wlg

A.C. Circuits

Lecture 12

A Prelude

- Complex Numbers

  - Euler's identity

    $e^{jx} = \cos x + j \sin x$

    $Re[e^{jx}] = \cos x$

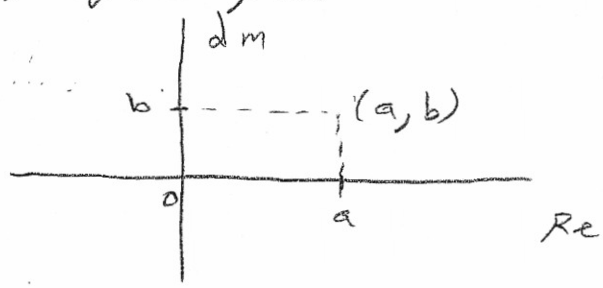
    $Im[e^{jx}] = \sin x$

  - rectangular form

    $a + jb$

    on a calulator; (a, b)

    graphic plot

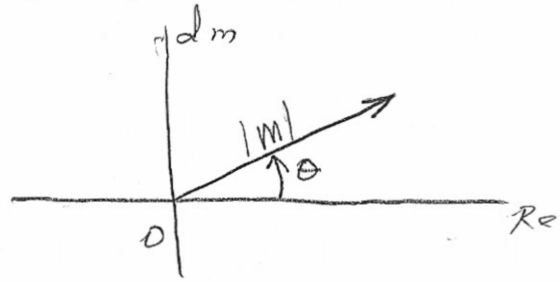


  - polar form

    $M \angle \theta$

    on a calculator; $(M \angle \theta)$

    graphic plot



- relationship between rectangular and polar

$$a + jb = \sqrt{a^2+b^2} \angle \tan^{-1}\left(\frac{b}{a}\right)$$

- the exponential form

$$e^{j\theta} \triangleq 1\angle\theta$$

by definition

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$|e^{j\theta}| = |\cos\theta + j\sin\theta|$$

$$= \sqrt{\cos^2\theta + \sin^2\theta}$$

$$|e^{j\theta}| = 1$$

* Of note;

$$e^{j(\omega t + \psi)} = e^{j\omega t} \cdot e^{j\psi}$$

$$Re\left[e^{j(\omega t+\psi)}\right] = \cos(\omega t + \psi)$$

* Of Note

$$e^{j90} = \cos 90 + j\sin 90 = j$$

$$e^{j90} = 1\angle 90 = j$$

* Of Note

$$j^2 = j \cdot j = 1\angle 90 \cdot 1\angle 90 = 1\angle 180^\circ = -1$$

$$j^2 = -1$$

Also,

$$j^3 = j^2 \cdot j = -j = 1\angle -90$$

$$j^4 = j^2 \cdot j^2 = (-1)(-1) = 1$$

- Calculator Exercises

Note; You need a TI 86, 89 vintage

Perform

$(20+j30)*(6+j25)$

Express in rectangular and polar form

On the calculator (set to rectangular)

(20,30)*(6,25) enter

Ans: $-630 + j680$ rectangular

$926.98\angle 132.8$ polar

Perform

$$\frac{320\angle 65}{6+j8}$$

On the calculator

$(320\angle 65) \div (6,8)$ enter

Ans: $31.3 + j6.58$ rectangular

$32\angle 11.87^\circ$ polar

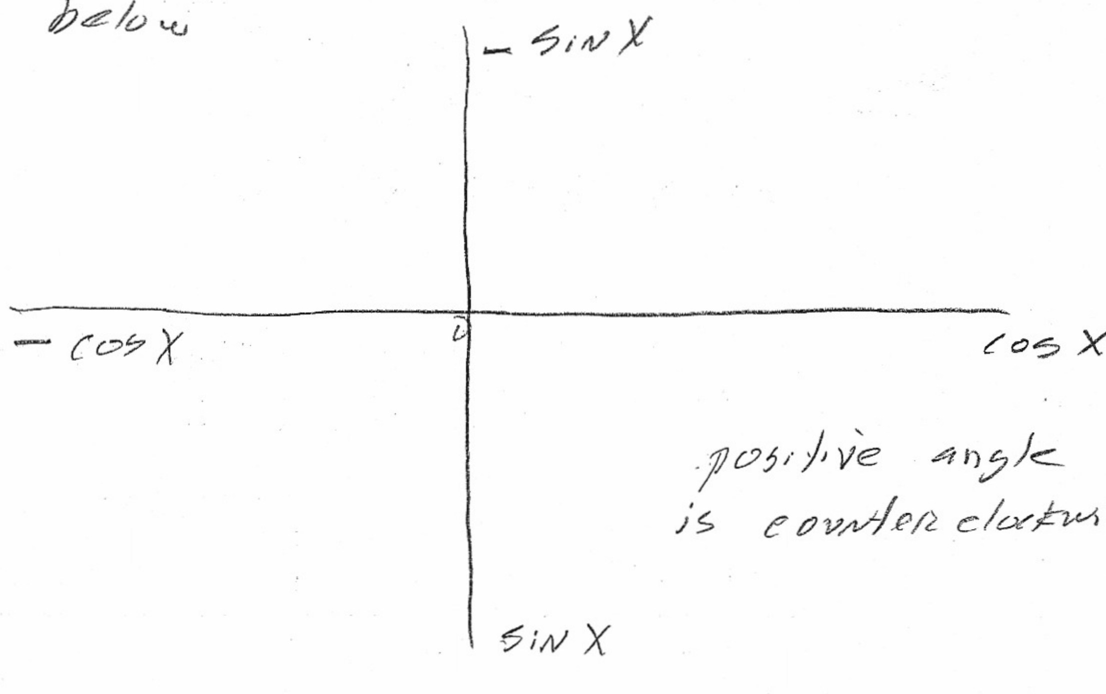
- converting cosine, sine functions

Suppose you have

<u>given</u>	<u>want</u>
$\cos(30^\circ)$	$\sin(?)$
$\sin(-60^\circ)$	$\cos(?)$
$\sin(\omega t + 60^\circ)$	$\cos(\omega t + ?)$

If you go back to basic trig relationships you can perform the above operations. However, there is a simple scheme for doing this, as presented below.

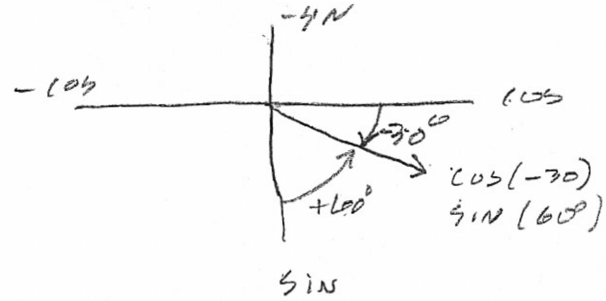
Express the quadrant axis as below



positive angle is counterclockwise

Given

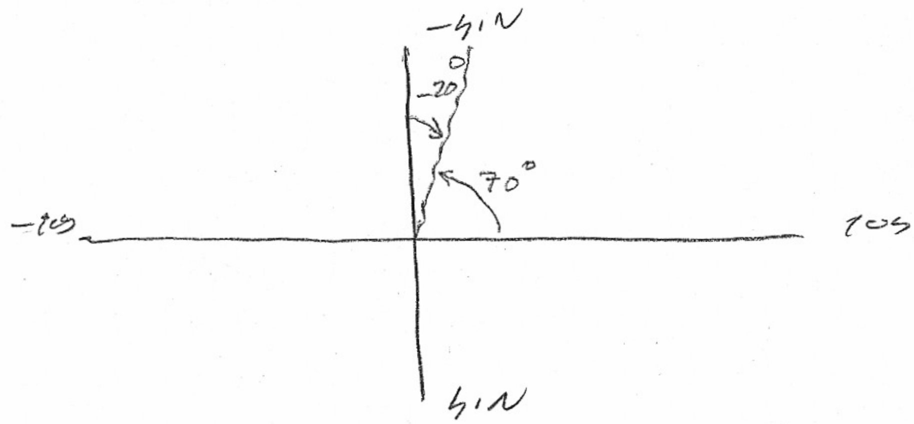
$\cos(-30^\circ) \xrightarrow{?} \sin(?)$



$\therefore \cos(-30^\circ) = \sin(60^\circ)$

Given

$-4\sin(\omega t - 20^\circ) \longrightarrow$ convert to cosine form



$-4\sin(\omega t - 20^\circ) = 4\cos(\omega t + 70^\circ)$

We need to know how to do the above when dealing with AC voltage and current sources expressed in the time domain.

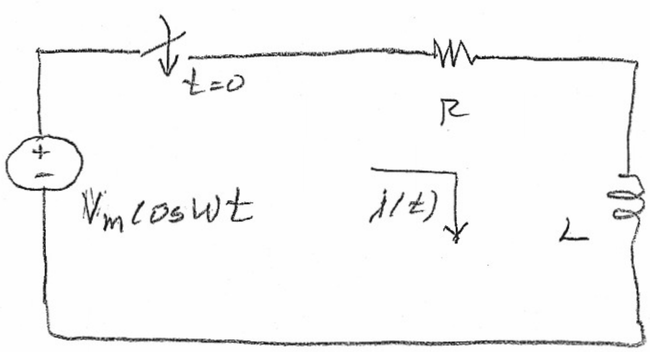
Phasors - And Background:

These notes give a very abbreviated background on the theory (analysis) of circuits with sinusoidal sources.

In AC circuits we are nearly always interested in the steady state solution.

Illustrating

Given



Solve for $i(t)$

$$R\,i(t) + L\frac{di}{dt} = V_m \cos \omega t$$

$$\frac{di}{dt} + \frac{R}{L}\,i(t) = \frac{V_m}{L}\cos \omega t \qquad (A)$$

Change the source to $\frac{V_m}{L} e^{j\omega t}$

find $\hat{i}(t)$, Take $\mathrm{Re}[\hat{i}(t)]$ to get the answer to (A)

$$\frac{d\hat{i}}{dt} + \frac{R}{L}\,\hat{i}(t) = \frac{V_m}{L} e^{j\omega t}$$

$$\hat{i} = \hat{i}_t + \hat{i}_{ss}$$

$\hat{i}_{ss}$: Assume

$$\hat{i}_{ss} = K e^{j\omega t}$$

$$\frac{d}{dt}[K e^{j\omega t}] + \frac{R}{L} K e^{j\omega t} = \frac{V_m}{L} e^{j\omega t}$$

$$K j\omega \cancel{e^{j\omega t}} + \frac{R}{L} K \cancel{e^{j\omega t}} = \frac{V_m}{L} \cancel{e^{j\omega t}}$$

$$K = \frac{V_m / L}{\frac{R}{L} + j\omega} = \frac{V_m}{R + j\omega L}$$

$$\hat{i}_{ss} = \left[\frac{V_m}{R + j\omega L}\right] e^{j\omega t} \qquad (B)$$

$$\hat{i}_{ss} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j\theta} \cdot e^{j\omega t}$$

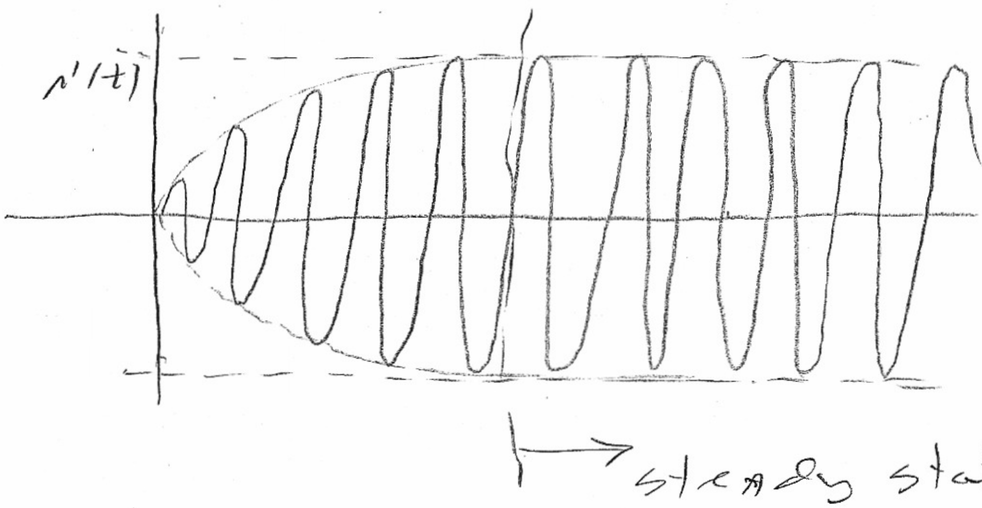
$$\hat{i}_{ss} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{j(\omega t - \theta)} \quad ; \quad \theta = \tan^{-1} \frac{\omega L}{R}$$

We want the real part of this

$$i_{ss} = \mathrm{Re}[\hat{i}_{ss}] = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \theta)$$

$$i_t = K e^{-\frac{R}{L}t}$$

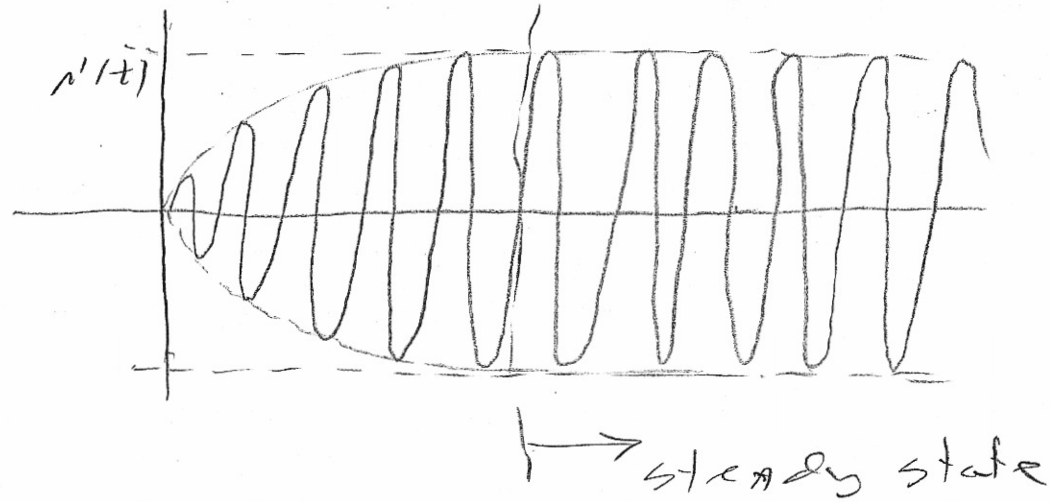
$$i(t) = \underbrace{K e^{-\frac{R}{L}t}}_{\text{transient part}} + \underbrace{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \theta)}_{\text{steady state part}}$$



In AC circuits we are mainly interested in the steady state. It turns out there is not a need to express as a differential equation. We express things in <u>phasor</u> form and basically use all the theorems and techniques we learned for <u>dc</u> circuits.

<u>So, what is a phasor?</u>

Let
$$v(t) = V_m \cos(\omega t + \theta)$$
then
$$v(t) = Re\left[V_m e^{j(\omega t+\theta)}\right]$$
$$= Re\left[V_m \cdot e^{j\theta} \cdot e^{j\omega t}\right]$$
$$v(t) = Re\left[\hat{V} e^{j\omega t}\right]$$



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So, what is a phasor?

Let $v(t) = V_m \cos(\omega t + \theta)$

then

$$v(t) = Re[V_m e^{j(\omega t + \theta)}]$$

$$= Re[V_m \cdot e^{j\theta} \cdot e^{j\omega t}]$$

$$v(t) = Re[\hat{V} e^{j\omega t}]$$

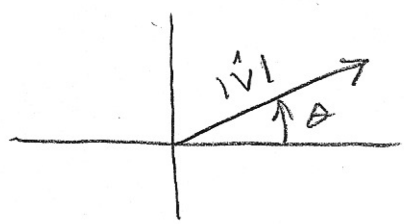
We define

$\hat{V}$ as a phasor

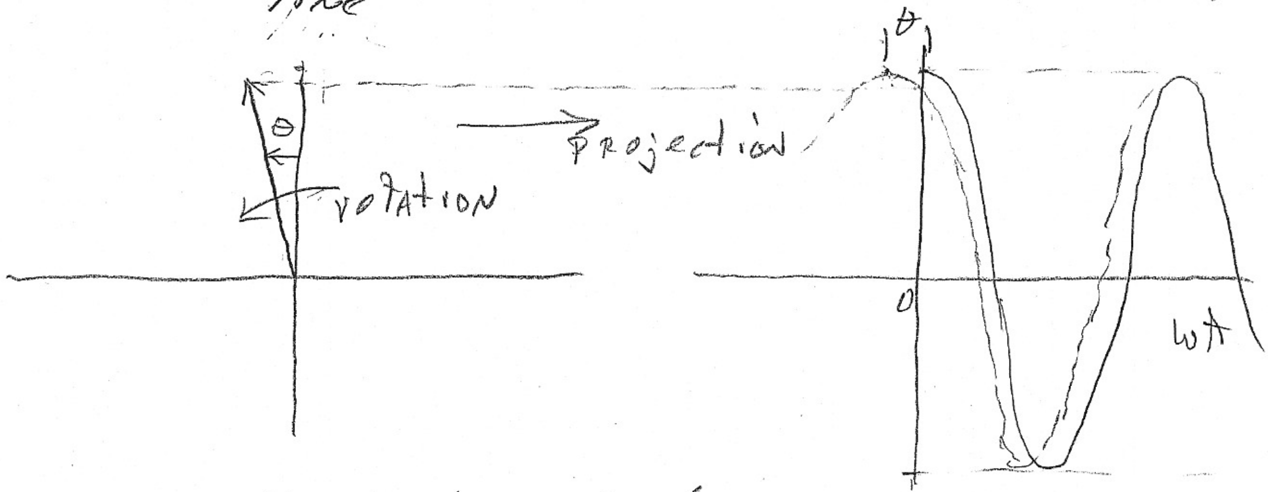
$$\hat{V} = V_m e^{j\theta} = V_m \angle\theta$$

$\hat{V}$ is not a vector. It does not obey the rules of vector analysis's (·product, ×, cross product)

$\hat{V}$ is a number at an angle



The sinor $[\hat{V} e^{j\omega t}]$ is a rotating line



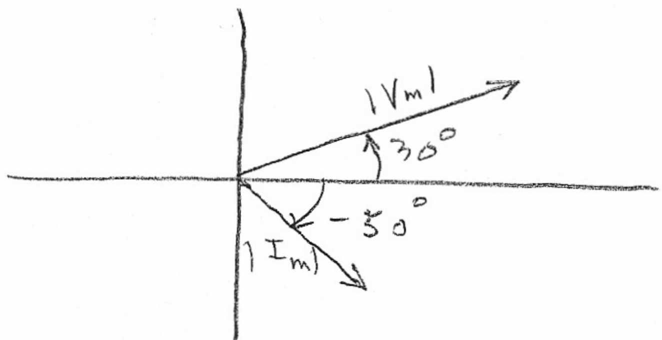
Illustrating A Sinor

Going from Phasor To Time Domain

Phasor		Time Domain
$V_m \angle \theta$	$\Longleftrightarrow$	$V_m \cos(\omega t + \theta)$

A Phasor Diagram

$$\vec{V} = V_m \angle 30^\circ, \quad \vec{I} = I_m \angle -50^\circ$$



Note: Voltage & current must have a different scale.

We will use the above in AC circuit analysis.

We now turn our attention to formulating the AC circuit as a phasor problem.

Phasor Relations & Circuit Elements

The resistor

Assume

$$i(t) = I_m \cos(\omega t + \theta_R) \qquad \hat{I} = I_m \angle \theta_R$$

$$v(t) = i(t) R$$

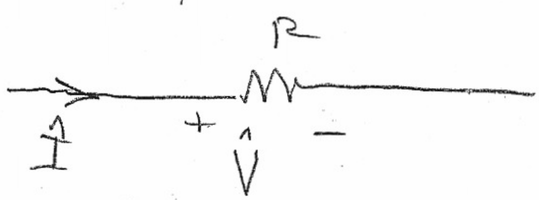
$$= I_m \cos(\omega t + \theta_R) R$$

$$= Re\left[I_m R e^{j\theta_R} \cdot e^{j\omega t} \right]$$

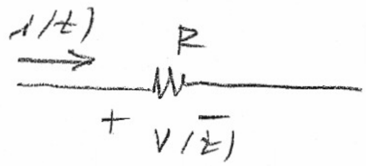
$$= Re\left[\hat{V} e^{j\omega t} \right]$$

$$\hat{V} = I_m \angle \theta_R R = \hat{I} R$$

∴ AC ohm's law for a resistor

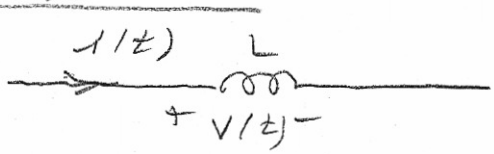


$$\hat{V} = R \hat{I}$$



$$v(t) = i(t) R$$

The Inductor



$$v(t) = L \frac{di}{dt}$$

Assume

$$i(t) = I_m \cos(\omega t + \theta_L) \qquad \hat{I} = I_m \angle \theta_L$$

$$v(t) = -\omega L I_m \sin(\omega t + \theta_L)$$

$$v(t) = -\omega L I_m \sin(\omega t + \theta_L)$$

$$= \omega L I_m \cos(\omega t + 90^\circ + \theta_L)$$

$$v(t) = Re\left[\omega L I_m e^{j(\omega t + 90^\circ + \theta_L)}\right]$$

$$= Re\left[\omega L I_m e^{j\theta_L} \cdot e^{j90} \cdot e^{j\omega t}\right]$$

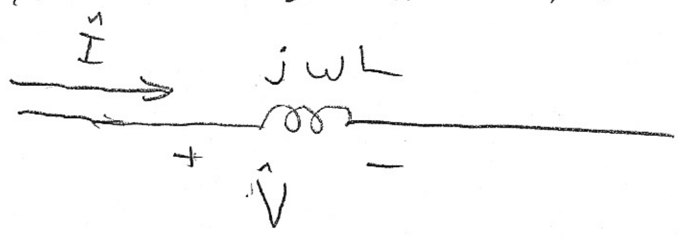
$$= Re\left[\hat{V} e^{j\omega t}\right]$$

$$\hat{V} = \omega L I_m \angle\theta_L \cdot j$$

$$\hat{V} = j\omega L I_m \angle\theta_L$$

$$\hat{V} = j\omega L \hat{I}$$

AC Ohm's Law for an inductor



$$\underline{\hat{V} = j\omega L \hat{I}}$$

<u>The Capacitor</u>

Let $v_c(t) = V_m \cos(\omega t + \theta_c)$

$$\hat{V}_c = V_m \angle\theta_c$$

Now recall

$$i(t) = C \frac{dv_c}{dt}$$

$$i(t) = -\omega C V_m \sin(\omega t + \theta_c)$$

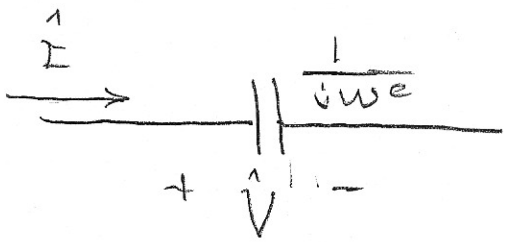
$$= \omega C V_m \cos(\omega t + 90^\circ + \theta_c)$$

$$= \mathrm{Re}\left[\omega C V_m e^{j90^\circ} \cdot e^{j\theta_c} \cdot e^{j\omega t}\right]$$

$$\hat{I} = j\omega C V_m \angle\theta_c = j\omega C \hat{V}$$

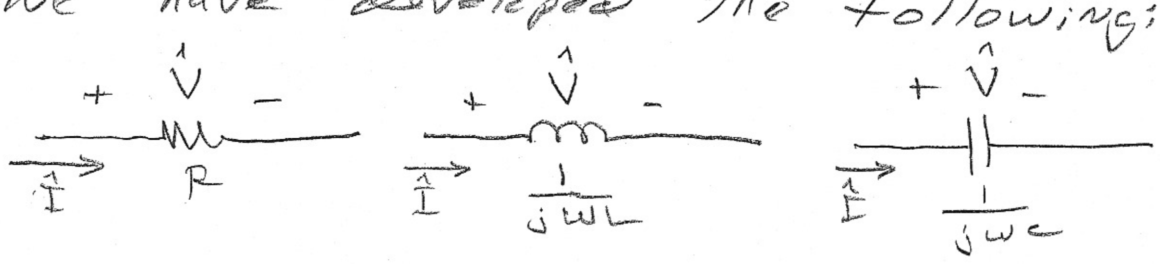
$$\hat{V} = \left[\frac{1}{j\omega C}\right] \times \hat{I}$$

Ohm's Law for the capacitor



<u>Impedance and Admittance</u>

We have developed the following:



$$\frac{\hat{V}}{\hat{I}} = R \qquad \frac{\hat{V}}{\hat{I}} = j\omega L \qquad \frac{\hat{V}}{\hat{I}} = \frac{1}{j\omega C}$$

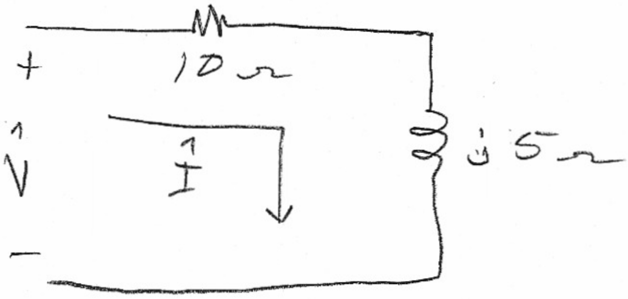
In a more general sense

we say

$$\frac{\hat{V}}{\hat{I}} = \hat{Z}$$

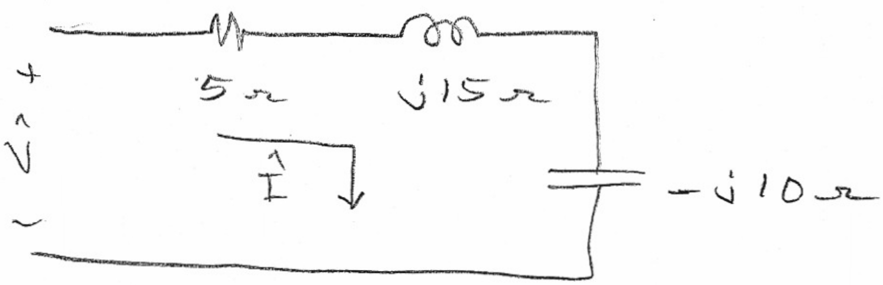
where $\hat{Z}$ is impedance, (units of ohms)

$$\hat{Z} = \underbrace{R}_{\substack{\text{resistance} \\ \text{(ohms)}}} + j\underbrace{X}_{\substack{\text{reactance} \\ \text{(ohms)}}}$$



$$\hat{V} = (10 + j5)\,\hat{I}$$

$$\hat{Z} = (10 + j5) \text{ ohms}$$



$$\hat{V} = (5 + j15 - j10)\,\hat{I} = (5 + j5)\,\hat{I}$$

$$\hat{Z} = (5 + j5)\ \Omega$$

We define admittance as

$$\hat{Y} = \frac{1}{\hat{Z}} = \underset{\substack{\text{conductance} \\ (S)}}{G} + j\underset{\substack{\text{susceptance} \\ (S)}}{B} \quad (S)$$

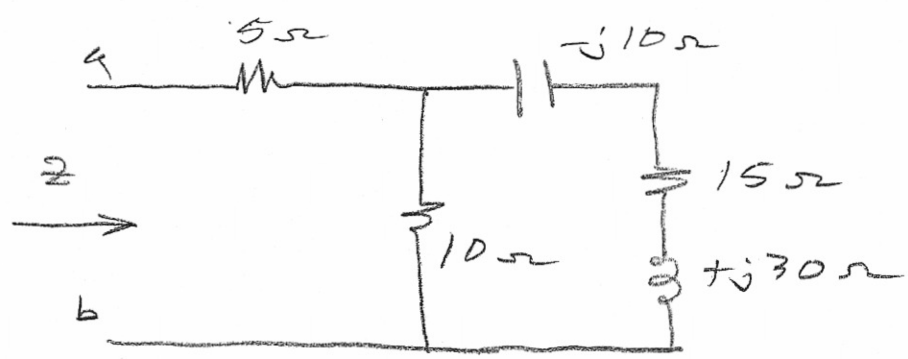
$Y \rightarrow$ Admittance (siemen, S)

$G \rightarrow$ conductance (siemen, S)

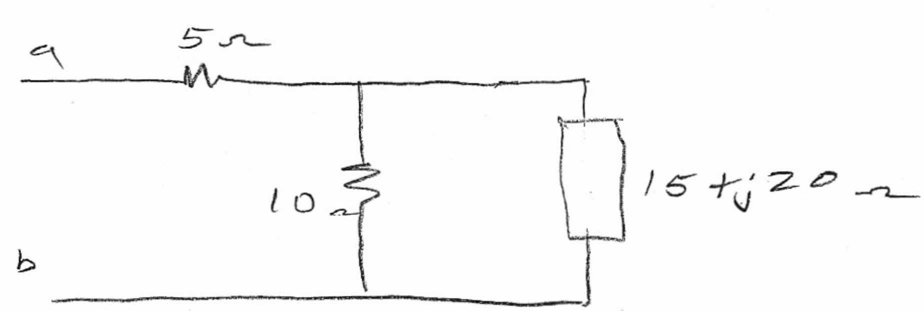
$B \rightarrow$ susceptance (siemen, S)

Example 1

Find the impedance seen looking into terminals a-b



This becomes



$$Z = 5 + 10 \| (15 + j20)$$

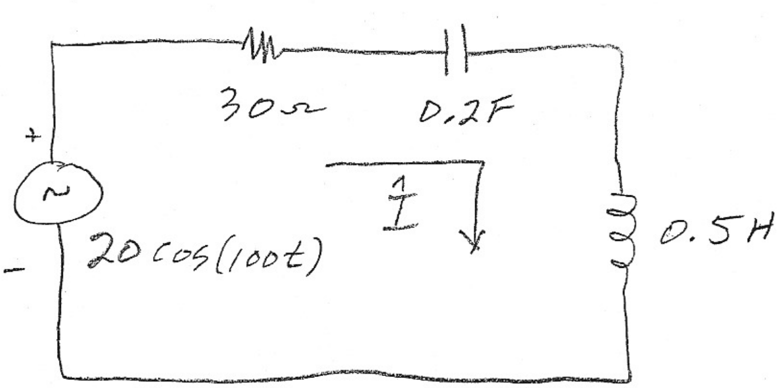
$$= 5 + \frac{10 \times (15 + j20)}{10 + 15 + j20}$$

$$= 5 + 7.56 + j1.95 \ \Omega$$

$$= 12.56 + j1.95 \ \Omega$$

$$\hat{Z} = 12.74 \angle 8.8^\circ \ \Omega$$

From this point it is an easy step to move to solution of A.C. Circuits. We do this in the next lesson. Think about the following.



Find the phasor current $\hat{I}$.