Lesson 17

Magnetically Coupled Circuits
Linear and Ideal Transformers

Notes for ECE 301

November, 2002

Prepared By
Walter L. Green
Professor Emeritus, Electrical Engineering
University of Tennessee
Knoxville, TN
Equations 17.27 and 17.28 correspond to the dot polarities, assumed voltage polarities and assumed current directions as shown in Figure 17.9.

![Figure 17.9: Configuration of circuit and assumed voltages and currents for \( \frac{V_2}{V_1} = N, \frac{I_2}{I_1} = \frac{1}{N} \).]

We establish the sign for the voltage and current ratios as follows:

**Sign for the voltage ratio,** \( \frac{V_2}{V_1} \),

Consider the ideal transformer shown in Figure 17.10.

![Figure 17.10: An assumed configuration for dots and polarity of \( V_1 \) and \( V_2 \).]
Now consider Figure 17.11.

\[ \frac{V_2}{V_1} = -N \]

Figure 17.11: Configuration where \( V_2/V_1 \) is negative.

The general case for \( \frac{V_2}{V_1} \) is:

- If \( V_1 \) and \( V_2 \) are both positive or both negative at the dotted terminals, +1 is used; otherwise use a - sign. This applies regardless if the assumed directions for \( I_1 \) and \( I_2 \).

The general case for \( \frac{I_2}{I_1} \) is:

- If \( I_1 \) and \( I_2 \) both enter or both leave dotted terminals a - sign is used in the current ratio. Otherwise a + sign we used.

There are very sound reasons for the above statements. Just as current cannot change instantaneously, flux also cannot change instantaneously in coupled circuits. We consider
the transformer with given
windings.

\[ I_1 \quad \text{remaining} \quad \text{circuit} \quad \text{remaining} \quad \text{circuit} \]

\[ \Phi_{12} \quad \text{I}_{1} \quad \text{I}_{2} \]

\[ 20:1 \quad 20:1 \]

**Figure 12.12:** Circuit used to explain flux linkage.

Basic idea, with current \( I_1 \) setting up (tending to set up) the flux \( \Phi_{12} \); there will be a voltage induced at the terminals of coil two that produces a current \( I_2 \) that sets up a flux \( \Phi_{21} \) that opposes the flux \( \Phi_{12} \). The current \( I_2 \) leaves the dot of coil \( 2 \).

We then say \( \frac{I_2}{I_1} = \frac{1}{n} \). If we apply a positive voltage at the dotted terminal of coil \( 1 \), then the dotted terminal of coil \( 2 \) will be positive. This might be easier to understand if we consider the task of making dots on the transformer.
An Illustration

Figure 17.13: Determining dots on an unknown transformer.

Suppose we have a transformer and all we have available are the terminals x-y of the primary and terminals a-b of the secondary. Our job is to make the appropriate dots for the primary and secondary. We start by using a switch and connecting a known positive voltage to terminal X. We automatically assume x is the dot terminal of the primary. We connect the red lead of a voltmeter to terminal "a". We suddenly close the switch on the primary side.

If the meter flips up-scale and back, we know a positive voltage was induced at terminal "a" and we make a dot there. If the meter needle flips to the left we mark "b" with the dot.
Example 17.3

\[ \begin{align*}
V_2 & = \frac{20\Omega}{-\jmath 40\Omega} \left( I_1 + \jmath V_1 \right) \\
& = \left( \begin{array}{cc}
1 & -4 \\
1 & -1
\end{array} \right) \left( \begin{array}{c}
V_1 \\
I_1
\end{array} \right)
\end{align*} \]

Figure 17.14: Circuit for Example 17.3

For the above circuit, solve for \( I_1, I_2, V_1, \) and \( V_2 \):

**Solution:**

We can write 4 equations in terms of 4 unknowns as follows

\[ \frac{V_2}{V_1} = -4 \implies 4V_1 + V_2 = 0 \]

\[ \frac{I_2}{I_1} = -\frac{1}{4} \implies I_1 - 4I_2 = 0 \]

\[ V_1 + 20I_1 = 50/20 \]

\[ -V_2 + (15 - \jmath 40)I_2 = 0 \]

\[ \begin{bmatrix}
4 & 1 & 0 & 0 \\
0 & 0 & 1 & 4 \\
1 & 0 & 20 & 0 \\
0 & -1 & 0 & (15 - \jmath 40)
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
50/20 \\
0
\end{bmatrix} \quad \text{Eq. 17.29} \]

\[ V_1 = 6.33/\underline{42^\circ}, \quad V_2 = 25.3/\underline{43.4^\circ} \, \text{V, rms} \]

\[ I_1 = 2.37/\underline{26.9^\circ} \, \text{A, rms}, \quad I_2 = 0.593/\underline{-153.2^\circ} \, \text{A, rms} \]
Example 17.4

Given the following ideal transformer:

![Circuit Diagram]

Figure 17.15: Circuit for Example 17.4.

Again, we have 4 equations and 4 unknowns.

\[
\frac{V_2}{V_1} = \frac{15}{3} = 5
\]

\[
\frac{V_2}{V_1} = 5 \quad \Rightarrow \quad 5V_1 - V_2 = 0
\]

\[
\frac{I_2}{I_1} = \frac{1}{5} \quad \Rightarrow \quad I_1 - 5I_2 = 0
\]

\[-j20I_1 - V_1 = 100\sqrt{3}
\]

\[V_1 + (30+j15)I_2 = 0
\]

\[
\begin{bmatrix}
V_1 & V_2 & I_1 & I_2 \\
5 & -1 & 0 & 0 \\
0 & 0 & 1 & -5 \\
-1 & 0 & -j20 & 0 \\
0 & 1 & 0 & (30+j15)
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
100\sqrt{3}
\end{bmatrix}
\]

\[V_1 = 6.9/\angle-17^\circ \text{ V, rms} \quad V_2 = 34.5/\angle-17^\circ \text{ V, rms}
\]

\[I_1 = 5.145/\angle36.5^\circ \text{ A, rms} \quad I_2 = 1.03/\angle36.5^\circ \text{ A, rms}
\]
Thevenin's Theorem & The Ideal Transformer:

Consider the following transformer configuration.

![Diagram of a transformer configuration]

Figure 17.16: Illustrating Thevenin reflected to the primary of the ideal transformer.

We open the primary to find the Thevenin voltage and impedance to the right of terminals a-b.

![Diagram of Thevenin's equivalent circuit]

Figure 17.17: For the circuit in Figure 17.17, \( I_1 = 0 \). When \( I_1 = 0 \), \( I_2 = 0 \); \( \frac{I_2}{I_1} \) and with \( I_1 = 0 \), \( I_2 = 0 \) (\( L_2 \), \( L_\text{m} \) are extremely large inducences, and \( I_2 = 0 \).
From Figure 17.17, $\hat{V}_2 = \hat{V}_L$.

$$\frac{V_2}{V_1} = n$$

So

$$\sqrt{V_1 = \frac{V_2}{n}} = V_{TH}$$

To find the reflected impedance, we short (de-adapt) the load voltage.

We apply a voltage of $V_1$ at a-b and find the ratio of $V_1/I_1 = Z_1$.

$$\frac{V_1}{I_1} = Z \text{ ref}$$

Use $V_1 = \frac{V_2}{n}$; $I_1 = nI_2$

$$\frac{V_1}{I_1} = \frac{V_2}{n} \times \frac{1}{nI_2} = \frac{1}{n^2} \times \frac{V_2}{I_2}$$

But

$$\frac{V_2}{I_2} = Z_2$$

$$\frac{V_1}{I_1} = \frac{Z_L}{n^2}$$
The equivalent circuit reflected to the primary is shown below.

Figure 17.16: Secondary reflected to the primary for transformer windings, polarities, and current directions of Figure 17.16.

Let us apply the above circuit to find \( I_1 \) and \( V_1 \) of Example 17.3. We have

\[
\frac{Z_1}{10} = \frac{15 - i40}{16}
\]

\[
I_1 = \frac{50120}{20 + \frac{15 - i40}{16}} = 2.37/26.8 \text{ A, rms} \quad \text{(check)}
\]

\[
V_1 = 50120 - 20I_1 = 60.33 - 42.4 \text{ V, rms}
\]

The above answers agree with the solutions of Example 17.3.
Reflecting the Primary to the Secondary for the Ideal Transformer

We start by considering the secondary open.

\[ V_e \text{LE} \]

Since \( I_2 = 0 \), \( I_1 = 0 \).

\[ V_1 = V_e \text{LE} \]
\[ V_2 = n V_1 = n V_e \text{LE} = V_{TH} \]

To find the reflected impedance

\[ \frac{I_X}{I_{TH}} = n \]

\[ Z_{ref} = Z_{TH} = \frac{V_{TH}}{I_{TH}} = \frac{V_2}{I_{TH}} \]

\[ V_2 = V_1 n \]
\[ I_X = n I_{TH} \]
\[ \frac{I_X}{I_{TH}} = n \]

\[ Z_{ref} = \frac{V_1 n^2}{I_X} \]

But \( V_1 = Z_x I_{TH} \Rightarrow Z_1 = \frac{V_1}{I_X} \)

\[ Z_{ref} = n^2 Z_1 \]

We now have the equivalent circuit as shown in Figure 17.19.
The following is true in general:

- The impedance reflected from the secondary to the primary is \( \frac{Z_2}{n} \) (where \( Z_2 \) is the secondary impedance).

- The impedance reflected from the primary to the secondary is \( nZ_p \) where \( Z_p \) is the primary impedance.

The above is true regardless of dot marking, voltage polarity, current direction.

- The voltage source of the secondary reflected to the primary is \( \frac{1}{n} \) \( V_{2s} \).

- The voltage source of the primary reflected to the secondary is \( nV_{1p} \).
Illustration A

What is the voltage source value and polarity reflected from the secondary to the primary?

\[ I_1 = 0 \Rightarrow I_2 = 0 \Rightarrow V_2 = -V_3 \]

But

\[ \frac{V_2}{V_1} = -\frac{1}{n} \quad \text{or} \quad V_1 = -\frac{V_2}{n} \]

Then

\[ V_1 = -\frac{(-V_3)}{n} = \frac{V_3}{n} \]

Illustration B

\[ V_1 = \frac{V_2}{n} = \frac{V_3}{n} \]

So the equivalent circuit is
Example 17.5

Find $I_1$ in the following ideal transformer circuit by (a) direct circuit analysis and (b) reflecting secondary to primary using information from Illustration B.

![Circuit Diagram]

**Figure 17.20**: Circuit for example 17.5.

\[
\frac{V_2}{V_1} = 4 \quad \Rightarrow \quad 4V_1 - V_2 = 0
\]

\[
\frac{I_2}{I_1} = \frac{1}{4} \quad \Rightarrow \quad I_1 - 4I_2 = 0
\]

\[
V_1 + 10I_1 = 100
\]

\[
-V_2 + 160I_2 = -160
\]

\[
\begin{bmatrix}
4 & -1 & 0 & 0 \\
0 & 0 & 1 & -4 \\
1 & 0 & 10 & 0 \\
0 & -1 & 0 & 160
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
100 \\
-160
\end{bmatrix}
\]

\[
I_1 = 3.12 \text{ A}
\]
Example 17.5

Reflecting the secondary as per Illustration B,

\[ I = \frac{100 - 40}{20} = 3 \, \text{A} \]

This checks with circuit analysis method.

Example 17.6

Find \( I_2 \) for the following circuit using (a) regular circuit analysis. (b) Reflect primary to secondary,

\[ 10 \Omega - i20 \Omega \]

Figure 17.21: Circuit for example 17.6.

(a) \( \frac{V_2}{V_1} = -2 \Rightarrow 2V_1 + V_2 = 0 \)
\[
\frac{I_2}{I_1} = \frac{1}{2} \Rightarrow I_1 - 2I_2 = 0
\]

\[
V_1 + (10 - j20)I_1 = 50\, \text{V}
\]

\[
V_2 + 20I_2 = 50\, \text{V}/40
\]

\[
\begin{bmatrix}
V_1 & V_2 & I_1 & I_2 \\
2 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 \\
1 & 0 & (10 - j20) & 0 \\
0 & 1 & 0 & 20
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
50\, \text{V} \\
50\, \text{V}
\end{bmatrix}
\]

\[
I_2 = 1.42 \, 166.2\, \text{A, rms}
\]

(b) Reflecting

\[
V_1 = 50\, \text{V}; \quad V_2 = -2V_1 = -100\, \text{V}
\]

Since \( V_2 \) is positive, down, Fig 17.21.
It must be have the source positive, up.

\[
I_2 = \frac{50\, \text{V}/40 + 100}{100 - j80} = 1.42 \, 166.2\, \text{A, rms}
\]

Checks with circuit analysis.
Consider the stereo amplifier and speaker configuration shown below.

![Circuit Diagram](image)

Figure 17.22: Circuit for Example 17.7.
Assume that the resistance of the amplifier is $180\,\Omega$. Assume that the open-circuit voltage of the amplifier is $V_g$ volts rms. Assume that the speaker has $8\,\Omega$.

(a) Without using a transformer, determine the power delivered to the speaker. This will be expressed in terms of $V_g$.

(b) Determine the value of $n$ for the transformer so that maximum power is delivered to the speaker.

(c) Determine the amount of power delivered to the speaker with the transformer in place.
The voltage across the speaker is

\[ V_0 = \frac{8 \times V_g}{8 + 180} \]

\[ P_o = \frac{V_0^2}{8} = \left[ \frac{64 V_g^2}{35344} \right] \times \frac{1}{8} \]

\[ P_o = 0.22635 V_g^2 \text{ mW} \]

(b) The resistance reflected is

\[ R_{\text{ref}} = \frac{8}{n^2} \]

We want this to equal 180

\[ \frac{8}{n^2} = 180 \]

\[ n = \sqrt{\frac{8}{180}} = 0.2108 \]

If \( N_2 = 5 \), \( N_1 = \frac{5}{0.2108} = 23.7 \) turns

(c) With the impedance match

\[ P_o = \frac{V_0^2}{180} = \frac{1}{180} \left[ \frac{180V_g}{180+180} \right]^2 = \frac{0.25V_g^2}{180} \]

\[ P_o = 1.4 V_g^2 \text{ mW} \]

Compare to

\[ P_o = 0.22635 V_g^2 \text{ mW} \]
Example 17.8

You are given the ideal transformer circuit shown below.

(a) Reflect the secondary impedance to the primary and find \( I_1 \).

(b) After finding \( I_1 \), find \( I_2 \).

(c) Find the average power delivered to the 10 \( \Omega \) load resistor.

(a) The impedance to the right of \( c-d \) is:

\[
Z_{cd} = \frac{80(100-j40)}{180-j40} = 46.7/\angle9.3^\circ \Omega
\]

\( Z_{\text{reflected}} = Z_R = \frac{46.7/9.3}{4^2} \)

\( Z_R = 2.92/\angle9.3^\circ \Omega \)

\( I_1 = \frac{100/10}{6+2.92/9.3} = 11.24/3.04^\circ \text{A, rms} \)

\( I_1 = 11.24/3.04^\circ \text{A, rms} \)
Example 17.8 (cont.)

(b) \[ \frac{I_2}{I_1} = \frac{1}{n} = \frac{1}{4} \]

\[ I_2 = \frac{I_1}{4} = \frac{11.24 \sqrt{3.04}}{4} = 2.81 \sqrt{3.04} \text{ A, rms} \]

\[ I_2 = 2.81 \sqrt{3.04} \text{ A, rms} \]

(c) We use the current division rule

\[ I_L = \frac{I_2 \times 80}{80 + 10 - j40} = \frac{(2.81 \sqrt{3.04}) \times 80}{90 - j40} \]

\[ I_L = 2.28 \sqrt{22.0} \text{ A, rms} \]

\[ P_{10} = |I_L|^2 \times 10 = (2.28)^2 \times 10 \]

\[ P_{10} = 52 \text{ W} \]
Example 17.9

Find the current $I_1$ in the circuit below using reflected impedance.

Figure 17.23: Circuit for example 17.9

The dots on the transformers make no difference in reflecting the resistance. Starting at the right we have

\[ Z_{gh} = \frac{160}{\left(\frac{4}{3}\right)^2} = \frac{160 \times 9}{16} = 90 \, \Omega \]

\[ Z_{ef} = \frac{60 \times 90}{60 + 90} = 36 \, \Omega \]

\[ Z_{cd} = 14 + 36 = 50 \, \Omega \]

\[ Z_{ab} = \frac{50}{5^2} = 2 \, \Omega \]

\[ I_1 = \frac{48 \, \text{V}}{2 + 2} = 12 \, \text{A}, \text{ rms} \]

\[ \therefore I_1 = 12 \, \text{A}, \text{ rms} \]
Example 17.10

You are given the circuit below that contains an ideal transformer. Determine the power delivered to the 10Ω resistor.

![Circuit diagram](image)

Figure 17.24: Circuit for Example 17.10.

We are given the choice in assigning $I_1$, $E_2$, $V$, and $V_2$ for the transformer. My choices are as shown in the following circuit.

![Circuit diagram](image)

As the circuit stands, we have 4 unknowns: $I_1$, $I_2$, $V$, and $V_2$. We therefore need 4 equations.
We write the equations as follows:

\[ \frac{I_2}{I_1} = \frac{1}{2} \quad \Rightarrow \quad I_1 - 2I_2 = 0 \]

\[ \frac{V_2}{V_1} = 2 \quad \Rightarrow \quad 2V_1 - V_2 = 0 \]

\[ 2I_1 + V_1 + 5(I_1 - I_2) = 46 \text{ Ld} \]

\[ V_1 + 7I_1 - 5I_2 = 46 \text{ Ld} \]

\[ 10I_2 + 5(I_2 - I_1) - V_2 = 0 \]

\[ -V_2 - 5I_1 + 15I_2 = 0 \]

\[
\begin{bmatrix}
V_1 & V_2 & I_1 & I_2 \\
0 & 0 & 1 & -2 \\
2 & -1 & 0 & 0 \\
1 & 0 & 7 & -5 \\
0 & -1 & -5 & 15
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
I_1 \\
I_2
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
46 \\
0
\end{bmatrix}
\]

\[ I_2 = 4 \text{ A, rms} \]

\[ P_10 = I_2^2 \times 10 = 160 \text{ W} \]

\[ P_{10} = 160 \text{ W} \]