Lesson 12
AC Circuits

The term AC circuits means the signal sources are sinusoidal. This is important because the power provided to our homes and generally to industrial sites is derived from sinusoidal or $\boxed{A}$lternating $\boxed{C}$urrent.

We start the lesson by considering the circuit shown in Figure 12.1.

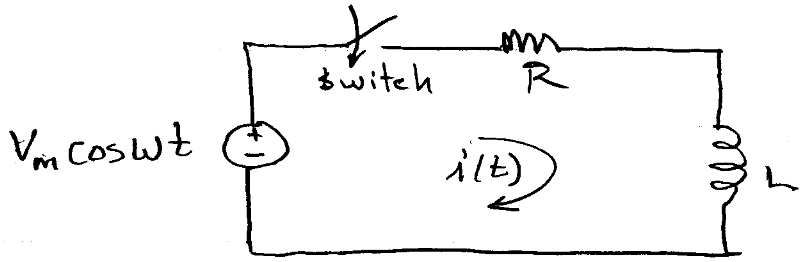


Figure 12.1: RL circuit with sinusoidal signal applied.

We assume the switch has been closed for a long time. That is, long enough for the transient state of the circuit to die out. The significance of being in steady state becomes clearer at a later point in the discussion.

Writing KVL for the circuit of Fig 12.1 gives

$$L\frac{di(t)}{dt} + Ri(t) = V_m\cos\omega t \qquad \text{Eq 12.1}$$

We can solve the steady state (also called the complimentary) solution by Assuming

$$i_{ss}(t) = A\cos\omega t + B\sin\omega t$$

We will find the solution to be of the form:

$$i_{ss}(t) = I\cos(\omega t + \phi) \qquad \text{Eq 12.2}$$

where I and ϕ will be in terms of ω, R, L and V_m.

At this time I think it is good to point out that if a R, L, C, linear ckt is supplied by sources with a single frequency, ω, all responses in the circuit, whether voltages or currents, will be of the form

$$X_i\cos(\omega t + \psi_i)$$

where X_i & ψ_i will be associated with the various voltages and currents in the circuit.

We return to Eq 12.1 but consider a different forcing function. First, let us recall from Euler's (blind mathematician) identity;

$$e^{jx} = \cos x + j \sin x \qquad \text{Eq 12.3}$$

$$\mathrm{Re}[e^{jx}] = \cos x$$
$$\mathrm{Im}[e^{jx}] = \sin x$$

We see that if we have an input of $V_m \cos(\omega t)$ that it is the same

$$\mathrm{Re}\left[V_m e^{j\omega t}\right] = V_m \cos \omega t \qquad \text{Eq 12.4}$$

Rather than solve the differential equation with $V_m \cos \omega t$, we will use $V_m e^{j\omega t}$ and take the $\mathrm{Re}(\cdot)$ of the solution. This may seem like we are going the long way about solving the problem but actually we gain an insight that is extremely valuable.

We bring ourselves to the point of solving the problem:

$$L\frac{di(t)}{dt} + Ri(t) = V_m e^{j\omega t} \qquad \text{Eq 12.5}$$

For the solution (steady state only) we assume

$$i_c(t) = \bar{I} e^{j\omega t} \qquad \text{Eq 12.6}$$

where $\bar{I}$ is a complex number to be determined (in this case it will be a function of V_m, R, ω, L.

We place Eq 12.6 into Eq 12.5 and find we have the following:

$$L\frac{d\{\bar{I} e^{j\omega t}\}}{dt} + R\{\bar{I} e^{j\omega t}\} = V_m e^{j\omega t}$$

OR

$$j\omega L\bar{I} e^{j\omega t} + R\bar{I} e^{j\omega t} = V_m e^{j\omega t} \qquad \text{Eq 12.7}$$

We note that $e^{j\omega t}$ can be divided out of Eq 12.7 and the result is;

$$\bar{I} = \frac{V_m}{R + j\omega L} \qquad \text{Eq 12.8}$$

OR

$$\bar{I} = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}\; e^{j\tan^{-1}\left(\frac{\omega L}{R}\right)}} \qquad \text{Eq 12.9}$$

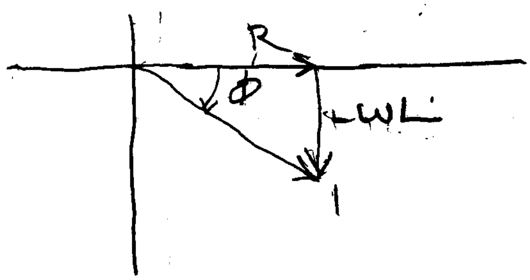
Eq 12.9 becomes

$$\bar{I} = \frac{V_m e^{-j\tan^{-1}\left(\frac{\omega L}{R}\right)}}{\sqrt{R^2+(\omega L)^2}} = \frac{V_m e^{j\phi}}{\sqrt{R^2+(\omega L)^2}} \quad \text{Eq 12.10}$$

with $\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$ Eq 12.11

The function $e^{j\phi}$ can be written as

$$e^{j\phi} = 1\angle\phi$$



Thus we can write

$$i_c(t) = \bar{I}e^{j\omega t} = \frac{V_m e^{j(\omega t+\phi)}}{\sqrt{R^2+(\omega L)^2}} \quad \text{Eq 12.12}$$

OR $i_{ss}(t) = \text{Re}\left(i_c(t)\right) = \dfrac{V_m}{\sqrt{R^2+(\omega L)^2}}\cos(\omega t+\phi)$ Eq 12.13

Eq 12.13 is the desired result.

An important result we see from Eq 12.12 is

$$i_c(t) = \bar{I} e^{j\omega t} = I_m e^{j\phi} \cdot e^{j\omega t} \quad \text{Eq 12.14}$$

where we see that

$$I_m e^{j\phi} = \frac{V_m}{R + j\omega L} \quad \text{Eq 12.15}$$

OR

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} e^{j(-\tan(\frac{\omega L}{R}))} \quad \text{Eq 12.16}$$

In the final result we can write that the forcing function to the given circuit is

$$V(t) = V_m \cos(\omega t + \theta) = R_e[V_m e^{j(\omega t + \theta)}) \quad \text{Eq 12.17}$$

as a complex number,

$$\boxed{V(t) = Re[V_m \angle\theta \, e^{j\omega t}]} \quad \text{Eq 12.18}$$

We can also write a similar equation for $i_{ss}(t)$:

From Eq 12.13,

$$i_{ss}(t) = I_m \cos(\omega t + \phi) \qquad \text{Eq } 12.19$$

where, in this case,

$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}$$

and

$$\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

We can also write Eq 12.19 as

$$\boxed{i_{ss}(t) = \mathrm{Re}\left[I_m \underline{/\phi}\, e^{j\omega t}\right]} \qquad \text{Eq } 12.20$$

In so far as solving for the undetermined coefficient, $\bar{I}$, we can drop $e^{j\omega t}$ and Re(-) in both Eq 12.18 and Eq 12.20.

We have left;

$$V_m \underline{/\theta} \quad \text{and} \quad I_m \underline{/\phi} \qquad \text{Eq } 12.21$$

Both $V_m \underline{/\theta}$ and $I_m \underline{/\phi}$ are called <u>phasors</u>;

We see that a phasor is characterized as a polar function in that it has magnitude and angle.

Further Development of the Concept of Phasor

Consider the following circuit:

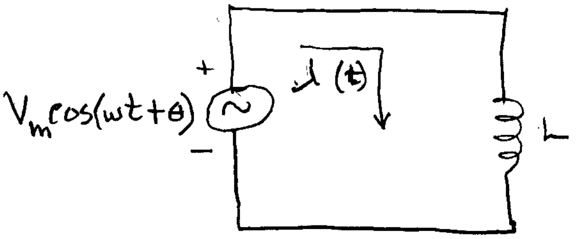


Fig 12.2: Inductor circuit

We drop $i_{ss}(t)$ and use $i(t)$ with the understanding that it is the steady state current.

For the above we have;

$$L\frac{di(t)}{dt} = V_m \cos(\omega t+\theta) \qquad \text{Eq. 12.22}$$

We use,

$$L\frac{di(t)}{dt} = V_m \angle\theta \, e^{j\omega t} \qquad \text{Eq. 12.23}$$

As per the method of undetermined coefficients used in the solution of Eq 12.23, we write

$$i(t) = I e^{j\omega t} \qquad \text{Eq 12.24}$$

We substitute this into Eq 12.23

$$j\omega \bar{I} e^{j\omega t} = V_m \angle\theta \; e^{j\omega t}$$

OR

$$j\omega \bar{I} = \bar{V} \qquad \text{Eq } 12.25$$

where

$$\bar{V} = V_m \angle\theta \qquad \text{(phasor)}$$

This gives,

$$\bar{I} = \frac{\bar{V}}{j\omega L} \qquad \text{Eq } 12.25$$

OR

$$\boxed{\frac{\bar{V}}{\bar{I}} = j\omega L} \qquad \text{Eq } 12.26$$

From the circuit diagram of Figure 12.2 we could have gone directly to Eq. 12.22.

Now consider that we have a purely capacitive circuit as below:

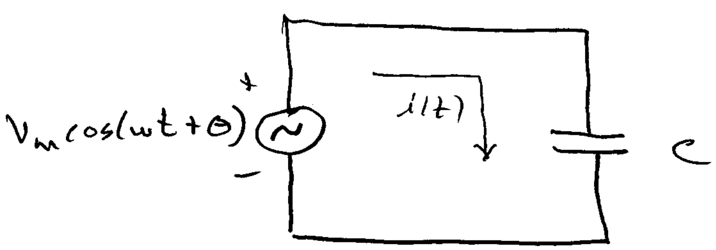


Figure 12.3: Purely capacitive circuit.

$$i(t) = C\frac{dV}{dt}$$

or

$$\bar{I}e^{j\omega t} = C\frac{d}{dt}[V_m e^{j(\omega t+\theta)}]$$

$$\bar{I}\cancel{e^{j\omega t}} = j\omega C\, V_m\angle\theta\, \cancel{e^{j\omega t}} \qquad \text{Eq } 12.25$$

$$\bar{I} = j\omega C\bar{V}$$

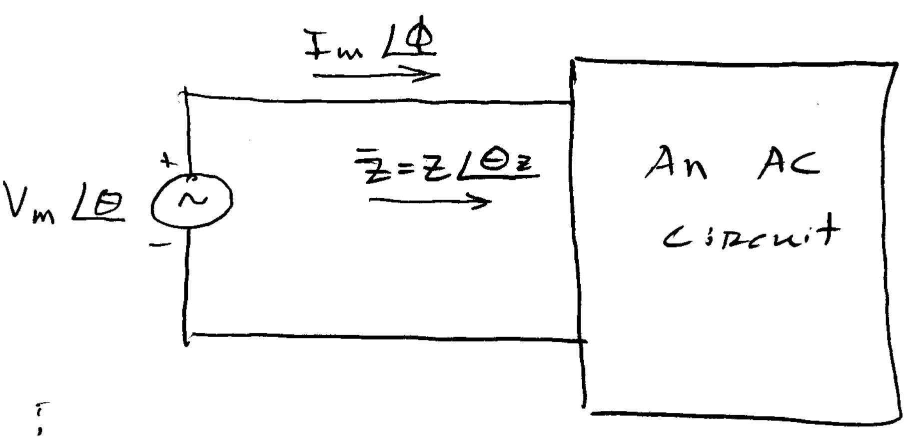
$$\frac{\bar{V}}{\bar{I}} = \frac{1}{j\omega C} \qquad \text{Eq. } 12.26$$

In sort of a natural way, we define the ratio of $\bar{V}/\bar{I}$ as impedance: We write

$$\frac{\bar{V}}{\bar{I}} = \bar{Z} \qquad \text{Eq } 12.27$$

The unit of $\bar{Z}$ is ohms since it is volts/Amps.

In the most general case we have



There is no doubt that it is easier to understand the above presentation by considering some simple examples.

Example 12.1

You are given the following circuit. Find $i_{ss}(t)$.

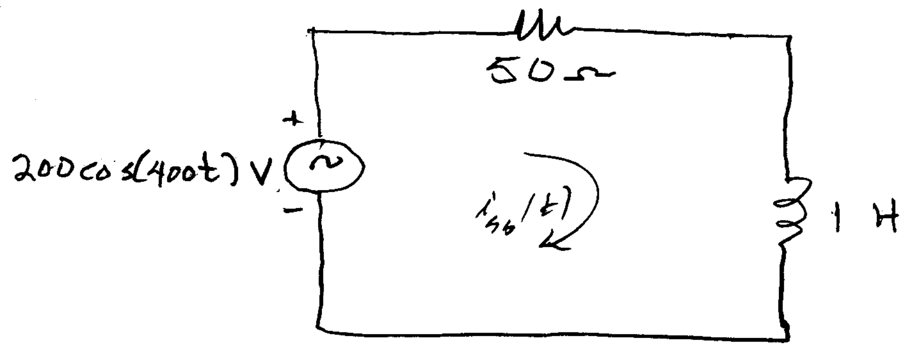


Figure 12.4: Original ckt for Ex 12.1.

The 1 H goes to a term → $j\omega L$.
In this case, $\omega = 400$, the source ω.
Using phasor notation our circuit becomes

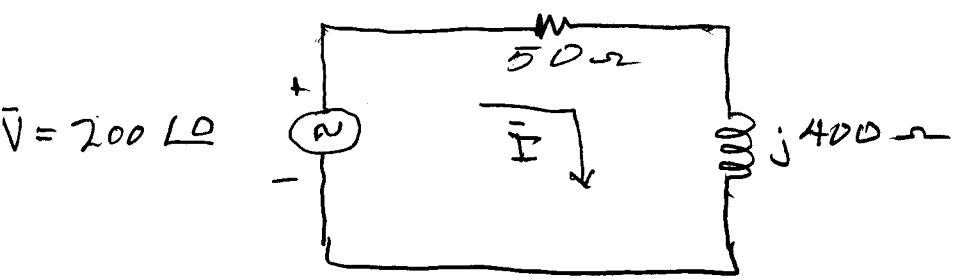


Figure 12.5: Phasor diagram for the ckt.

We can now write

$$\bar{I} = \frac{\bar{V}}{\bar{Z}} = \frac{200\angle 0}{50 + j400} \qquad \text{Eq } 12.28$$

using a calculator this gives

$$\boxed{\bar{I} = 0.496\angle -82.9} \qquad \text{Eq. } 12.29$$

Therefore, the steady state current is

$$i_{ss}(t) = 0.496\cos(400t - 82.9^\circ) \qquad \text{Eq } 12.30$$

In most applications, we stop at Eq. 12.29, <u>the phasor solución.</u>

<u>Impedance In A General Sense</u>

When we look into the terminals of any AC circuit we see an impedance;

$$Z(\omega) = R(\omega) + jX(\omega) \qquad \text{Eq } 12.30$$

We usually drop the (ω) and write

$$Z = R + jX \qquad \text{Eq } 12.31$$

We denote R as resistance, of course. We denote X as reactance, a new term. So we "think" as

$$Z = \text{Impedance} = \text{Resistance} + j\,\text{Reactance}.$$

We say that a load is inductive if the angle associated with Z is positive. We might then write

$$Z = R + jX_L \qquad \text{Eq 12.32}$$

where R is some equivalent resistance and $jX_L = j\omega L$ is some equivalent reactance.

If the phase angle of Z is negative we could write;

$$Z = R + jX_C \qquad \text{Eq 12.33}$$

where R is some equivalent resistance and

$$jX_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} \qquad \text{Eq 12.34}$$

OR

$$X_C = -\frac{1}{\omega C} \qquad \text{Eq 12.35}$$

Example 12.2

Given the following "load" for an AC circuit:

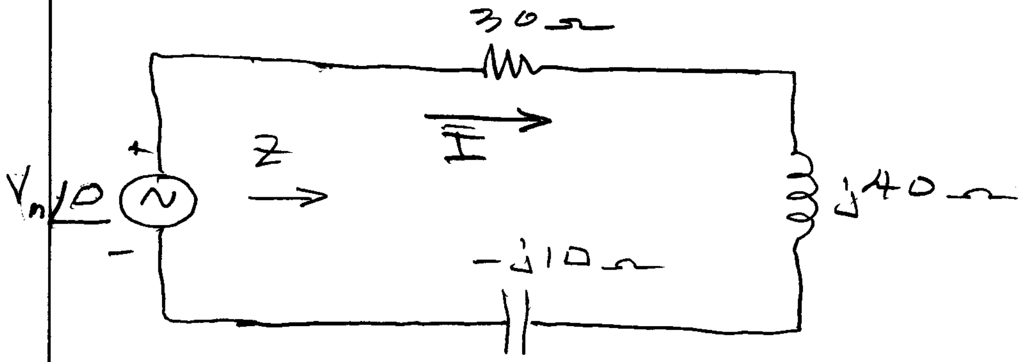


Figure 12.6: Circuit for Example 12.2.

We see that

$$Z = 30 + j40 - j10 = 30 + j30$$

$$Z = 30 + j30 = 30\sqrt{2}\,\angle 45^\circ\ \Omega$$

We note for this case, we have an inductive load. We could think of it as a circuit with a resistor and inductor

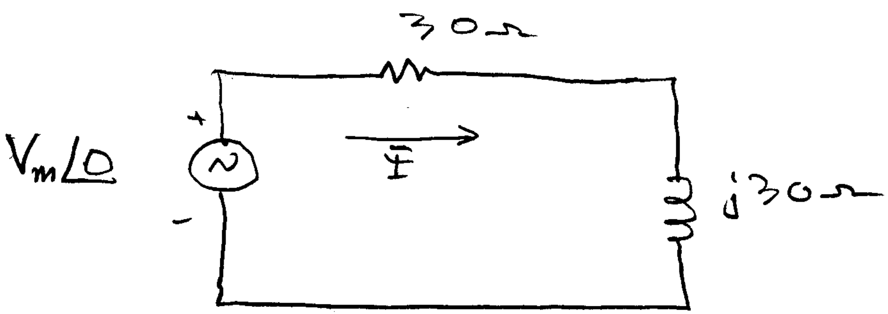


Figure 12.7: Equivalent ckt of Fig 12.6

Now $\omega L = 30$. We would normally know ω so we could calculate L. Finally, we note

$$\frac{\bar{V}}{\bar{I}} = 30\sqrt{2}\,\angle 45^\circ \qquad \text{Eq } 12.36$$

OR

$$\bar{I} = \frac{\bar{V}}{30\sqrt{2}}\,\angle -45^\circ = I_m\,\angle -45^\circ \qquad \text{Eq } 12.37$$

Keep in mind that $\bar{V} = V_m \angle 0$.

If we represent the two phasor $\bar{I}$ and $\bar{V}$ in a phasor plane we have

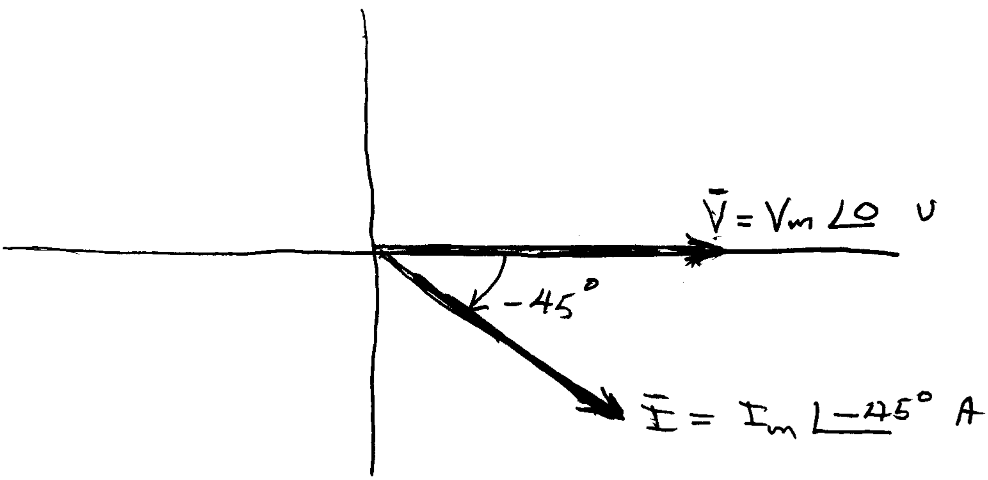


Figure 12.8: Phasor diagram for Ex 12.2

We conclude that, if an AC circuit has an inductive load, the voltage leads the current.

Example 12.3

Consider the following circuit

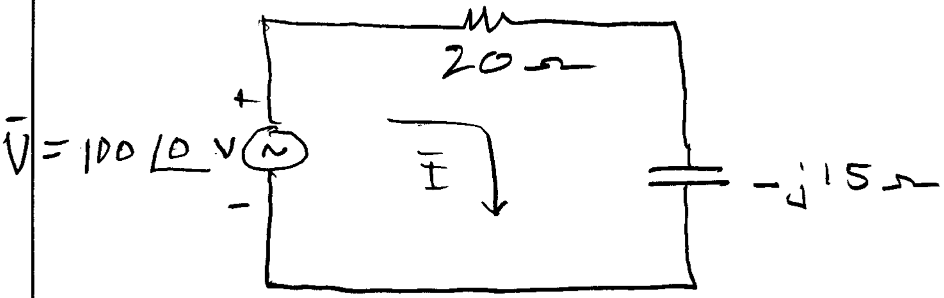


Figure 12.9: Circuit for Example 12.3.

$$\bar{I} = \frac{100\angle 0}{20 - j15} = 4\angle 36.9 \text{ A}$$

Now, if we present the phasor diagram of $\bar{V}$ and $\bar{I}$ we note that $\bar{I}$ <u>leads</u> $\bar{V}$:

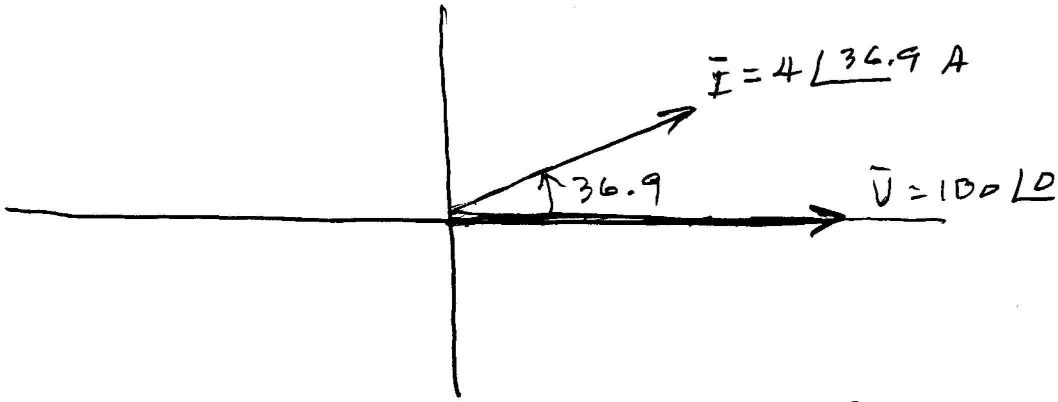


Figure 12.10: Phasor diagram of $\bar{V}$ and $\bar{I}$ for Example 12.3

We write

ELI the ICE man

E I L
Voltage leads the current in an inductive ckt
I E C
Current Leads the voltage in a capacitive ckt.