LESSON 17

MAGNETICALLY COUPLED CIRCUITS
(THE TRANSFORMER)
In my assessment, the three most important uses of transformers are:

- Isolating one circuit from another (in terms of the signal level and obtain fewer cases of noise).
- Impedance matching.

Properly selecting (by design or off the shelf purchase) can match $R_L$ to $R_S$ for maximum power transfer.

- Changing levels of voltage and current in power distribution.

The presentation here is very introductory. Basically, when a time-varying signal is applied to a coil of wire that has another coil of wire nearby, two magnetic fluxes are set up by the first coil and two flux by the second coil. This is illustrated in Figure 17.1.
Figure 17.11 Illustrating Flux in a transformer.

From physics and our forefathers we know

$$\Phi_1 = \Phi_{11} + \Phi_{12}$$  \hspace{1cm} \text{Eq. 17.1}

and

$$\Phi_2 = \Phi_{22} + \Phi_{21}$$  \hspace{1cm} \text{Eq. 17.2}

$\Phi_1$ is the magnetic flux associated with

coil one due to current $I_1(t)$.  

$\Phi_2$ is the magnetic flux associated with

coil two due to current $I_2(t)$.  

Reviewing the property of the inductor

as shown in Figure 17.2,

Figure 17.2: The basic inductor.

we recall:

$$V(t) = N \frac{d\Phi}{dt}$$  \hspace{1cm} \text{Eq. 17.3}
where \( N \) is the number of turns of wire of the inductor and \( \Phi(t) \) is the magnetic flux linking the coil, due to current \( i(t) \).

We can further write:

\[
V(t) = N \frac{d\Phi}{dt} \frac{d}{di} \quad \text{Eq. 17.4}
\]

This is justified since \( \Phi \) changes with time because of the time varying current. We lump \( N \frac{d\Phi}{dt} \) into a term we define as self-inductance and we have

\[
V(t) = L \frac{di}{dt} \quad \text{Eq. 17.5}
\]

which is the familiar equation for the voltage across an inductor.

Referring back to Figure 17.1, assume coil 2 is open (\( i_2 = 0 \)). The current \( i_1 \) causes a flux \( \Phi_1 \) which has two components. \( \Phi_1 = \Phi_{11} + \Phi_{12} \). \( \Phi_{11} \) links only coil 1 while \( \Phi_{12} \) links both coil 1 and coil 2.

We have

\[
V_1 = N_1 \frac{d\Phi_{11}}{dt} = N_1 \frac{d\Phi_{11}}{di_1} \frac{di_1}{dt} = L_1 \frac{di_1}{dt} \quad \text{Eq. 17.6}
\]
and also

\[ V_2 = N_2 \frac{d \Phi_2}{dt} \frac{di_1}{dt} = M_{21} \frac{di_1}{dt} \quad \text{Eq. 17.7} \]

\( M_{21} \) is called the mutual inductance with subscript meaning the voltage induced in coil 2 due to current \( i_1 \) in coil 1.

Now suppose that coil 1 was opened (\( i_1 = 0 \)). We have

\[ V_1 = N_1 \frac{d \Phi_1}{dt} \frac{d i_2}{dt} = M_{12} \frac{d i_2}{dt} \quad \text{Eq. 17.8} \]

and

\[ V_2 = N_2 \frac{d \Phi_2}{dt} \frac{d i_2}{dt} = L_2 \frac{d i_2}{dt} \quad \text{Eq. 17.9} \]

We are assuming a linear system and we more or less have supplied superposition above. Now if both coils are closed we can write

\[ V_1 = L_1 \frac{d i_1}{dt} + M_{12} \frac{d i_2}{dt} \quad \text{Eq. 17.10} \]

\[ V_2 = L_2 \frac{d i_2}{dt} + M_{21} \frac{d i_1}{dt} \quad \text{Eq. 17.11} \]

It is shown in most basic circuit books that \( M_{12} = M_{21} = M \).
The above equation become

\[ V_1(t) = L_1 \frac{di_1(t)}{dt} + M \frac{di_2(t)}{dt} \quad \text{Eq. 17.12} \]

\[ V_2(t) = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} \quad \text{Eq. 17.13} \]

We use a dot marking to tell us the polarity of the voltage induced from one coil to the other.

Consider the following with respect to mutually induced voltages.

![Diagram of two coils](image)

**Figure 17.3a**

The current \( i_2 \) entering the dot of coil 2 will cause the induced voltage, \( M \frac{di_2}{dt} \), in coil 1, to be positive at the dot of coil 1.

![Diagram of two coils](image)

**Fig 17.3b**

Lower dot in coil 2 is positive because \( i_1 \) is entering the dot of coil 1. Hence the voltage is \(-M \frac{di_1}{dt}\).
At this point we are not interested in solving the differential equations of Eq 17.12 and 17.13. We are interested in the AC steady state, so earlier equations become

\[ V_1 = j\omega L_1 \hat{I}_1 + j\omega M \hat{I}_2 \quad \text{Eq 17.14} \]
\[ V_2 = j\omega L_2 \hat{I}_2 + j\omega M \hat{I}_1 \quad \text{Eq 17.15} \]

These equations would pertain to the situation shown in Figure 17.4.

![Figure 17.4](image)

Figure 17.4: Showing circuit that matches Equations 17.14 & 17.15.

Now consider the circuit below which includes impedance,

![Figure 17.5](image)

Figure 17.5: A circuit for illustrating how to write equations for a coupled circuit.
From Figure 17.5 we have

\[ z_1 I_1 + j \omega L_1 I_1 - j \omega M I_2 = \hat{V}_1 L_1 \]

and

\[ z_2 \hat{I}_2 + j \omega L_2 \hat{I}_2 - j \omega M I_1 = -\hat{V}_2 L_2 \]

The above equations can be solved for \( \hat{I}_1 \) and \( \hat{I}_2 \).

**Example 17.1**

Given the transformer circuit shown below, solve for the phasor currents \( \hat{I}_1 \) and \( \hat{I}_2 \).

![Circuit Diagram](image)

Figure 17.6: Circuit for Example 17.1.

We have

\[ (20 + j10) \hat{I}_1 - j5 \hat{I}_2 = 110 L_2 \quad \text{Eq. 17.12} \]

\[ -j5 \hat{I}_1 + (j15 - j10 + 30) \hat{I}_2 = 0 \quad \text{Eq. 17.17} \]

Solving the above for \( \hat{I}_1 \) and \( \hat{I}_2 \)

\[ \hat{I}_1 = 4.78 / -25.4 \quad \text{A, rms} \]

\[ \hat{I}_2 = (0.785 / 55.2 \quad \text{A, rms} \]
The core materials of transformers:

The core, that is, the material around which the wire of the transformer is wound, materializes a number of air, wood, plastic, bakelite, iron, steel, determine the operating characteristics and how we analyze the transformer.

We recall from earlier work that

\[ L = \frac{N^2 \mu A}{l} \quad \text{Eq. 17.18} \]

defines the inductance of a coil.

where

- \( \mu \) = permeability of the core
- \( N \) = number of turns of wire making up the coil
- \( A \) = cross-sectional area of the coil
- \( l \) = length of the coil.

Normally, the treatment of transformers is divided into two groups:

(a) Linear transformers

(b) Ideal transformers

We briefly consider the linear transformer and then spend the remaining time on the ideal transformer.
The Linear Transformer:

A transformer is classified as being linear if the core materials on which the coils are wound are magnetically linear. Some materials that are magnetically linear and are used for core materials for transformers include air, paper (cardboard), plastic. Probably air core transformers are the most frequently encountered and they are used often in communication circuits. The method of analysis is the same as presented earlier, we might add that the mutual inductance and self inductances are related by

\[ K = \frac{M}{\sqrt{L_1 L_2}} \quad \text{Eq. 17.19} \]

\( K \) is called the coefficient of coupling in terms of the magnetic flux;

\[ K = \frac{\Phi_{12}}{\Phi_1} = \frac{\Phi_{12}}{\Phi_{11} + \Phi_{12}} \quad \text{Eq. 17.20} \]

and

\[ K = \frac{\Phi_{21}}{\Phi_2} = \frac{\Phi_{21}}{\Phi_{21} + \Phi_{22}} \quad \text{Eq. 17.21} \]
If all the flux of coil 1 links coil 2 and vice versa, then $K=1$ and we have perfect coupling. This comes close to happening when the coil are on air core and wrapped on top of one another as shown below.

![Image of a transformer with primary and secondary coils]

Figure 17.7: A tightly wound air core transformer, $K=1$.

Once again, the type of analysis applied to Example 17.1 is usually used for linear transformers.

Example 17.2

Obtain the input impedance for the linear transformer shown below.

![Image of a circuit with input impedance]

Figure 17.8: Circuit for Example 17.2
To find the input impedance, often called the driving point impedance, we reactivate all sources and solve for

\[ Z_{in} = \frac{V_i}{I_i} \]

The circuit becomes,

\[ (j\omega L_1 + Z_1)I_1 + j\omega M I_2 = V_i \]

\[ j\omega M I_1 + (j\omega L_2 + Z_2)I_2 = 0 \]

by Cramer's rule,

\[ I_1 = \frac{\begin{vmatrix} V_i & j\omega M \\ 0 & j\omega L_2 + Z_2 \end{vmatrix}}{\begin{vmatrix} j\omega L_1 + Z_1 & j\omega M \\ j\omega L_2 + Z_2 & j\omega M \end{vmatrix}} \]

From which,

\[ \frac{V_i}{I_1} = j\omega L_1 + Z_1 + \frac{\omega^2 M^2}{j\omega L_2 + Z_2} = j\omega L_1 + Z_1 + Z_R \]

where \( Z_R = Z_{\text{reflected}} = \frac{\omega^2 M^2}{j\omega L_2 + Z_2} \)

This introduces the concept of reflected impedance of a transformer.
IDEAL TRANSFORMERS

Basically, an ideal transformer is one in which \( K = 1 \). In constructing the ideal transformer, it will have a large number of turns and a core with a high permeability. One assumes that because of the high permeability, the flux lines from one winding are completely coupled to the second winding.

To have high permeability, ideal transformers have cores constructed from magnetic material. The cores are usually constructed from alloys of iron and steel.

To set up the equations for the ideal transformer, consider the linear transformer below.

![Diagram of transformer](image)

**Figure 17.9:** A basic circuit for the linear transformer.
The equations corresponding to Figure 17.9 are:

\[ V_1 = j\omega L_1 I_1 + j\omega M I_2 \]  \hspace{1cm} \text{Eq. 17.22}

\[ V_2 = j\omega M I_1 + j\omega L_2 I_2 \]  \hspace{1cm} \text{Eq. 17.23}

From Eq. 17.22 we have

\[ I_1 = \frac{V_1 - j\omega M I_2}{j\omega L_1} \]

Substitute this into Eq. 17.23:

\[ V_2 = \frac{j\omega M [V_1 - j\omega M I_2]}{j\omega L_1} + j\omega L_2 I_2 \]

For perfect coupling \( M = \sqrt{L_1 L_2} \) and we use this in the previous equation.

\[ V_2 = \frac{\sqrt{L_1 L_2}}{L_1} V_1 - j\omega \frac{\sqrt{L_1 L_2}}{L_1} I_2 + j\omega \sqrt{L_2} I_2 \]

\[ V_2 = \sqrt{\frac{L_2}{L_1}} \hat{V}_1 = n \hat{V}_1 \]  \hspace{1cm} \text{Eq. 17.23}

\[ V_2 = n \hat{V}_1 \]

OR

\[ \frac{V_2}{V_1} = n \]  \hspace{1cm} \text{Famous Equation}  \hspace{1cm} \text{Eq. 17.24} \]
You may wonder why

\[ \sqrt{\frac{L_2}{L_1}} = n. \quad \text{Eq. 17.75} \]

Recall:

\[ L_2 = \frac{N_2^2 M_2 A_2}{l_2}; \quad L_1 = \frac{N_1^2 M_1 A_1}{l_1} \]

With the ideal transformer

\[ A_2 = A_1, \quad M_2 = M_1, \quad l_2 = l_1 \]

Thus,

\[ \frac{L_2}{L_1} = \left( \frac{N_2}{N_1} \right)^2 = (n)^2 \]

or

\[ \sqrt{\frac{L_2}{L_1}} = n \]

We call \( n \) the turns ratio.

Because we have a very large number of turns and the high permeability for both coil, we assume that

\[ L_2/L_1, \; M \to \infty \]

but that \( \frac{L_2}{L_1} = n^2 \).
For the ideal transformer we assume:

(a) Coercive coefficient, $k, = 1$.
(b) $L_1, L_2, M \to \infty$.
(c) Primary and secondary resistance of the coils are zero: $R_1 = 0, R_2 = 0$

Having a lossless transformer means the power supplied to the primary will be available at the secondary,

\[ V_1 i_1 = V_2 i_2 \quad \text{Eq. 17.26} \]

Using Eq. 17.24 we have

\[ \frac{I_2}{I_1} = \frac{1}{n} \quad \text{Eq. 17.27} \]

This equation, along with

\[ \frac{V_2}{V_1} = n \]

are the fundamental equations of the ideal transformer.