Lesson 15

Basic RLC Circuits
Emphasis On The Second Order System Transient Response

Notes for ECE 301

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ECE 301
LECTURE NOTES 15
RLC CIRCUITS

We now consider the RLC circuit. We will only take the cases of parallel RLC and series RLC for this course. Generally, it can be "tricky" to find various initial conditions for RLC circuits. We keep the following in mind.

The Inductor

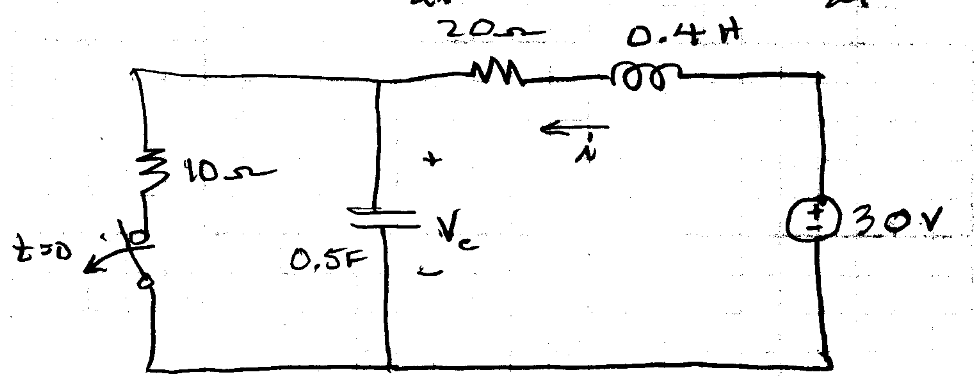
- $i_L(0^+) = i_L(0^-)$
- An inductor looks like a short circuit in steady state.

The capacitor

- $V_C(0^+) = V_C(0^-)$
- The capacitor looks like an open circuit in steady state.

Example 15.1

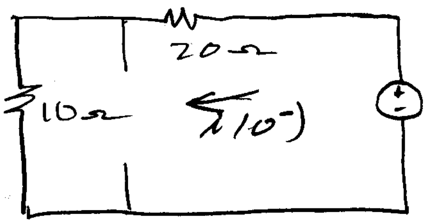
The switch in the circuit below has been closed for a very long time. Find: $i(0^-)$, $i(0^+)$, $i(\infty)$, $\frac{di(0^+)}{dt}$, $v(0^-)$, $v(0^+)$, $\frac{dv(0^+)}{dt}$, $v(\infty)$



You may wonder why we would want to determine all these conditions. The answer is, depending on the configuration of the RLC circuit, we generally need them.

For $t<0$

(inductor is a short), (capacitor is open ckt)

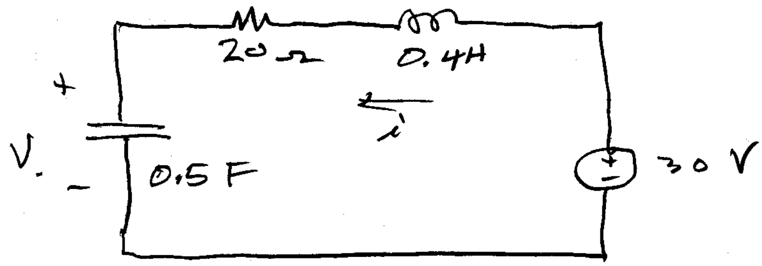


$$i(0^-) = i(0^+) = \frac{30}{30} = 1\,A$$

$$V_c(0^-) = V_c(0^+) = 1A \times 10\Omega = 10\,V$$

That wasn't too bad.

For $t>0$



$i(\infty) = 0$ (capacitor acts like an open ckt)

$v(\infty) = 30\,V$ (capacitor charges up)

We have

$$L\frac{di}{dt} + Ri(t) + V_c(t) = 30$$

$$\frac{di}{dt} + \frac{20}{.4}i + \frac{V_c(t)}{.4} = \frac{30}{.4}$$

$$\frac{di(t)}{dt} + 50i(t) + 2.5V_c(t) = 75$$

$$\frac{di(0^+)}{dt} = 75 - 50i(0^+) - 2.5V_c(0^+)$$

OR
$$\frac{di(0^+)}{dt} = 75 - 50 \times 1 - 2.5 \times 10 = 0$$

$$\frac{di(0^+)}{dt} = 0$$ (if you think about this, relative to the original ckt. you know it is correct)

Since
$$i_c(t) = C\frac{dV_c}{dt}$$
where
$$i_c(0^+) = i(0^-) = 1A$$

$$\frac{dV_c(0^+)}{dt} = \frac{1}{0.5} = \underline{2V} \qquad QED$$

So you see, you need to be on top of things in order to determine I.C.'s in general.

Now we turn our attention to the parallel RLC circuit, driven by a current source.

The Parallel RLC Circuit

We consider the circuit shown in Fig 15.1.

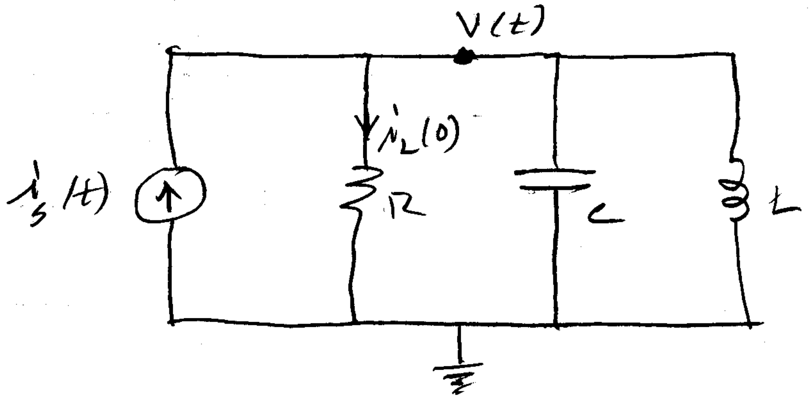


Figure 15.1; The parallel RLC circuit.

Using nodal analysis we have;

$$\frac{V(t)}{R} + C\frac{dV}{dt} + \frac{1}{L}\int_{x=0}^{x=t} V(x)\,dx + i_L(0) = i_s(t) \quad \text{Eq 15.1}$$

Taking the derivative of both sides and dividing by C gives

$$\frac{d^2V(t)}{dt^2} + \frac{1}{RC}\frac{dV(t)}{dt} + \frac{V(t)}{LC} = \frac{1}{C}\frac{di_s}{dt} \quad \text{Eq 15.2}$$

Before considering this equation further, we first consider the series RLC circuit.

The Series RLC Circuit

We consider the following circuit:

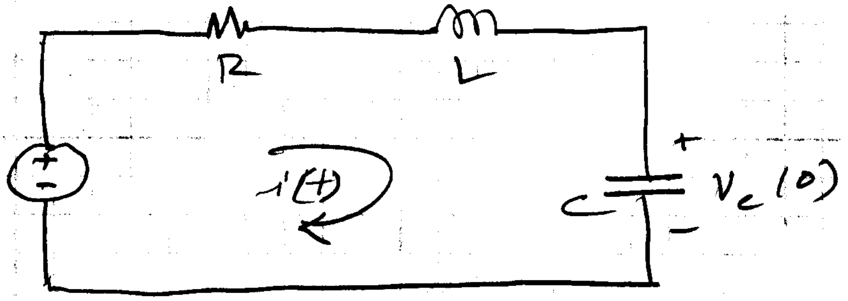


Figure 15.2: The series RLC circuit.

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{x=0}^{x=t} i(x)\,dx + V_c(0) = V_s(t) \qquad \text{Eq } 15.3$$

Taking the derivative of both sides and dividing by L gives,

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} = \frac{1}{L}\frac{dV_s(t)}{dt} \qquad \text{Eq } 15.4$$

Notice that the form of Equation 15.4 is the same as Equation 15.2, only the dependent variables are different, one v(t), the other, i(t).

Before working example problems, we consider a special way of representing a 2nd order differential equation.

The advantage of using the "special way" is that it carries over from one branch of engineering to another. Also, it is the standard way of representing 2nd order systems in the feedback control field.

The Canonical (standard) Form For 2nd Order LTI (linear, time-invariant) Differential Equation

In a general sense, a linear time-invariant differential equation can be expressed as;

$$\frac{d^2x(t)}{dt^2} + a_1 \frac{dx(t)}{dt} + a_2 x(t) = a_3 f(t) \quad \text{Eq 15.5}$$

In the standard form this becomes,

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = \omega_n^2 f(t) \quad \text{Eq 15.6}$$

The assumption has been made in Eq 15.6 that $a_2 = a_3 = \omega_n^2$. Generally, this may not be true. However, in order for $x(\infty)$ to go to 1 when $f(t)$ is a unit step input, it is necessary for $a_2 = a_3$. Otherwise, the unit step input will not produce a level of one in steady state for $x(t)$. This is no big deal but should be observed in passing.

If we take the Laplace transform of Eq 15.6, and form the transfer function, we have

$$\frac{X(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \qquad \text{Eq } 15.7$$

Perhaps it is appropriate to define transfer function at this point.

Definition:

A transfer function is defined as the ratio of the Laplace transform of the output (response) to the input (excitation) with all initial conditions equal to zero. Transfer functions are only defined for linear, time-invariant systems.

We will not be spending time with transfer functions in this class but one should have an appreciation on how they relate to differential equations.
We now return to the differential equation expressed in standard form.

This was given earlier in Eq 15.6, repeated here;

$$\frac{d^2x(t)}{dt^2} + 2\xi\omega_n\frac{dx(t)}{dt} + \omega_n^2 x(t) = \omega_n^2 f(t) \qquad \text{Eq. 15.6}$$

The characteristic equation is,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \qquad \text{Eq 15.8}$$

The roots of this equation are

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

which can be written as

$$s_1, s_2 = -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2} \qquad \text{Eq 15.9}$$

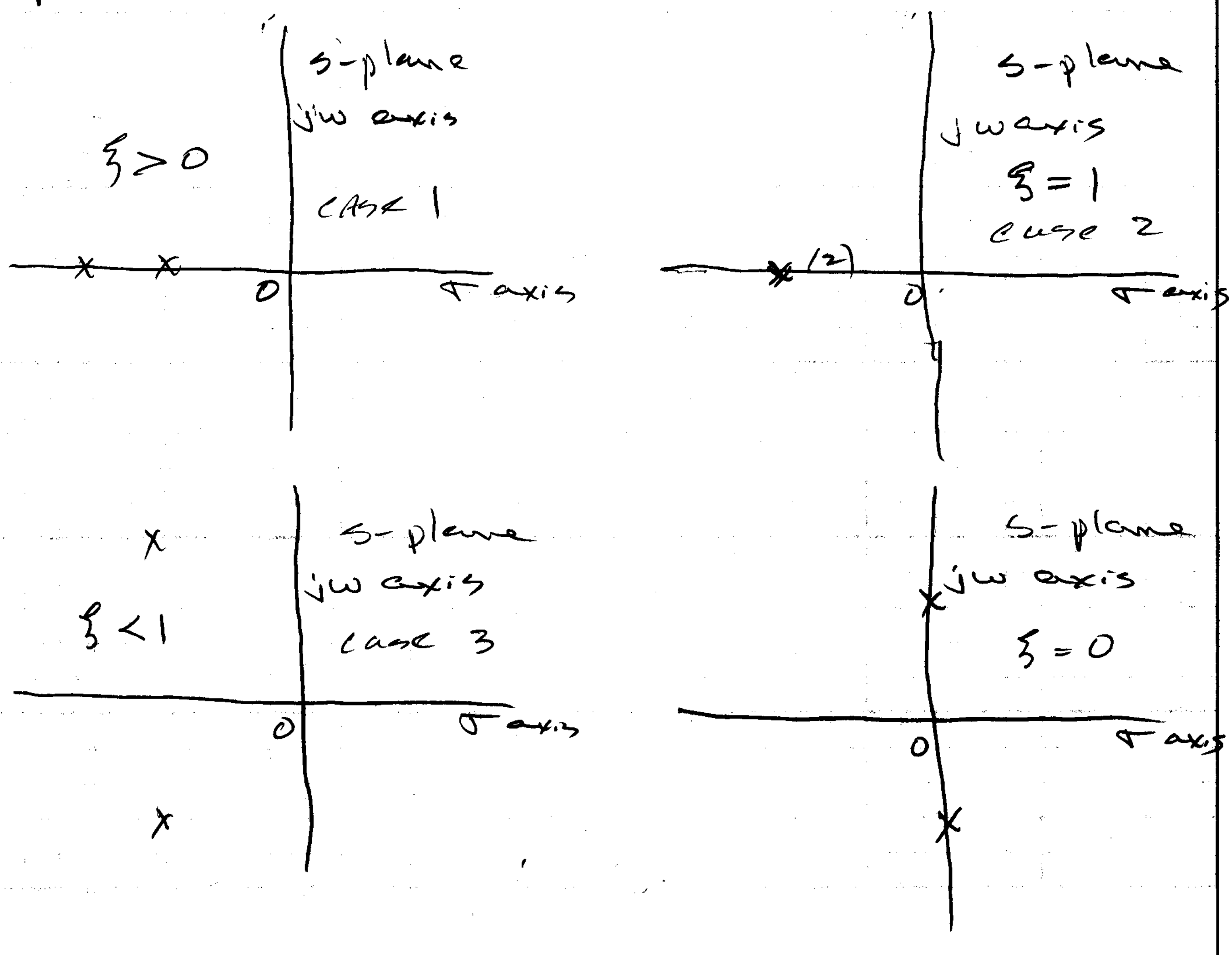
The roots s_1, s_2 fall into three cases:

Case 1: $\xi > 1$ overdamped
roots are real and unequal

Case 2: $\xi = 1$ critically damped
roots are repeated

Case 3: $\xi < 1$ underdamped

One might add a forth case with $\xi = 0$. In this case the roots are imaginary, no real part. They are said to be on the $j\omega$ axis in the s-plane. The above can be described graphically as follows:



The "x" in the previous diagrams signifies a root of $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. These roots come from the denominator of the transfer function given in Eq 15.7 and they are called poles. It is customary to signify poles in the s-plane using "X".

For case 1, if the forcing function, $f(t)$, in Eq 15.6 is a unit step, the response is generally of the shape in Figure 15.3 below.

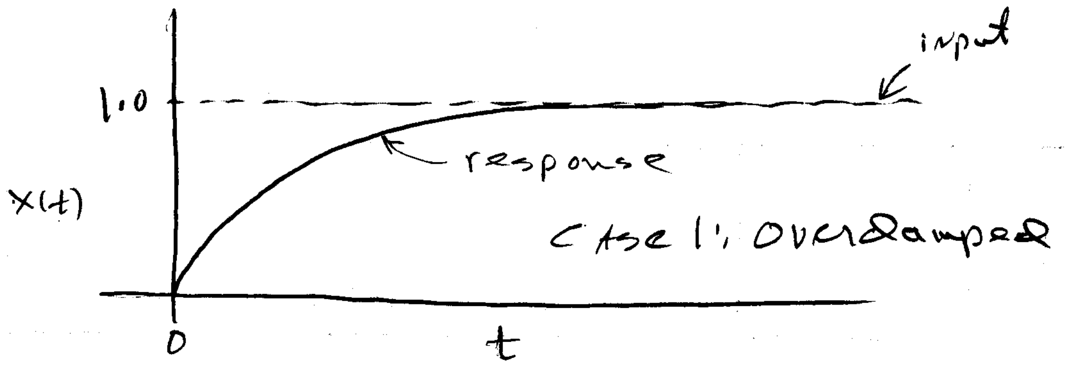


Figure 15.3: Step response of an overdamped circuit (system).

Actual time values along the time axis will depend on ω_n.

The response for the case when $\xi = 1$ does not appear to be greatly different from that of $\xi > 1$. However, in this case the responses is the fastest response we can have without having overshoot. We contrast this with $\xi > 1$ in the sketch of Figure 15.4.

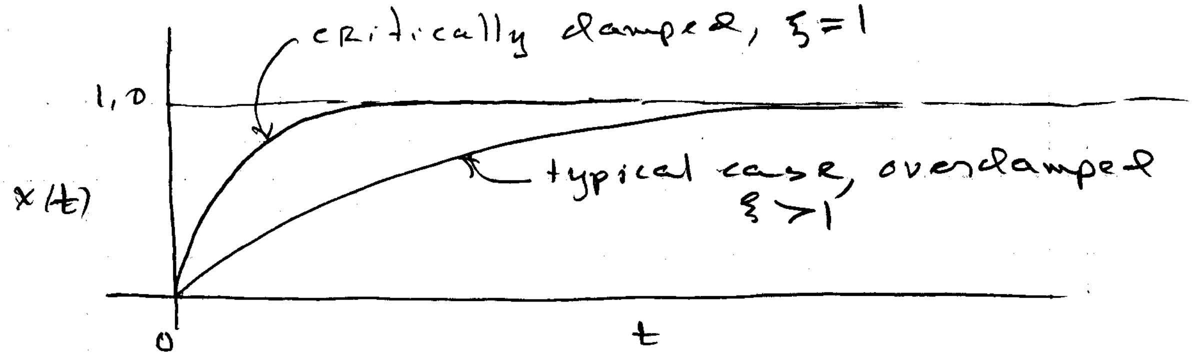


Figure 15.4: Illustrating Critical Damping.

Figure 15.5 shows the case for $\xi = 0.5$ ($\xi < 1$, underdamped)

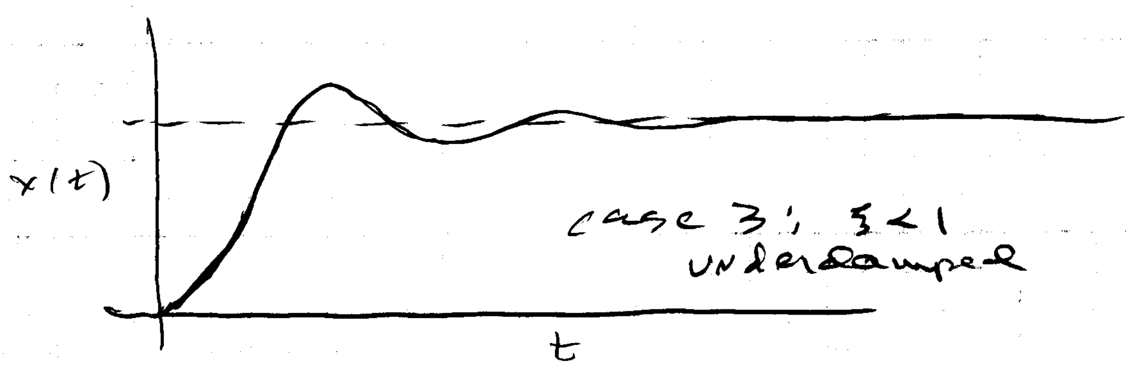


Figure 15.5: A typical response of a 2nd order system, $\xi < 1$, underdamped.

It is instructive, to at least once, to go through the mathematics for the case when $\xi < 1$. We have

$$s_1, s_2 = -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

which can be written as

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \qquad \text{Eq } 15.10$$

We often write this as

$$s_1, s_2 = -\xi\omega_n \pm j\omega_d \qquad \text{Eq } 15.11$$

$\underline{\xi}$ = damping coefficient

$\underline{\omega_n}$ = undamped natural resonant frequency

$\underline{\omega_d}$ = damped natural resonant frequency

$\underline{\omega_n}$ is the frequency at which the response resonants, oscillates, "rings" when we have no damping ($\xi=0$)

ω_d is the frequency at which the response resonants, "rings" when $0 < \xi < 1$.

15.13

If the forcing function to Eq 15.6 is a step of value E, and $\xi < 1$, underdamped, $x(t)$ becomes

$$x(t) = E + Ae^{(-\xi\omega_n + j\omega_d)t} + Be^{-(\xi\omega_n - j\omega_d)t} \quad \text{Eq 15.12}$$

We can get the above into a mix of sine, cosine, cosine alone, or sine alone. We consider a mix of sine, cosine first:

Using Euler's Identity

$$e^{jx} = \cos x + j\sin x$$

$$x(t) = E + Ae^{-\xi\omega_n t}[\cos\omega_d t + j\sin\omega_d t] + Be^{-\xi\omega_n t}[\cos\omega_d t - j\sin\omega_d t]$$

We can write this as;

$$x(t) = E + K_1 e^{-\xi\omega_n t}\cos\omega_d t + K_2 e^{-\xi\omega_n t}\sin\omega_d t \quad \text{Eq 15.13}$$

We evaluate K_1 & K_2 using initial conditions of the problem.

Before presenting an example let us express Eq 15.12 in a sine only form. The term will include an angle as we shall see.

Return to Equation 15.13 and rewrite as follows:

$$x(t) = E + \left(\sqrt{K_1^2+K_2^2}\right) e^{-\zeta \omega_n t} \left[\frac{K_1}{\sqrt{K_1^2+K_2^2}} \cos \omega_d t + \frac{K_2}{\sqrt{K_1^2+K_2^2}} \sin \omega_d t\right]$$

Eq. 15.14

We want the term in the brackets of Eq 15.14 to be of the form $\sin(\omega_d t + \Theta)$. To get it in this form we resort to trig identities. In particular, we have:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad \text{Eq 15.15}$$

In Eq 15.14 we want

$$\frac{K_1}{\sqrt{K_1^2+K_2^2}} = \sin A = \sin \Theta \qquad \text{Eq 15.16}$$

and

$$\frac{K_2}{\sqrt{K_1^2+K_2^2}} = \cos A = \cos \Theta \qquad \text{Eq 15.17}$$

Consider the following triangle:

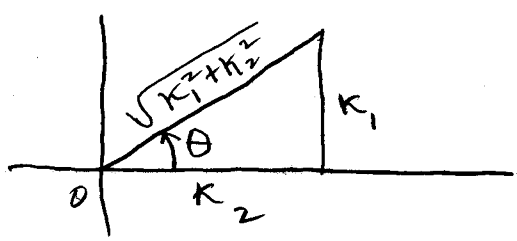


Figure 15.6: Triangle relationships.

This triangle defines the results we want. We can write Eq 15.14 as

$$x(t) = E + (K_1^2 + K_2^2)\, e^{-\xi \omega_n t}\left[\sin\theta \cos\omega_d t + \cos\theta \sin\omega_d t\right]$$

OR

$$x(t) = E + \sqrt{(K_1^2 + K_2^2)}\, e^{-\xi \omega_n t}\left[\sin(\omega_d t + \theta)\right] \quad \text{Eq 15.18}$$

K_1 and K_2 are determined using initial conditions associated with the 2nd order differential equation.

Suppose that we say that for this case, all initial conditions are zero; $E = 1$.
In particular, that $x(0) = 0$, $\frac{dx(0)}{dt} = 0$

From this we find;

$$K_1 = -1, \quad K_2 = \frac{-\xi}{\sqrt{1-\xi^2}} \quad \text{Eq 15.19}$$

Now ξ cannot be negative for a stable system. This means that both K_1 and K_2 are negative numbers. The triangle shown in Figure 15.6 cannot be in the first quadrant. It must be in the third quadrant as shown in Figure 15.7.