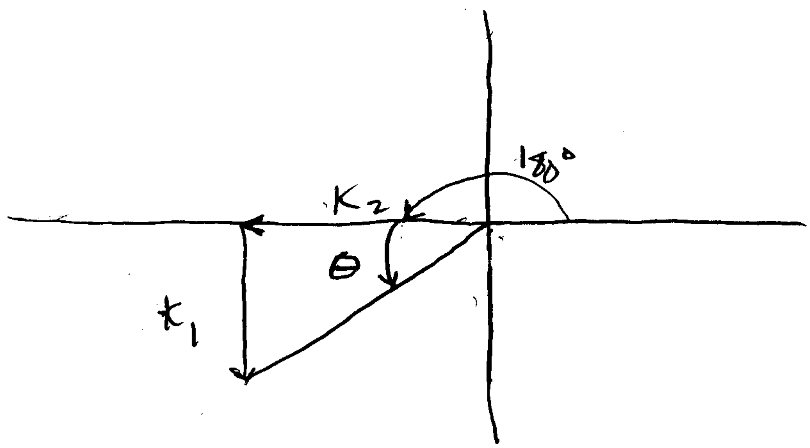


Figure 15.7: Correct triangle relationship for K_1 and K_2

Therefore, the sin portion of Eq 15.18 must be $\sin(\omega_d t + 180 + \theta)$ which is equal to $-\sin(\omega_d t + \theta)$.

With K_1 and K_2 having the values shown in Eq 15.19, we see

$$\sqrt{K_1^2 + K_2^2} = \frac{1}{\sqrt{1-\xi^2}} \qquad \text{Eq } 15.20$$

So Eq 15.18 becomes,

$$x(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_n\sqrt{1-\xi^2}\,t + \theta\right) \qquad \text{Eq } 15.21$$

We see that $x(0) = 0$, as it should, and we see that $x(\infty) = 1$, as it should. The θ in Eq 15.21 can be written

$$\cos\theta = \xi \;, \qquad \tan\theta = \frac{\sqrt{1-\xi^2}}{\xi} \qquad \text{Eq } 15.22$$

To verify Eq 15.21 we select $\xi = 0.5$, which we see later will give ≈16% overshoot for a step input. We select $\omega_n = 2$ rad/sec. Then

$$x(t) = 1 - 1.1547 e^{-t} \sin(1.732t + 60^\circ) \quad \text{Eq } 15.23$$

A MATLAB program for testing the response is shown below:

C:\MATLAB6p1\work\second_order.m
November 6, 2002

```
% A MATLAB program that solves and plots an underdamped
% 2nd order system response.  The program is second_order.m
% Written November 6, 2002;  Office Computer, wlgreen


t = 0:.01:8;
v = 1-1.1547*exp(-t).*sin(1.732*t+(pi/3));
plot(t,v)
grid
ylabel('v (output)')
xlabel('t (sec)')
title(' Second Order Sysem Step Response')
```

Figure 15.8: MATLAB program for 2nd order system step response.

The response from the program is shown in Figure 15.9.

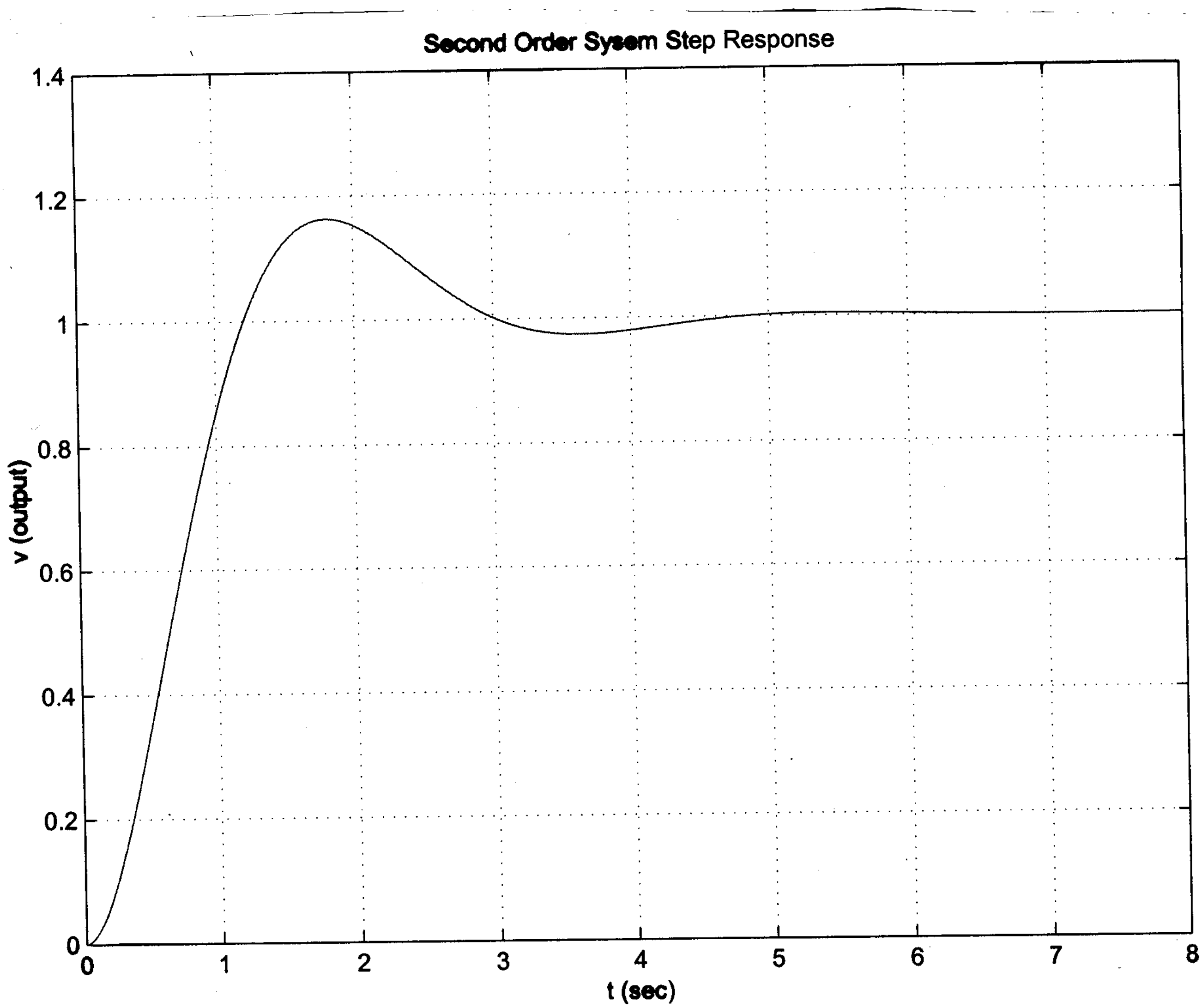


Figure 15.9: Step response of a second order system: $\zeta = 0.5$, $\omega_n = 2$.

Above response for;

$$v(t) = 1 - 1.1547 e^{-t} \sin(1.732t + 60^\circ)$$

For any given 2nd order differential equation we can find ξ and ω_n.

The general nature of the response of Eq 15.21 is shown in Fig 15.5. We want to know the time to the first peak of this response. If we take

$$\frac{dx(t)}{dt} = 0$$

and solve for t_p we find

$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}} \qquad \text{Eq } 15.24$$

We can also find that the amount of overshoot is

$$\boxed{O.S. = (100)\, e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}\,\%} \qquad \text{Eq } 15.25$$

Eq 15.24 and 15.25 are extremely important in systems. The pretty much characterize the step response of a system.

The previous notes can be considered as general background for 2nd order circuits. We now turn to some examples. We start with cases in which the circuit is overdamped. Then we graduate to critical damped and finally, underdamped.

Example 15.2

You are given the circuit shown in Figure 15.10.

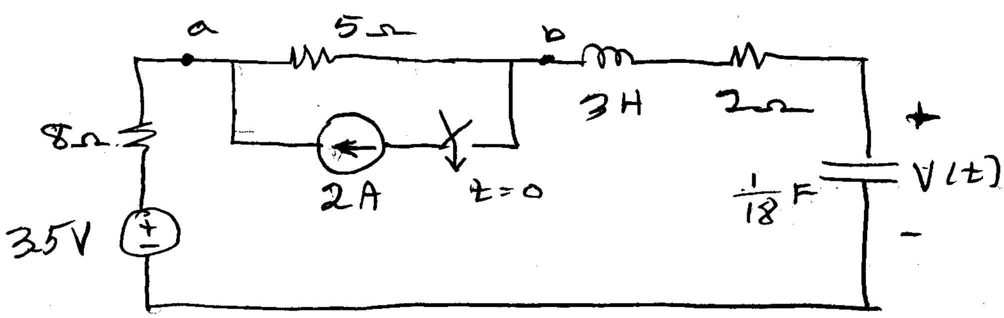


Figure 15.10: Circuit for Example 15.2

For t<0

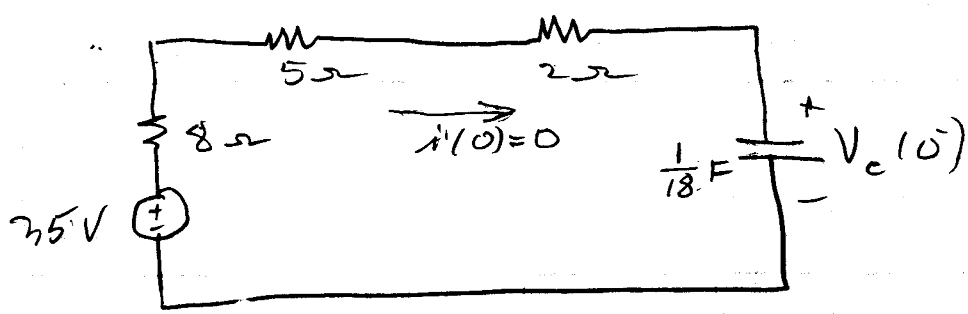


Figure 15.11: Circuit of Example 15.2, t<0.

We see that;

$$i(0) = 0, \quad V(0^-) = 35\text{ V} = V(0^+)$$

$t > 0$

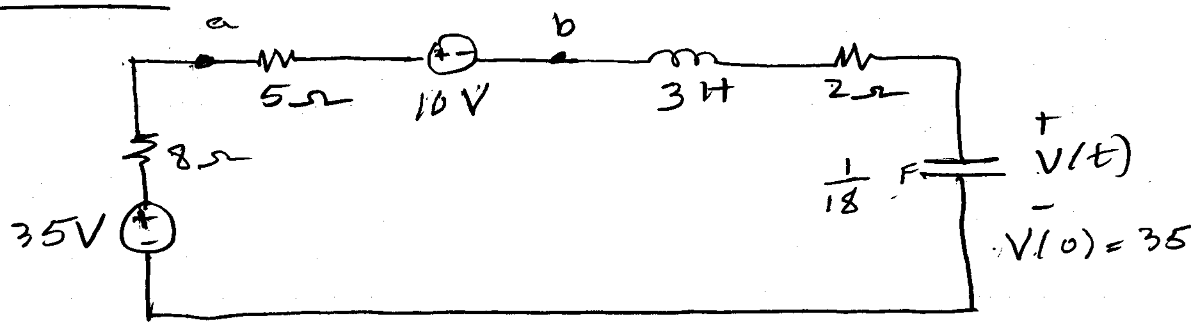


Figure 15.12: Circuit of Example 15.2, $t > 0$.

The above circuit can be simplified as shown in Figure 15.13.

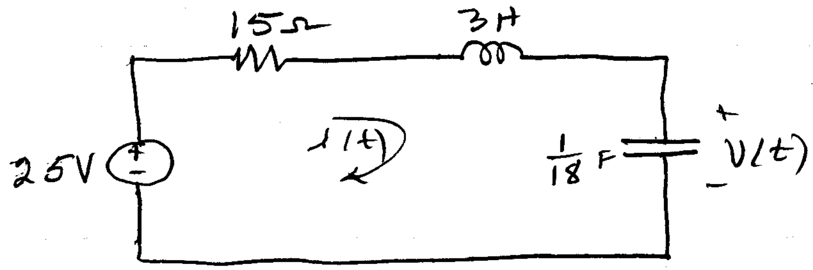


Figure 15.13: Circuit in simplified form for Example 15.2.

We write the following equations:

$$15\,i(t) + 3\frac{di}{dt} + v(t) = 25 \qquad \text{Eq 15.26}$$

$$i(t) = \frac{1}{18}\frac{dv}{dt} \qquad \text{Eq 15.26A}$$

or

$$\frac{15}{18}\frac{dv}{dt} + \frac{3}{18}\frac{d^2v}{dt} + v(t) = 25 \qquad \text{Eq 15.27}$$

$$\frac{d^2 V(t)}{dt^2} + 5\frac{dV(t)}{dt} + 6V(t) = 150 \qquad \text{Eq } 15.28$$

The particular solution is

$$V_p(t) = 25$$

For the transient solution, we must the roots from the characteristic equation:

$$s^2 + 5s + 6 = (s+2)(s+3) = 0$$

Thus

$$V_t(t) = Ae^{-2t} + Be^{-3t} \qquad \text{Eq } 15.29$$

Then

$$V(t) = 25 + Ae^{-2t} + Be^{-3t} \qquad \text{Eq } 15.30$$

We now evaluate A & B using initial conditions. We know that

$$i(0^-) = i(0^+) = 0$$

$$V(0^-) = 30V = V(0^+) \qquad \text{Eq } 15.31$$

We need $V(0^+)$, $\frac{dV(0^+)}{dt}$ to use in Equation 15.27 to solve for A & B.

From Eq 15.23A we have

$$\frac{dV(0)}{dt} = 18\,i(0) = 0 \qquad \text{Eq } 15.32$$

We are now in a position to solve for A & B in Eq 15.29.

We have

$$v(0) = 35 = 25 + Ae^{-2\times 0} + Be^{-3\times 0}$$

or

$$\boxed{A + B = 10} \qquad \text{Eq 15.33}$$

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t}$$

or

$$\frac{dv(0)}{dt} = 0 = -2A - 3B \qquad \text{Eq 15.34}$$

so

$$A + B = 10$$
$$2A + 3B = 0$$

which gives;

$$A = 30, \quad B = -20$$

Then

$$\boxed{v(t) = 25 + 30e^{-2t} - 20e^{-3t}} \qquad \text{Eq 15.35}$$

Example 15.3

The following problem is taken from "Basic Engineering Circuit Analysis", J. David Irwin, John Wiley & Sons, pp 226.

The switch in the network of Figure 15.14 has been in position 1 for a very long time. At $t=0$, the switch is changed to position 2. Find $V_o(t)$ for $t \geq 0$.

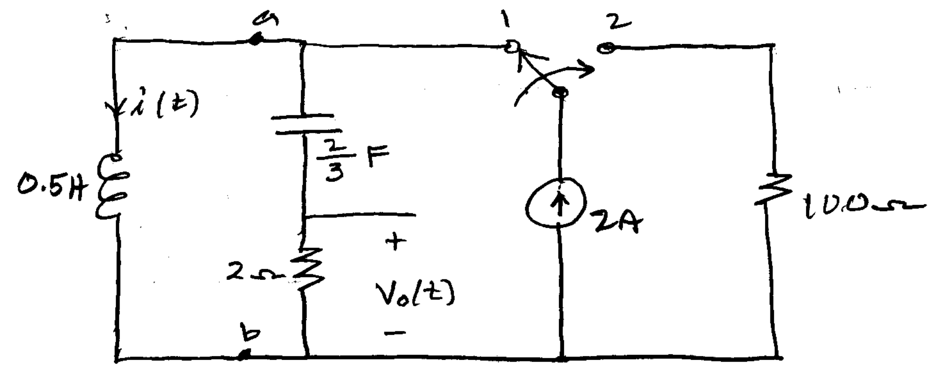


Figure 15.14: Circuit for Example 15.3.

For $t<0$, we note $\boxed{i(0) = 2 \text{ Amps.}}$

Since there is no resistance in the branch with the inductor, $V_{ab}=0$. Also, the voltage across the capacitor is zero since no current is flowing through the 2Ω resistor, hence $V_o(0)=0$ and since $V_{ab}(0)=0$, we know the capacitor voltage at $t=0$ is zero.

For $t>0$

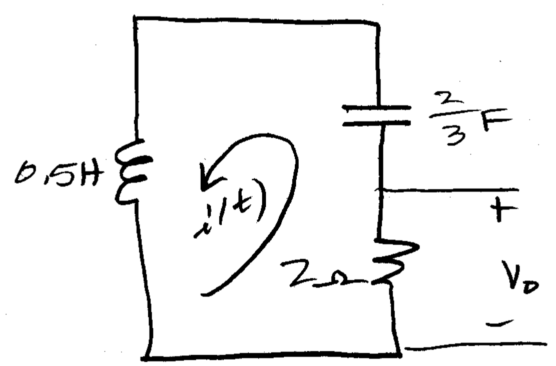


Figure 15.15: Circuit for example 15.3 for $t\geq 0$.

We can write;

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(t)\,dt = 0 \qquad \text{Eq } 15.36$$

for the circuit of Figure 15.15.

Taking the derivative of this equation gives

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C} = 0$$

or

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{i(t)}{LC} = 0 \qquad \text{Eq } 15.37$$

Substituting in numerical values gives

$$\frac{d^2 i(t)}{dt^2} + 4\frac{di(t)}{dt} + 3i(t) = 0$$

The characteristic equation is;

$$s^2 + 4s + 3 = (s+3)(s+1) = 0$$

Then

$$i(t) = Ae^{-3t} + Be^{-t} \qquad \text{Eq } 15.38$$

We need

$$i(0),\ \frac{di(0)}{dt}$$

for initial conditions. We have $i(0) = 2A$. From Eq 15.33 we have

$$\frac{di(0)}{dt} = -\frac{R}{L}i(0) - \cancelto{0}{\frac{1}{RC}\int_0^0 i(0)\,dt}$$

Eq 15.39

so,

$$\frac{di(0)}{dt} = -4i(0) = -8$$

Then from Eq 15.38,

$$i(0) = 2 = Ae^{-3\times 0} + Be^{-0} = A+B$$

$$\boxed{A+B = 2} \qquad \text{Eq } 15.40$$

and

$$\frac{di(t)}{dt} = -3Ae^{-3t} - Be^{-t}$$

$$\frac{di(0)}{dt} = -8 = -3A - B$$

$$\boxed{3A + B = 8} \qquad \text{Eq } 15.41$$

Solving Eq 15.40 and 15.41 gives

$$A = 3, \; B = -1$$

$$i(t) = 3e^{-3t} - e^{-t} \; A \qquad \text{Eq } 15.42$$

Now

$$V_o(t) = -2i(t)$$

$$\boxed{V_o(t) = 2(e^{-t} - 3e^{-3t})} \qquad \text{Eq } 15.43$$

Example 15.4

You are given the following circuit.

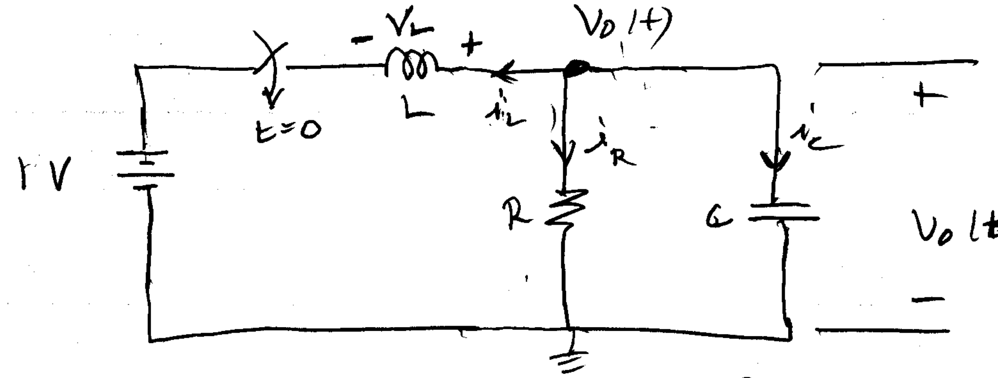


Figure 15.16: Circuit for Example 15.4.

All initial conditions are zero.

$R = 5\,\Omega$, $C = 0.04\,F$, $L = 1\,H$.

We will use nodal analysis to set-up the differential equation. We have

$$i_R + i_C + i_L = 0$$

OR

$$\frac{V_o(t)}{R} + C\frac{dV_o(t)}{dt} + i_L(t) = 0 \qquad \text{Eq } 15.44$$

Since $V_L(t) = L\dfrac{di_L}{dt}$

$$i_L(t) = \frac{1}{L}\int_0^t V_L(t)\,dt + i_L(0) \qquad \text{Eq } 15.45$$

15.28

We note by KVL that

$$V_o(t) - V_L(t) - 1 = 0$$

OR

$$V_L(t) = V_o(t) - 1 \qquad \text{Eq } 15.46$$

with $i_L(0) = 0$, we substitute the above into Eq 15.42 to find

$$i_L(t) = \frac{1}{L}\int_0^t (V_o(t) - 1)\,dt \qquad \text{Eq } 15.47$$

Now substitute Eq 15.47 into Eq 15.44. This gives

$$\frac{V_o(t)}{R} + C\frac{dV_o(t)}{dt} + \frac{1}{L}\int_0^t (V_o(t) - 1)\,dt = 0$$

We take the derivative of the above with respect to t:

$$\frac{1}{R}\frac{dV_o(t)}{dt} + C\frac{d^2V_o(t)}{dt^2} + \frac{(V_o(t) - 1)}{L} = 0$$

OR

$$\frac{d^2V_o(t)}{dt^2} + \frac{1}{RC}\frac{dV_o}{dt} + \frac{V_o(t)}{LC} = \frac{1}{LC}$$

Eq 15.48

We now put in numerical values to find the 2nd order diff. equation.

$$\frac{d^2 V_o(t)}{dt^2} + 5\frac{dV_o}{dt} + 25V_o(t) = 25 \qquad \text{Eq } 15.49$$

We compare this to Eq 15.6 formatted for $V_o(t)$ rather than $x(t)$ and we have

$$\frac{d^2 V_o(t)}{dt} + 2\xi\omega_n\frac{dV_o}{dt} + \omega_n^2 V_o(t) = \omega_n^2 u(t)$$

Eq 15.50

We have already solved the above equation for zero initial conditions. This was Eq 15.21 (p 15.16) which is repeated below:

$$V_o(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_n\sqrt{1-\xi^2}\,t + \theta\right)$$

" Eq 15.51

Comparing Equation 15.49 and 15.51 gives

$\omega_n = 5$ rad/sec $\quad \xi = 0.5$

Substituting into Eq 15.51 gives

$$\boxed{V_o(t) = 1 - 1.16e^{-2.5t}\sin(4.33t + 60^\circ)}$$

Eq 15.52

If we want to sketch this waveform we recall that from Eq 15,24 (p 15.19)

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{Eq 15,24}$$

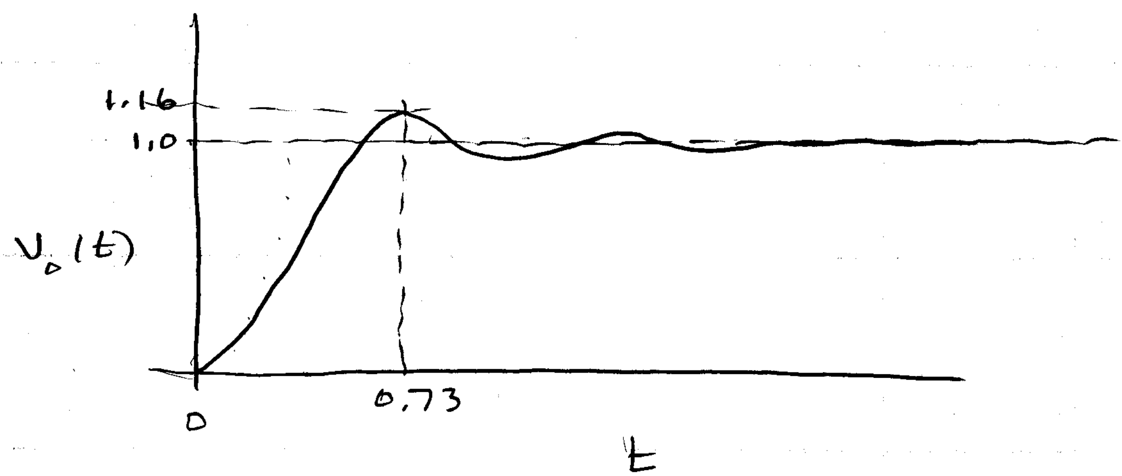
and

$$O.S. = 100\, e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\,\% \qquad \text{Eq 15,25}$$

Using our value of ζ and ω_n we have

$$t_p = 0.73 \text{ sec} \qquad O.S. = 16.3\%$$

So our response can now be sketched.



One can easily use MATLAB to verify this response.