ECE 301

Transient Analysis:

These notes are only intended to supplement the material given in the text. In some cases the material will be given in a different order than the text presentation, However, in the final analysis, the same material is covered. In any case, you should read the text in addition to these notes.

The general meaning of the word transient is well understood, this general definition applies also to circuits. We assume that the transient response of a circuit is what happens to various voltages or currents when the circuit changes from one state of excitation to another state of excitation. For example, consider the RC circuit shown in Figure 14.1

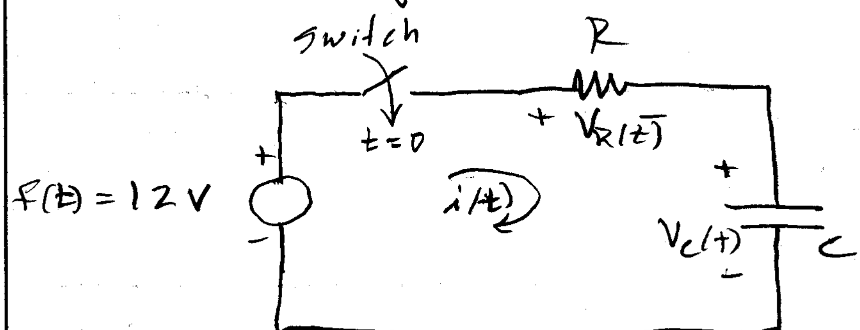


Figure 14.1: An RC circuit.

Initially the voltage across the capacitor is zero (we assume this). After the switch has been closed for a very long time, the voltage across the capacitor becomes 12V. We illustrate this in Figure 14.2

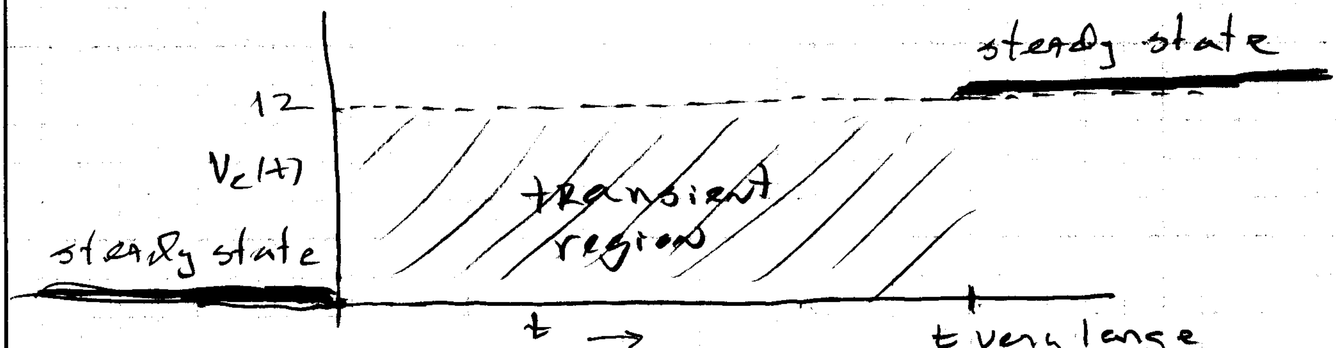


Figure Fig 14.2: Illustrating transient region.

We know $V_c(\infty) = 12$ volts because $i(t)$ through the capacitor is $i(t) = C\frac{dv}{dt}$ and we know $i(\infty) = 0$ because the capacitor acts as an open-circuit as $t \rightarrow \infty$. This means $i(\infty) = 0$ and all of the 12 V must be across the capacitor.

Again we refer to the circuit of Figure 14.1 but with the condition that $f(t) = 0$ and $V_c(0) = V_c$. The circuit for this case becomes as shown in Figure 14.3.

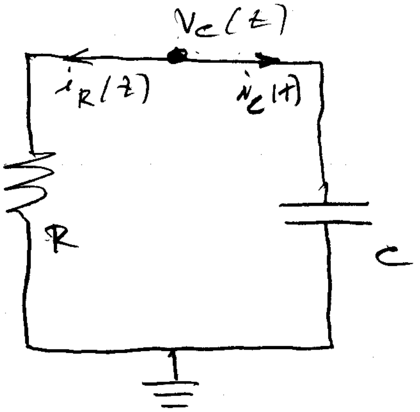


Figure 14.3: The source free RC circuit.

From this circuit we can write;

$$i_c + i_R = 0$$

We know

$$i_c(t) = C\frac{dV_c}{dt} \quad ; \quad i_R = \frac{V_c}{R}$$

so

$$C\frac{dV_c}{dt} + \frac{V_c}{R} = 0 \qquad \text{Eq } 14.1$$

or

$$\frac{dV_c}{dt} + \frac{V_c}{RC} = 0 \qquad \text{Eq } 14.2$$

From the standpoint of mathematics, this equation is called a first order, linear, time invariant d.e. Since we have no forcing function, we call the solution to this equation the

natural response → complimentary solution
→ transient solution
or transient response

There are several ways to solve this equation. We will use separation of variables here:

$$\frac{dV_c(t)}{V_c(t)} = -\frac{dt}{RC} \qquad \text{Eq } 14.3$$

Integrate Eq 14.3

$$\int_{t=0}^{t=t} \frac{dV_c(t)}{V_c(t)} = -\frac{1}{RC}\int_0^t dt \qquad \text{Eq } 14.4$$

$$\ln V_c(t) + K = -\frac{t}{RC}$$

$$\ln(V_c(t)) = -\frac{t}{RC} + K_1$$

$$V_c(t) = e^{-\frac{t}{R}+K_1} = e^{K_1} \cdot e^{-\frac{t}{RC}} = K e^{-\frac{t}{RC}}$$

We evaluate K by noting that $V_c(0) = V_c$

Therefore

$$V_c(t)\Big|_{t=0} = V_c = K e^{-\frac{t}{RC}}\Big|_{t=0} = K \qquad \text{Eq } 14.5$$

Thus;

$$V_c(t) = V_c e^{-\frac{t}{RC}} \qquad \text{Eq } 14.6$$

This is called the free response of the series RC circuit.

The product, RC, has a special significance. Consider the sketch in Figure 14.4

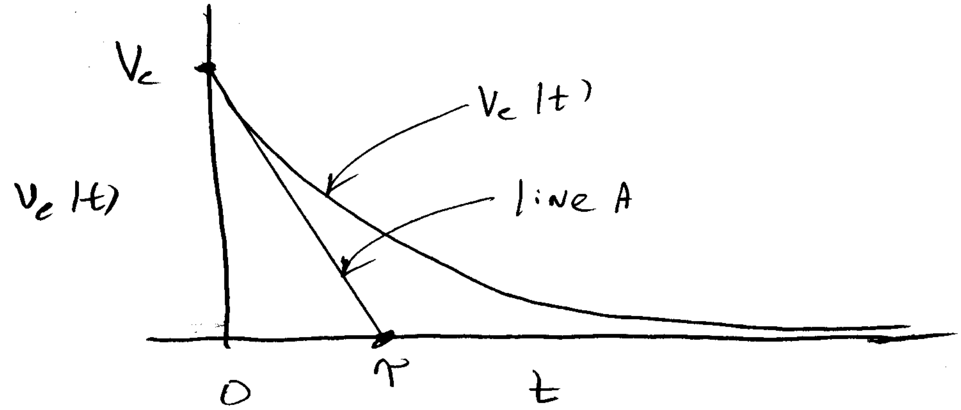


Figure 14.4: Unforced RC circuit response

Line A in the diagram represents a linear line of the form $y = mt + b$. We assume the slope of the line is defined by

$$m = \left.\frac{dV_c}{dt}\right|_{t=0} = -\frac{V_c}{RC}e^{-\frac{t}{RC}}\Bigg|_{t=0} = -\frac{V_c}{RC} \qquad \text{Eq } 14.7$$

Then

$$y = -\frac{tV_c}{RC} + m$$

We evaluate m by using the set $(x_0, y_0) = (0, V_c)$ so $m = V_c$. We have

$$y = -\frac{tV_c}{RC} + V_c \qquad \text{Eq } 14.8$$

We define the point (time) at which this linear line intersects the t axis as τ. We define τ as one time constant. We see from Eq. 14.8 that with $y = 0$,

$$0 = -\frac{t V_c}{RC} + V_c$$

OR

$$t = \boxed{\tau = RC} \qquad \text{Eq. 14.9}$$

Having define τ in this manner gives us a way to evaluate the state of the response in terms of τ.

Normally, one consider the circuit to be in steady state at

$$\boxed{t = 4\tau} \qquad \text{Eq 14.10}$$

Actually

$$V_c(4\tau) = .01832 V_c \qquad \text{Eq 14.11}$$

and we say that the response is within 2% of the final value.

The Forced RC Circuit

Now consider the RC circuit of Figure Figure 14.5.

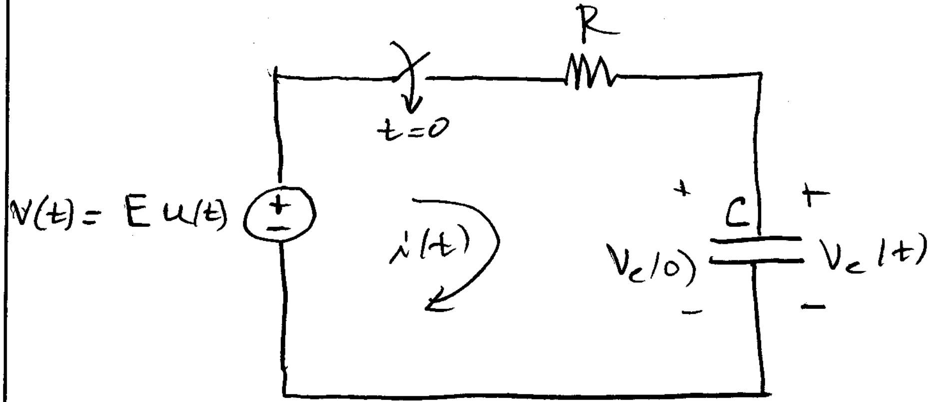


Figure 14.5; The forced series RC circuit.

V(t) is a step input, starting at t=0, of amplitude E. The differential equation of the circuit in terms of $V_c(t)$ is

$$RC\frac{dV_c(t)}{dt} + V_c(t) = E\,u(t) \qquad \text{Eq. } 14.12$$

or

$$\frac{dV_c(t)}{dt} + \frac{V_c(t)}{RC} = \frac{E}{RC}\,u(t) \qquad \text{Eq. } 14.13$$

We assume $V_c(t)$ express as

$$V_c(t) = V_{cN}(t) + V_{cp}(t) \qquad \text{Eq. } 14.14$$

V_{cN} = Natural (transient) response.

V_{cp} = particular (steady state) response.

We first find V_{ep} by noting that the forcing function is a constant. We assume

$$V_{ep} = K$$

and substitute this in Eq 14.13 to find

$$\frac{d(K)}{dt} + \frac{K}{RC} = \frac{E}{RC} \qquad \text{Eq } 14.15$$

$$K = E.$$

We use this in Eq. 14.14, later.

First we find V_{cN}, which is the solution to

$$\frac{dV_{cN}}{dt} + \frac{V_{cN}(t)}{RC} = 0 \qquad \text{Eq. } 14.16$$

We solved this problem earlier using separation of variables. We solve it a different way below.

Assume

$$V_{cN}(t) = K_N e^{st} \qquad \text{Eq } 14.17$$

There are some good reasons for assuming this form and they are explained in a first course in differential equations.

Substituting Eq 14.17 into Eq 14.16 gives

$$\frac{d[Ke^{st}]}{dt} + \frac{Ke^{st}}{RC} = 0$$

OR

$$sKe^{st} + \frac{Ke^{st}}{RC} = 0 \qquad \text{Eq } 14.18$$

giving;

$$s + \frac{1}{RC} = 0 \qquad \text{Eq } 14.19$$

OR

$$s = -\frac{1}{RC} \qquad \text{Eq } 14.20$$

Equation 14.19 is called the characteristic equation of the D.E. and Eq. 14.20 gives the characteristic root, which is often called the eigenvalue.

We are led to

$$V_{CN}(t) = Ke^{-\frac{t}{RC}} \qquad \text{Eq } 14.21$$

Using this in Eq. 14.14, along with the expression for V_{CP}, gives

$$V_C(t) = E + Ke^{-\frac{t}{RC}} \qquad \text{Eq } 14.22$$

It is at this point, not at Eq 14.21, that we evaluate K using initial conditions.

From the physics of the circuits we know that $V_c(\infty)$ (steady state) will give

$$V_c(\infty) = E \qquad \text{Eq } 14{,}23$$

and we know that at $t=0$

$$V_c(0) = \text{initial voltage on the capacitor} \qquad \text{Eq } 14{,}24$$

We can say then, from Eq 14,22,

$$V_c(0) = E + K e^{-\frac{t}{RC}} \Big|_{t=0}$$

or

$$V_c(0) = E + K$$

and

$$K = V_c(0) - E$$

then

$$V_c(t) = E + (V_c(0) - E) e^{-\frac{t}{RC}} \qquad \text{Eq } 14.23$$

We can also view the above, using $V_c(\infty) = E$, as

$$\boxed{V_c(t) = V_c(\infty) + [V_c(0) - V_c(\infty)] e^{-\frac{t}{RC}}} \qquad \text{Eq } 14.24$$

We view this as the general expression for $V_c(t)$, for any initial conditions.

If we had the differential equation,

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = \frac{f(t)}{\tau}$$

then

$$x(t) = x(\infty) + [x(\infty) - x(0)]\, e^{-\frac{t}{\tau}}$$

A special case of interest for Eq 14.24 is when $V_c(0) = 0$. We then have

$$V_c(t) = V_c(\infty)\left[1 - e^{-\frac{t}{RC}}\right] \qquad \text{Eq } 14.25$$

For $V_c(\infty) = E$ (the input a constant value)

$$V_c(t) = E\left(1 - e^{-\frac{t}{RC}}\right) \qquad \text{Eq } 14.26$$

A sketch of this response is shown in Figure 14.6.

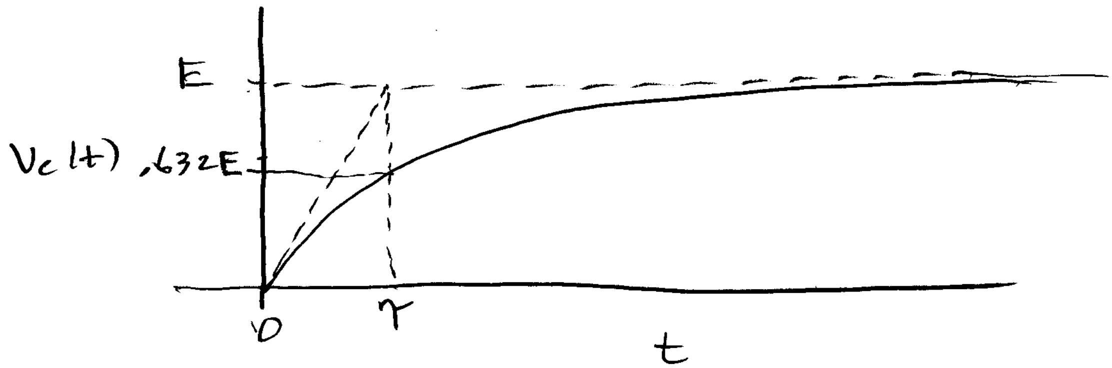


Figure 14.6: Response of a series RC circuit with zero I.C. and a step input of $v(t) = E u(t)$.

14.12

Figure 14.6 shows that a line with an intial slope of

$$\frac{dV_c(t)}{dt} = \frac{d}{dt}\left[E - Ee^{-\frac{t}{RC}}\right]$$

giving

$$\frac{dV_c(0)}{dt} = \frac{E}{RC},$$

intersects the line, $V_c(t) = E$ at τ. When this intersection points is projected down toward the time axis, it crosses $V_c(t)$ at

$$V_c(t) = V_c(\tau) = E\left(1 - e^{-\frac{t}{RC}}\right)\Big|_{t=\tau} \qquad \text{Eq } 14.26$$

or

$$V_c(\tau) = E\left(1 - e^{-1}\right) = 0.632\,E \qquad \text{Eq } 14.27$$

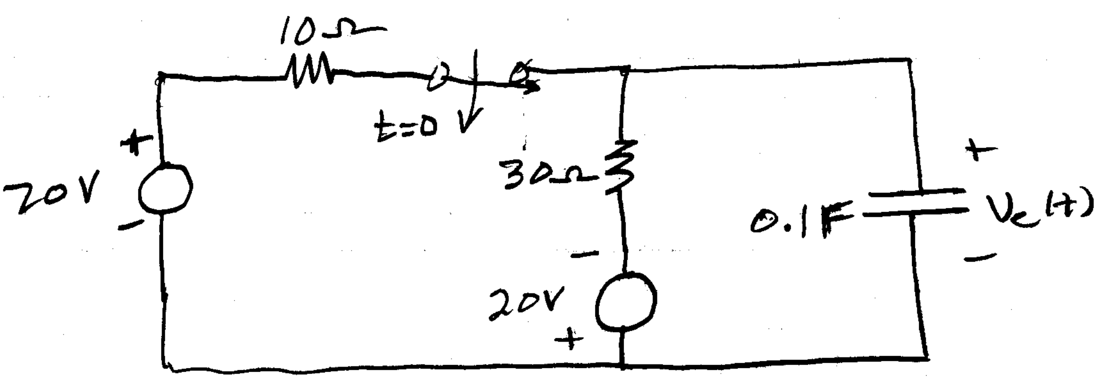
This is significant. If we are given, or expermentally determined, the step response capacitor voltage of a series RC circuit, we can determine the time constant, τ, by dropping down from the final value of $V_c(t)$ to $.632 \times$ final value, project horizontally until we strike the response curve; the project down to the time axis and we can determine τ as what we read on the time axis. Knowing τ, we know RC. If we know R we can determine C.

A fundamental thing to remember about a capacitor is that the voltage across it cannot change instantaneously. This means then when switching occurs we remember to use;

$$V_c(0^+) = V_c(0^-)$$

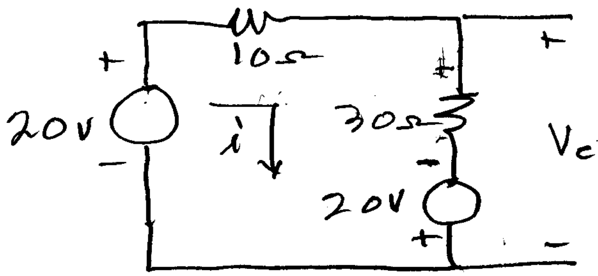
Example 14.1

You are given the circuit shown below. The switch has been closed for a very long time. Determine $V_c(t) \geq 0$.



Solution

The capacitor acts like an open circuit because it is fully charge if the switch has been closed for a very long time. Thus,

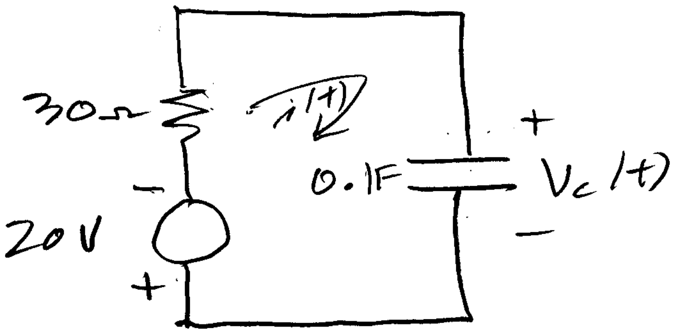


$$i = \frac{40}{40} = 1\ A$$

So

$$V_c(0^-) = -20 + 30 = 10\ V$$

After the switch is opened we can view the circuit as



We can write the D.E. as

$$30\,C\,\frac{dV_c}{dt} + V_c(t) = -20$$

OR

$$\frac{dv_c}{dt} + \frac{v_c}{3} = -\frac{20}{3}$$

$$V_{cp} = -20$$

$$V_{cN} = Ke^{-\frac{t}{3}}$$

$$v_c(t) = -20 + Ke^{-\frac{t}{3}}$$

We know $V_c(0^+) = 10\,V$, so

$$10 = -20 + Ke^{-\frac{t}{3}}\Big|_{t=0}$$

$$K = 30$$

$$\boxed{V_c(t) = -20 + 30e^{-\frac{t}{3}}}$$

We could have used Eq. 14,24 directly:

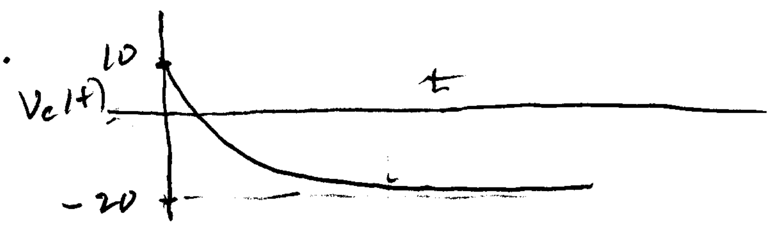
$$v_c(t) = V_c(\infty) + [V_c(0) - V_c(\infty)]e^{-\frac{t}{RC}}$$

and wrote

$$V_c(t) = -20 + [10-(-20)]e^{-\frac{t}{3}}$$

$$V_c(t) = -20 + 30e^{-\frac{t}{3}}\ V$$

You should be able to sketch $V_c(t)$ for $t \geq 0$.



The Series L-R Circuit

We consider first the unforced case.

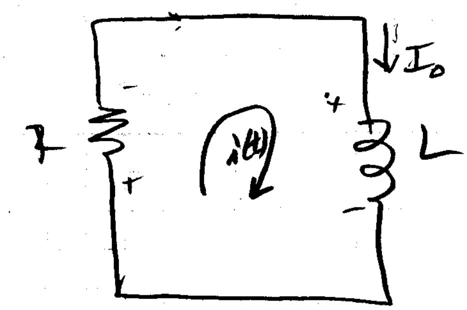


Figure 14.7: The unforced LR circuit.

From application of KVL we have

$$Ri(t) + L\frac{di(t)}{dt} = 0$$

OR

$$\frac{di}{dt} + \frac{i(t)}{L/R} = 0 \qquad \text{Eq. 14.28}$$

It is easy to show that

$$i(t) = I_0 e^{-\frac{t}{L/R}}$$

We define L/R as τ, the circuit time constant and write,

$$i(t) = I_0 e^{-\frac{t}{\tau}} \qquad \text{Eq. 14.29}$$

The thing to keep in mind with inductors is that the

- the current cannot change instantaneously
- in steady state the inductor looks like a short circuit.

Now consider the R-L circuit with a forcing function as shown in Figure 14.8.

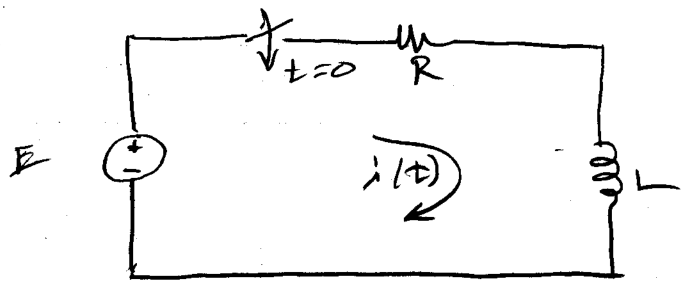


Figure 14.8: Forced R-L circuit.

We have,

$$L\frac{di(t)}{dt} + Ri(t) = E$$

OR

$$\frac{di}{dt} + \frac{i(t)}{L/R} = \frac{E}{L} \qquad \text{Eq } 14.30$$

We again have, similarly to the RC circuit;

$$i(t) = i_N(t) + i_p(t)$$

$$i_p(t) = K = \frac{E}{R}$$

$$i_N(t) = K_N e^{st} = K_N e^{-\frac{t}{L/R}}$$

$$i(t) = \frac{E}{R} + K_N e^{-\frac{t}{\tau}} \qquad \tau = \frac{L}{R}$$

We need to evaluate K_N.

We assume $i(0) = I_0$. Then

$$i(t)\Big|_{t=0} = \left[\frac{E}{R} + K_N e^{-\frac{t}{\tau}}\right]\Big|_{t=0}$$

$$i(0) = I_0 = \left[\frac{E}{R} + K_N e^{0}\right]$$

so

$$K_N = I_0 - \frac{E}{R}$$

then

$$\boxed{i(t) = \frac{E}{R} + \left[I_0 - \frac{E}{R}\right] e^{-\frac{t}{\tau}}} \quad \text{Eq 14.31}$$

Since

$$i(\infty) = \frac{E}{R}, \quad i(0) = I_0$$

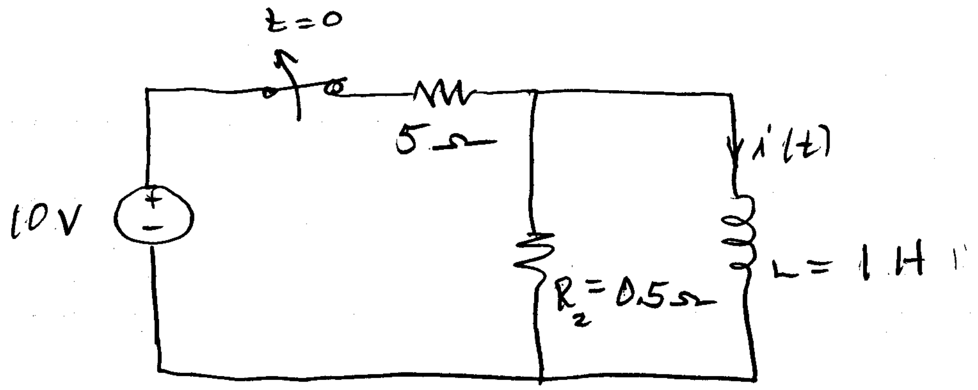
we have

$$\boxed{i(t) = i(\infty) + \left[i(0) - i(\infty)\right] e^{-\frac{t}{\tau}}} \quad \text{Eq 14.32}$$

We see the similarity between current in the R-L circuit (Eq 14.32) and voltage across the capacitor (Eq 14.24). The question might arise as to how we can find the voltage across the inductor for the R-L circuit or the capacitor current for the R-C circuit. We now turn our attention to this question.

Example 14.2

The switch in the circuit below has been closed long enough for the current $i(t)$ to be constant. At $t=0$, the switch is opened. Find $i(t) \geq 0$.

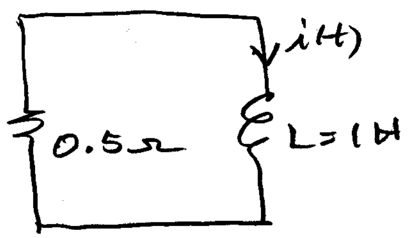


$i(t)$ for $t<0$; $i = \frac{10}{5} = 2A = i(0^-)$

So we know;

$$i(0^+) = i(0^-) = 2A$$

For $t>0$ the circuit becomes;



We know $i(t) = i(\infty) + [i(0) - i(\infty)]\, e^{-\frac{t}{\tau}}$

$i(\infty) = 0$, $i(0) = 2A$, $\tau = \frac{L}{R} = 2$ sec

$$i(t) = 2e^{-0.5t}$$