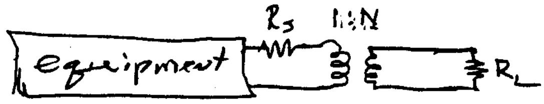
LESSON 17

MAGNETICALLY COUPLED CIRCUITS
(THE TRANSFORMER)

Lecture 17
Transformers

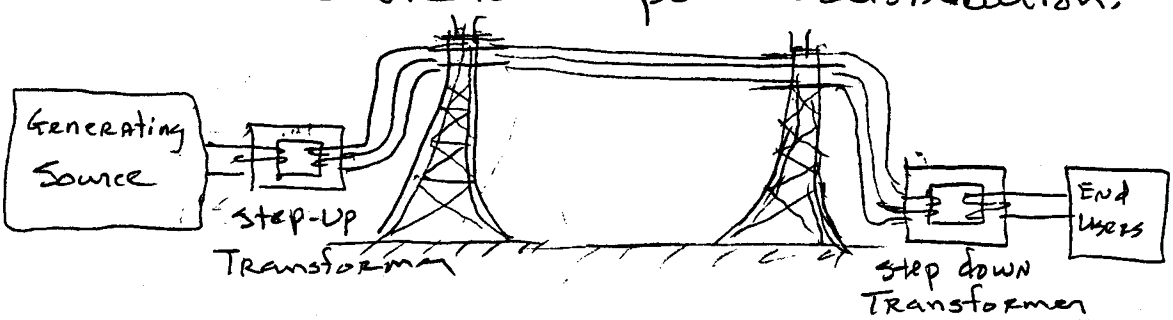
In my assessment, the three most important uses of transformers are:

- isolating one circuit from another in terms of dc signal level and certain cases of noise.
- Impedance matching.



Properly selecting (by design or off the shelf purchase) N can match R_L to R_s for maximum power transfer.

- Changing levels of voltage and current in power distribution.



The presentation here is very introductor. Basically, when a time varying signal is applied to a coil of wire that has another coil of wire nearby, two magnetic flux are set up by the first coil and two flux by the second coil. This is illustrated in Figure 17.1.

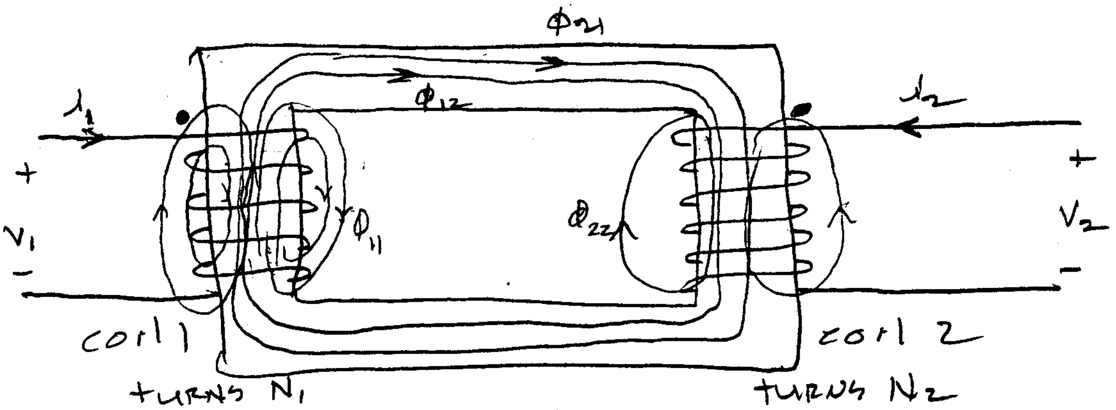


Figure 17.1: Illustrating Flux in a transformer.

From physics and our forefathers we know

$$\Phi_1 = \Phi_{11} + \Phi_{12} \qquad \text{Eq } 17.1$$

and

$$\Phi_2 = \Phi_{22} + \Phi_{21} \qquad \text{Eq } 17.2$$

Φ_1 is the magnetic flux associated with coil one due to current $i_1(t)$.

Φ_2 is the magnetic flux associated with coil two due to current $i_2(t)$.

Reviewing the property of the inductor as shown in Figure 17.2,

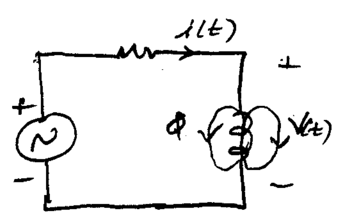


Figure 17.2: The basic inductor.

We recall;

$$v(t) = N\frac{d\Phi}{dt} \qquad \text{Eq } 17.3$$

where N is the number of turns of wire of the inductor and $\Phi(t)$ is the magnetic flux linking the coil, due to current $i(t)$.

We can further write;

$$v(t) = N\frac{d\Phi}{di} \times \frac{di}{dt} \qquad \text{Eq } 17.4$$

This is justified since Φ changes with time because of the time varying current. We lump $N\frac{d\Phi}{di}$ into a term we define as self inductance and we have

$$v(t) = L\frac{di}{dt} \qquad \text{Eq } 17.5$$

which is the familiar equation for the voltage across an inductor.

Referring back to Figure 17.1, assume coil 2 is open ($i_2=0$). The current i_1 causes a flux Φ_1 which has two components. $\Phi_1 = \Phi_{11} + \Phi_{12}$. Φ_{11} links only coil 1 while Φ_{12} links both coil 1 and coil 2.

We have

$$V_1 = N_1\frac{d\Phi_1}{dt} = N_1\frac{d\Phi_1}{di_1}\frac{di_1}{dt} = L_1\frac{di_1}{dt} \qquad \text{Eq } 17.6$$

and also

$$V_2 = N_2 \frac{d\phi_{12}}{di_1} \frac{di_1}{dt} = M_{21} \frac{di_1}{dt} \qquad \text{Eq } 17.7$$

M_{21} is called the mutual inductance with subscript meaning the voltage induced in coil 2 due to current i_1 in coil 1.

Now suppose that coil 1 is opened ($i_1 = 0$). We have

$$V_1 = N_1 \frac{d\phi_{21}}{di_2} \frac{di_2}{dt} = M_{12} \frac{di_2}{dt} \qquad \text{Eq } 17.8$$

and

$$V_2 = N_2 \frac{d\phi_2}{di_2} \frac{di_2}{dt} = L_2 \frac{di_2}{dt} \qquad \text{Eq } 17.9$$

We are assuming a linear system and we more or less have applied superposition above. Now if both coil are closed we can write

$$V_1 = L_1 \frac{di_1}{dt} + M_{12} \frac{di_2}{dt} \qquad \text{Eq } 17.10$$

$$V_2 = L_2 \frac{di_2}{dt} + M_{21} \frac{di_1}{dt} \qquad \text{Eq } 17.11$$

It is shown in most basic circuit books that $M_{12} = M_{21} = M$.

The above equation become

$$v_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \qquad \text{Eq } 17.12$$

$$v_2(t) = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \qquad \text{Eq } 17.13$$

We use a dot marking to tell us the polarity of the voltage induced from one coil to the other.

Consider the following with respect to mutually induced voltages.

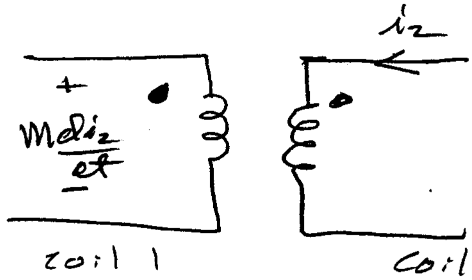


Figure 17.3a

The current i_2 entering the dot of coil 2 will cause the induced voltage, $M\frac{di_2}{dt}$, in coil 1, to be positive at the dot of coil 1.

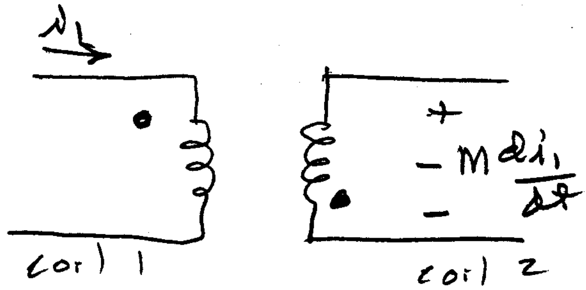


Fig 17.3b

Lower dot in coil 2 is positive because i_1 is entering the dot of coil 1. Hence the voltage is $-M\frac{di_1}{dt}$.

at this point we are not interested in solving the differential equations of Eq 17.12 and 17.13. We are interested in the AC steady state. So earlier equations become

$$\hat{V}_1 = j\omega L_1 \hat{I}_1 + j\omega M \hat{I}_2 \qquad \text{Eq 17.14}$$

$$\hat{V}_2 = j\omega L_2 \hat{I}_2 + j\omega M \hat{I}_1 \qquad \text{Eq 17.15}$$

These equations would pertain to the situation shown in Figure 17.4

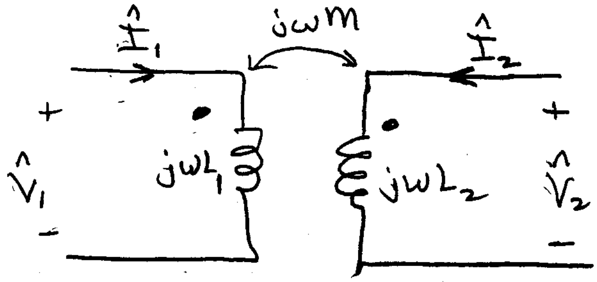


Figure 17.4: Showing circuit that matches Equations 17.14 & 17.15.

Now consider the circuit below which includes impedance.

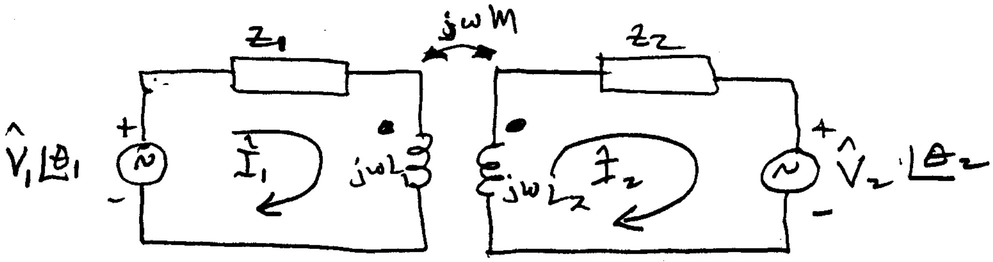


Figure 17.5: A circuit for illustrating how to write equations for a coupled circuit.

From Figure 17.5 we have

$$Z_1 \hat{I}_1 + j\omega L_1 I_1 - j\omega M \hat{I}_2 = \hat{V}_1 \angle\theta_1$$

and

$$Z_2 \hat{I}_2 + j\omega L_2 I_2 - j\omega M I_1 = -\hat{V}_2 \angle\theta_2$$

The above equations can be solved for $\hat{I}_1$ and $\hat{I}_2$.

Example 17.1

Given the transformer circuit shown below. Solve for the phasor currents $\hat{I}_1$ and $\hat{I}_2$.

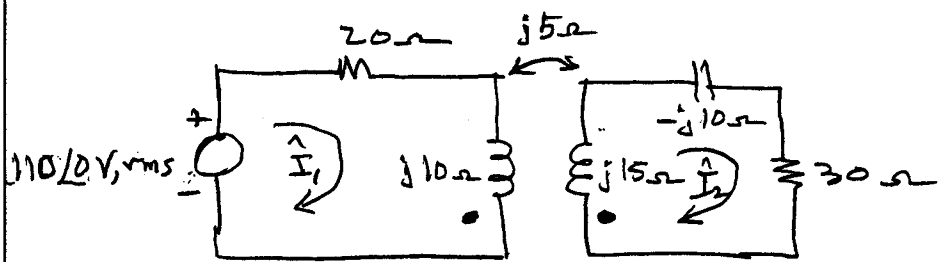


Figure 17.6: Circuit for Example 17.1.

We have

$$(20 + j10)\hat{I}_1 - j5 I_2 = 110\angle 0 \qquad \text{Eq } 17.16$$

$$-j5 I_1 + (j15 - j10 + 30) I_2 = 0 \qquad \text{Eq } 17.17$$

Solving the above for $\hat{I}_1$ and $\hat{I}_2$

$$\hat{I}_1 = 4.78\angle -25.4 \text{ A, rms}$$

$$\hat{I}_2 = 0.785\angle 55.2 \text{ A, rms}$$

The core materials of transformers:

The core, that is, the material around which the wire of the transformer is wound ~~can be a number of~~ such as ~~materials such as~~ air, wood, plastic, bakelite, iron, steel, determine the operating characteristics and how we analyze the transformer.

We recall from earlier work that

$$L = \frac{N^2 \mu A}{l} \qquad \text{Eq. 17.18}$$

defines the inductance of a coil where

μ = permeability of the core

N = number of turns of wire making up the coil

A = cross section area of the coil

L = length of the coil.

Normally, the treatment of transformers is divided into two groups:

(a) Linear transformers

(b) ideal transformers

We briefly consider the linear transformer and then spend the remaining time on the ideal transformer.

The Linear Transformer:

A transformer is classified as being linear if the core materials on which the coils are wound are magnetically linear. Some materials that are magnetically linear and are used for core materials for transformers include air, paper (cardboard), plastic. Probably air core transformers are the most frequently encountered and they are used often in communication circuits. The method of analysis is the same as presented earlier. We might add that the mutual inductance and self inductances are related by

$$K = \frac{M}{\sqrt{L_1 L_2}} \qquad \text{Eq } 17.19$$

K is called the coefficient of coupling in terms of the magnetic flux;

$$K = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}} \qquad \text{Eq } 17.20$$

and

$$K = \frac{\phi_{21}}{\phi_2} = \frac{\phi_{21}}{\phi_{21} + \phi_{22}} \qquad \text{Eq } 17.21$$

If all the flux of coil 1 links coil 2 and vice-versa, then $K=1$ and we have perfect coupling. This comes close to happening when the coils are on an air core and wrapped on top of one another as shown below.

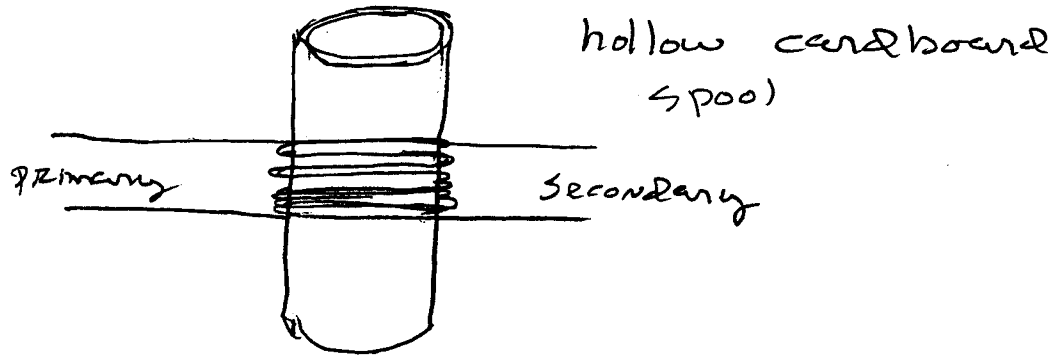


Figure 17.7: A tightly wound air core transformer. $K \approx 1$.

Once again, the type of analysis applied to Example 17.1 is usually used for linear transformers.

Example 17.2

Obtain the input impedance for the linear transformer shown below.

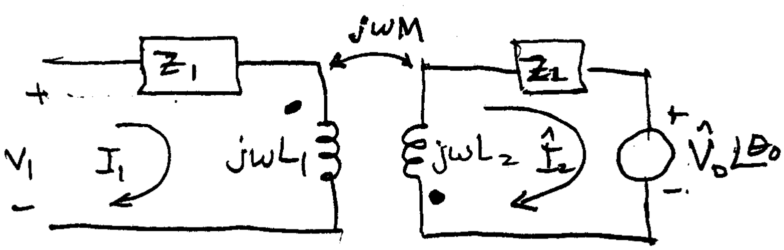
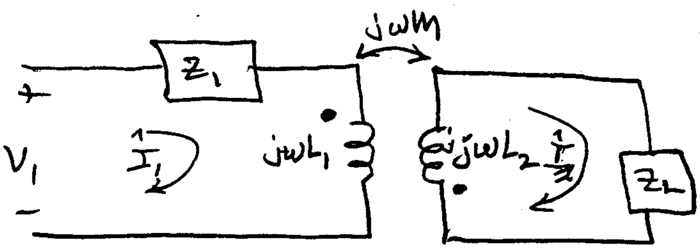


Figure 17.8: Circuit for Example 17.2

To find the input impedance, often called 17.11
the driving point impedance, we deactivate all sources and solve for

$$Z_{IN} = \frac{\hat{V}_1}{\hat{I}_1}$$

The circuit becomes,



$$(j\omega L_1 + Z_1)I_1 + j\omega M I_2 = \hat{V}_1$$

$$j\omega M I_1 + (j\omega L_2 + Z_L)I_2 = 0$$

by Cramer's rule;

$$\hat{I}_1 = \frac{\begin{vmatrix} V_1 & j\omega M \\ 0 & j\omega L_2 + Z_2 \end{vmatrix}}{\begin{vmatrix} j\omega L_1 + Z_1 & j\omega M \\ j\omega M & j\omega L_2 + Z_L \end{vmatrix}}$$

From which

$$\frac{V_1}{I_1} = j\omega L_1 + Z_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_L} = j\omega L_1 + Z_1 + Z_R$$

where $Z_R = Z_{Reflected} = \dfrac{\omega^2 M^2}{j\omega L_2 + Z_L}$

This introduces the concept of reflected impedance of a transformer.

IDEAL TRANSFORMERS

Basically, an ideal transformer is one in which $K=1$. In construction, the ideal transformer will have a large number of turns and a core with a high permeability. One assume that because of the high permeability, the flux lines from one winding are completely coupled to the second winding. To have high permeability, ideal transformers have cores constructed from magnetic material. The cores are usually constructed from alloys of iron and steel.

To set up the equations for the ideal transformer, consider the linear transformer below.

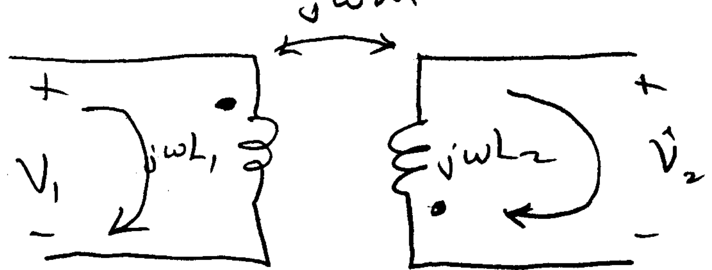


Figure 17.9: A BASIC circuit for the linear transformer

The equations corresponding to Figure 17.9 are;

$$\hat{V}_1 = j\omega L_1 \hat{I}_1 + j\omega M \hat{I}_2 \qquad \text{Eq } 17.22$$

$$\hat{V}_2 = j\omega M \hat{I}_1 + j\omega L_2 \hat{I}_2 \qquad \text{Eq } 17.23$$

From Eq 17.22 we have

$$I_1 = \frac{V_1 - j\omega M I_2}{j\omega L_1}$$

Substitute this into Eq 17.23:

$$\hat{V}_2 = \frac{\cancel{j\omega} M [\hat{V}_1 - j\omega M \hat{I}_2]}{\cancel{j\omega} L_1} + j\omega L_2 \hat{I}_2$$

For perfect coupling $M = \sqrt{L_1 L_2}$ and we use this in the previous equation.

$$\hat{V}_2 = \frac{\sqrt{L_1 L_2}}{L_1} \hat{V}_1 - \cancel{j\omega \frac{\cancel{L_1} L_2}{\cancel{L_1}} \hat{I}_2 + j\omega L_2 \hat{I}_2}$$

$$\hat{V}_2 = \sqrt{\frac{L_2}{L_1}}\, \hat{V}_1 = n V_1 \qquad \text{Eq } 17.23$$

$$\hat{V}_2 = n \hat{V}_1$$

OR $\boxed{\dfrac{V_2}{V_1} = n}$ Famous Equation Eq 17.24

You may wonder why

$$\sqrt{\frac{L_2}{L_1}} = n. \qquad \text{Eq } 17,25$$

Recall;

$$L_2 = \frac{N_2^2 \mu_2 A_2}{l_2}; \quad L_1 = \frac{N_1^2 \mu_1 A_1}{l_1}$$

With the ideal transformer

$$A_2 = A_1,\ \mu_2 = \mu_1,\ l_2 = l_1$$

Thus,

$$\frac{L_2}{L_1} = \left(\frac{N_2}{N_1}\right)^2 = (n)^2$$

OR

$$\sqrt{\frac{L_2}{L_1}} = n$$

We call n the turns ratio.
Because we have a very large number of turns and a high permeability for both coil, we assume that

$$L_2, L_1, M \rightarrow \infty$$

but that $\frac{L_2}{L_1} = n^2$.

For the ideal transformer we assume:

(a) coupling coefficient, k, $= 1$.

(b) $L_1, L_2, M \longrightarrow \infty$.

(c) Primary and secondary resistance of the coils are zero: $R_1 = 0$, $R_2 = 0$

Having a lossless transformer means the power supplied to the primary will be available at the secondary,

$$V_1 i_1 = V_2 i_2 \qquad \text{Eq } 17.26$$

Using Eq 17.24 we have

$$\boxed{\frac{I_2}{I_1} = \frac{1}{n}} \qquad \text{Eq } 17.27$$

This equation, along with

$$\boxed{\frac{V_2}{V_1} = n}$$

are the fundamental equations of the ideal transformer.