*v = 2000',  'v(0) = 0', 'Dv(0) = 0')

ans =

20-20*exp(-10*t)-200*exp(-10*t)*t

>>
```