Lesson 18

An Introduction To Frequency Response

Attention!! Read the message Board.

Notes for ECE 301

November, 2002

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ECE 301
wlg
Lesson 18
Fall, 2002

Frequency Response

What does it mean when someone says "this system has a good frequency response"? Of course we need to define good. What do we mean when we say "good student", "good child", "good food."? Each would has it's own index for qualifying good. And each one of us, perhaps, would have a different definition of good. Take me, I never though caviar was good food nor do I think the fancy chopped liver spread (pâté) the French whip-up, is good food. I am more of a deep South ham and eggs, pinto beans, fried okra and corn bread boy!

Before I tell you what I think good frequency response means to me, I will first discuss frequency response of a system and why it is important in all fields of engineering.

Consider the diagram of Figure 18.1.

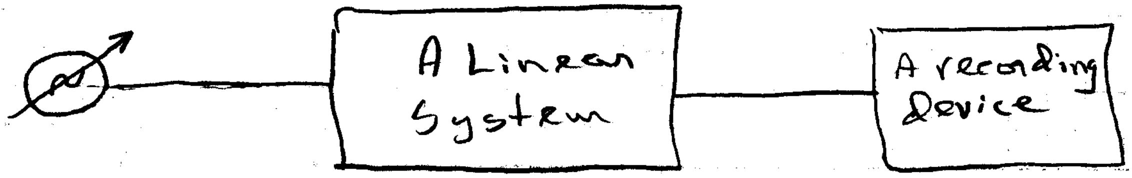


Figure 18.1: A simple system configuration for frequency response testing.

A characteristic of a linear system is that if we apply a sinusoidal input to the system, in steady state, the output of the system will also be a sinusoid of the same frequency as the input. The difference between the output and input sine wave is that the output, generally, will have a different amplitude and different phase than that of the input - that's all. But, this difference in amplitude and phase is important when it comes to discussing system performance.

Before we talk more about performance let us turn our attention to the actual process of obtaining a frequency response.

Since this is a basic electrical engineering course, I will restrict the system I discuss to electric circuits. But, frequency response applies to systems across the board. For example, it is not difficult to talk about the frequency response of the spring-mass-damper system of Figure 18.2.

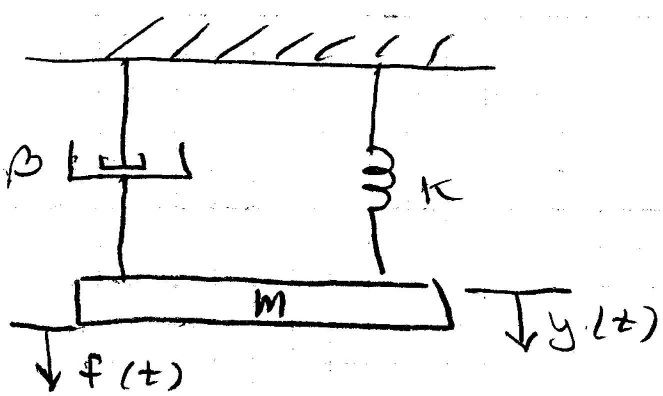


Figure 18.2; Spring-Mass-Damper System: Frequency response applies here.

In the NASA space program, frequency response was intensely used to qualify the integrity of a space vehicle, the SATURN V for example.

My discussion we apply to circuits but the techniques can readily be adapted to other systems.

Consider the system of Figure 18.3

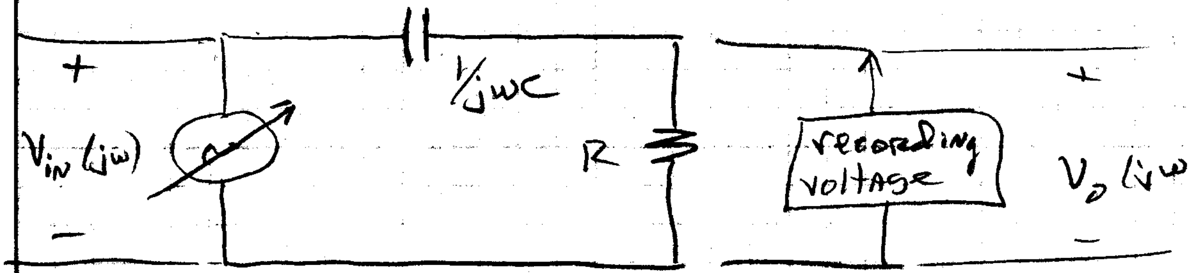


Figure 18.3: Apply concept of Frequency response to a circuit.

Consider the input to be $V_{in}(j\omega)$ and the output $V_o(j\omega)$. We are considering here AC steady state and we want to find $V_o(j\omega)$. Using the voltage division we have

$$V_o(j\omega) = \frac{V_{in}(j\omega)\, R}{R + 1/j\omega C} \qquad \text{Eq 18.1}$$

We see that if $|V_{in}(j\omega)|$ remains fixed as ω changes, then $V_o(j\omega)$ will be zero at $\omega = 0$ and $|V_{in}(j\omega)|$ as $\omega \to \infty$ (infinity)

$$V_o(j\omega) = \frac{V_{in}(j\omega)\,(j\omega RC)}{1 + j\omega RC} \qquad \text{Eq 18.2}$$

We could calculate $V_o(j\omega)$ for various values of ω in the range $0 \leq \omega \leq \infty$ or we could make measurements in a laboratory to obtain this information. To illustrate the calculations, let $V_{in}(j\omega) = 1\angle 0$, $R=1$, $C=1$ to simplify matters. We are then working with:

$$V_o(j\omega) = \frac{1}{1 + 1/j\omega} \qquad \text{Eq } 18.3$$

Let $\omega = 0.5$, 1 and 4 and determine $V_o(j\omega)$. The results are presented in Table 18.1

	$\lvert V_o(j\omega)\rvert$	$\angle V_o(j\omega)$
$\omega = 0$	0	0
$\omega = 0.5$	0.447	63.4°
$\omega = 1$	0.707	45°
$\omega = 4$	0.97	14°
$\omega = \infty$	1	0

Table 18.1: Calculations for $\frac{j\omega}{1+j\omega}$.

We can make plots of $\angle V_o(j\omega)$ vs ω and $|V_o(j\omega)|$ vs ω. Figure 18.4 shows $|V_o(j\omega)|$ vs ω, for the values we have and we extrapolate for values in between

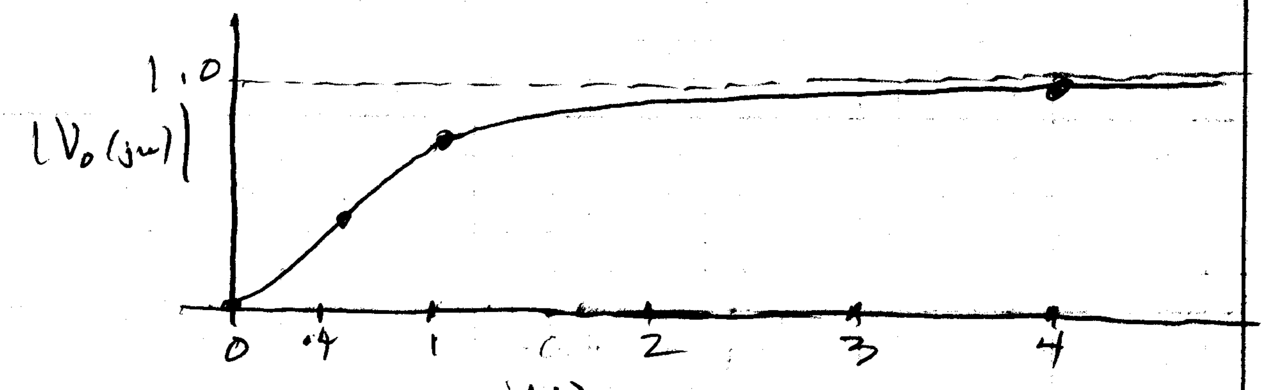


Figure 18.4; showing $|V_o(j\omega)| = \dfrac{|j\omega|}{|1+j\omega|}$.

We call the plot in Figure 18.4 the frequency response of the network in Figure 18.3.

To simplify matter we usually replace $j\omega \rightarrow s$, to avoid messy j algebra. We can always switch back to $j\omega$ after the analysis of the system is completed.

We can extend our horizon in understanding frequency response by defining and discussing transfer function.

A Definition of transfer function:

Consider the following diagram

$$R(j\omega) \longrightarrow \boxed{G(j\omega)} \longrightarrow C(j\omega)$$

We define

$$\frac{C(j\omega)}{R(j\omega)} = G(j\omega) \qquad \text{Eq } 18.4$$

In words; the transfer function, $G(j\omega)$, of a system (circuit) is the ratio of the response to excitation when this ratio is in terms of $j\omega$ and all initial conditions are zero.

In a more general sense, a transfer function is defined as the ratio of the Laplace transform of the response to excitation of a system when all initial conditions are zero. Transfer functions are defined only for time invariant, linear systems.

The words (in defining transfer function) may be more difficult to understand that the process of finding a transfer function.

Consider the system given in Figure 18.5.

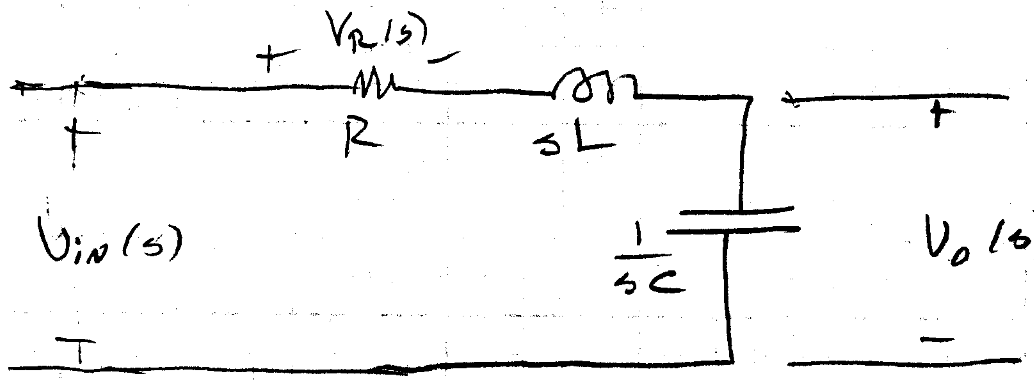


Figure 18.5: A circuit defined in terms of complex impedance using the Laplace transform variable, s.

We have, using voltage division,

$$V_o(s) = \frac{V_{in}(s)\,(1/sC)}{R + sL + 1/sC} \quad \text{Eq } 18.5$$

OR

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1}{s^2LC + sRC + 1} \quad \text{Eq } 18.6$$

Equation 18.6 gives the transfer function between the voltage across the capacitor (the output) and the input voltage ($V_{in}(s)$). There are other transfer function we can define for this network. For example, we can find the transfer function, $V_R(s)/V_{IN}(s)$ as (using voltage division)

$$V_R(s) = \frac{V_{IN}(s)\ R}{R + sL + \frac{1}{sC}}$$

OR

$$\frac{V_R(s)}{V_{IN}(s)} = \frac{sRC}{s^2LC + sRC + 1} \qquad \text{Eq 18.7}$$

Now the nice thing about the above analysis is that we can replace $s \to j\omega$ and obtain the frequency response between a particular output and the input. For example, Equation 18.6 gives

$$\frac{V_o(j\omega)}{V_{IN}(j\omega)} = G(j\omega) = \frac{\frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} \qquad \text{Eq 18.8}$$

For given values of R, L, C we can obtain the frequency response for $G(j\omega)$ by making calculations (say $\omega = \omega_1, \omega_2, \omega_3 ---- \omega_{125}$) and determining $|G(j\omega)|$ and $\underline{/G(j\omega}$. I would not want to make the 125 calculations with my TI 86 – think of the time. However, we have computer software, for example, MATLAB that is perfect for this task.

Before we talk about the task anymore, let us return to the concept first. What are we doing here? Well, the transfer function, $G(s)$, is a complex variable function in the variable s. In general we can express this function by factoring the numerator and denominator of $G(s)$. That is

$$G(s) = \frac{N(s)}{D(s)} \qquad \text{Eq 18.9}$$

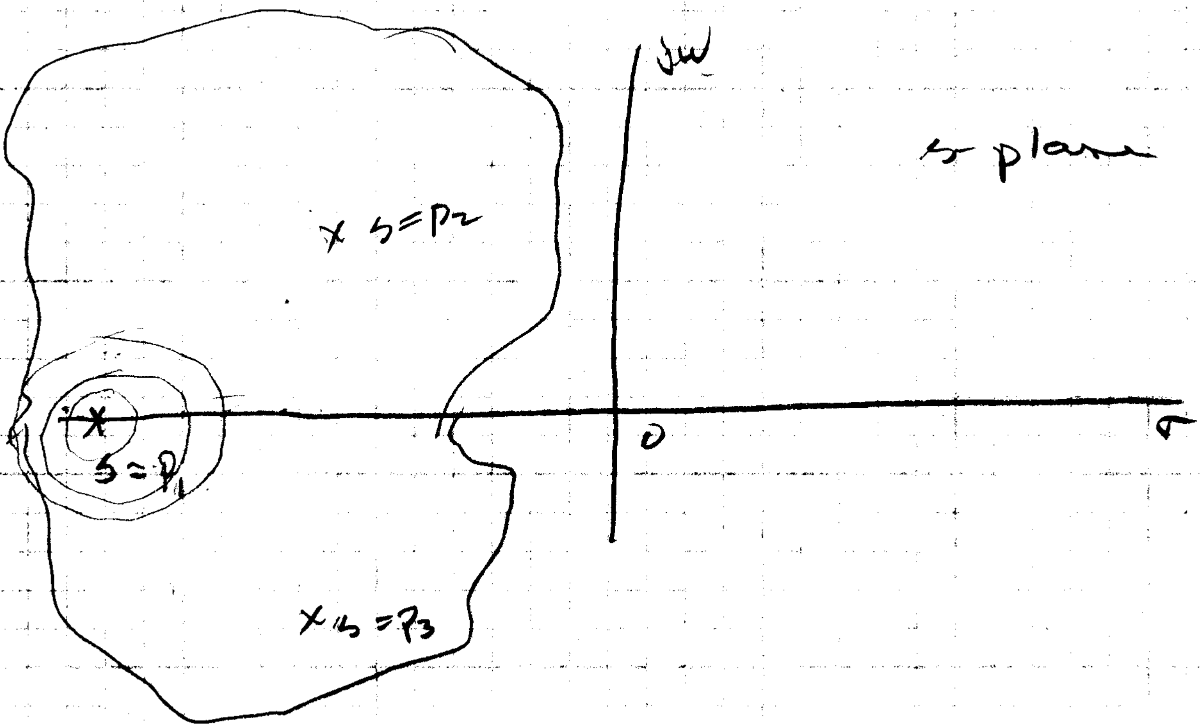
where $N(s)$ is the numerator polynomial and $D(s)$ is the denominator polynomial.

We can factor $N(s)$ and $D(s)$ into the individual roots:

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{s^{z}(s-p_1)(s-p_2)\cdots(s-p_n)} \quad \text{Eq 18.10}$$

We say that $s = z_1$, $s = z_2$, --- $s = z_m$ are the zeros of the transfer function. We call them zeros because if, say, $s = z_1$ $G(z_1) = 0$, etc. We call the factors (roots) of $D(s)$ the poles of the transfer function. For example, if $s = p_2$, $|G(p_2)| = \infty$. The reason $s = p_1$, $s = p_2$, --- $s = p_n$ are called poles is a little more difficult to explain than zeros. However, think of a large canvas (tent) covering the s-plane over which the values of $s = p_1$, $s = p_2$, ... $s = p_n$ are located. This is sketched in Figure 18.6. As we evaluate the function $G(s)$ in the neighborhood of say $s = p_1$, think of $|G(s)|$

being represented by an axis coming out of the surface of the paper. 18,12



As we evaluate |G(s)| around concentric circles in the neighborhood of s=p1, it is similar to raising a pole underneath the canvas and sort of like a tent, so that is how Walter Green thinks of poles. Note that at exactly s=p1, the pole would go all the way to Heaven!

The problem of preparing the frequency response is so important that enormous time is devoted to the problem. For example, a graphical procedure called "Bode" (after a distinguished engineer/scientist from Bell labs, 1930's) has been developed. Plotting the Bode actually gives you more appreciation for the frequency response but time does not allow us to cover this in ECE301. We will use MATLAB to plot our frequency response functions.

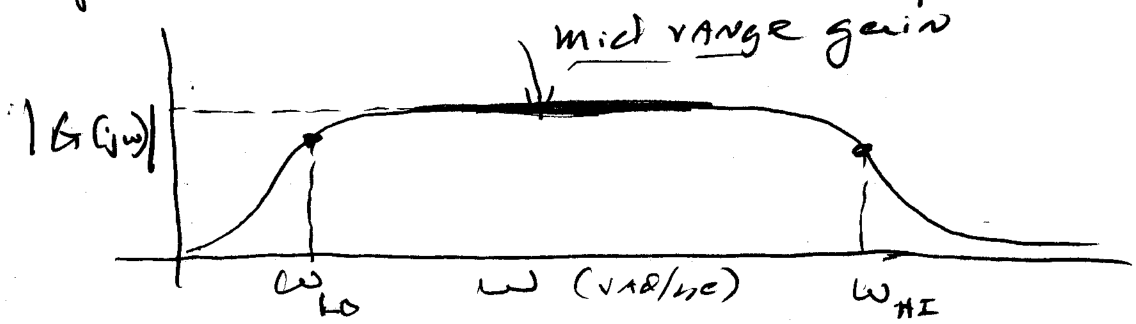
In the signal processing toolkit of MATLAB, we find a host of functions that aid us in frequency response. These include <u>freqs</u>, <u>bode</u>, and <u>freqresp</u>. We will use freqs and bode in these notes.

Before we start running frequency response with the computer, let us look at a few things that are used in describing a frequency response.

<u>Bandwidth</u> of a system. The bandwidth of a system is defined as the range of ω (or f) over which the response is 0.707 of the midrange response (sometimes called the pass band)

<u>Example:</u>

The following might represent the frequency response of a stereo amplifier,



If we assume the mid range gain is 1, then ω_{LO} & ω_{HI}, called the lower and upper cut-off frequencies, are at $|G(j\omega_{LO})| = |G(j\omega_{HI})| = 0.707$.

The bandwidth, ω_{BW}, $= \omega_{HI} - \omega_{LO}$

We often talk about the .707 points and also called them the -3 dB points.

$$\# \text{dB} = 20 \log |G(j\omega)|_{\omega=\omega_i}$$

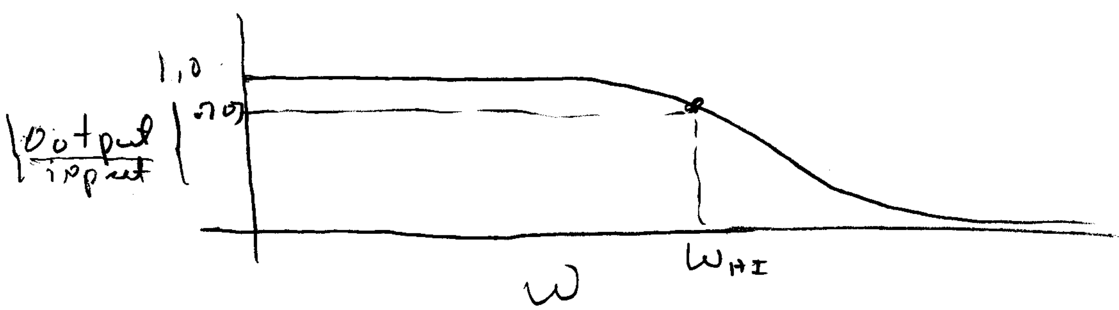
If $|G(j\omega_i)| = |G(j\omega_{LO})| = |G(j\omega_{HI})| = 0.707$ then

$$20 \log_{10}(0.707) = -3 \text{ dB}$$

ω_{LO} & ω_{HI} are called the cut-off points and also called the -3 dB points.

<u>Example</u>

The frequency response of a particular DC torque motor is shown below



For the dc motor, there is no low frequency cut-off. So the bandwidth is ω_{H}.
So we say, here, that the bandwidth is

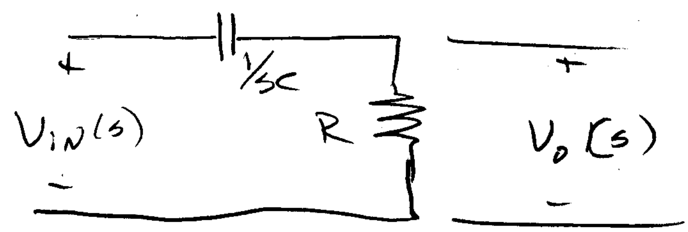
$$\omega_{BW} = \omega_{HI}.$$

Transfer Functions

Let us turn our attention next to transfer function and then we will come back to frequency response.

Example 1

Find the transfer function for the following network:

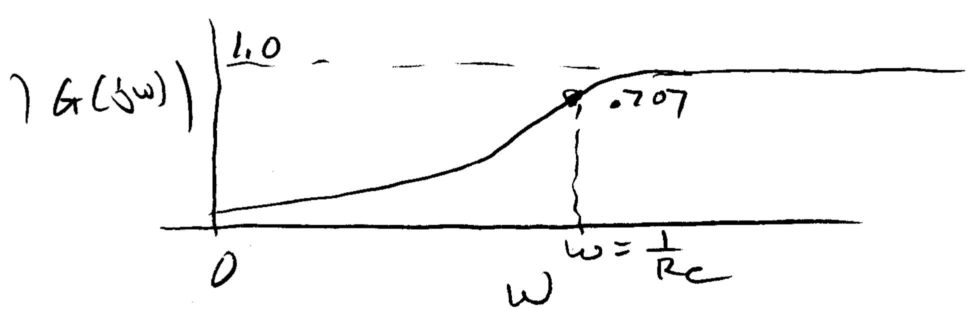


$$\frac{V_o(s)}{V_{IN}(s)} = \frac{R}{R + 1/sC} = \frac{sRC}{1 + sRC}$$

As a frequency response

$$\frac{V_o(j\omega)}{V_{IN}(j\omega)} = G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

For some given values of R & C we would have



This is a high pass filter (it passes high frequencies but attenuates low frequencies).

The cut-off frequency is at $\omega = \frac{1}{RC}$. We show this as follows:

This defines cut-off

$$\frac{|j\omega RC|}{|1+j\omega RC|} = \boxed{0.707 = \frac{1}{\sqrt{2}}}$$

$$\frac{\omega RC}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{2}}$$

square both sides,

$$\frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2} = \frac{1}{2}$$

$$1+\omega^2 R^2 C^2 = 2\omega^2 R^2 C^2$$

$$1-\omega^2 R^2 C^2 = 0$$

$$\omega^2 = \frac{1}{R^2 C^2}$$

$$\omega = \frac{1}{RC} = \text{cutoff}$$

Question: Let $\omega_{co} = \omega$ cutoff $= 10$ radians/sec for the above high pass filter. If an input signal of $v(t) = 10\cos 10t$ is applied to this filter, what will be the magnitude of the output sinusoid?

Answer; We are at $\omega = 10$ which is the cutoff frequency.

Thus the input signal will be multiplied by 0.707. The output amplitude will be $0.707 \times 10 = 7.07$.

$\frac{1}{10 \times}$ $\frac{1}{RC}$

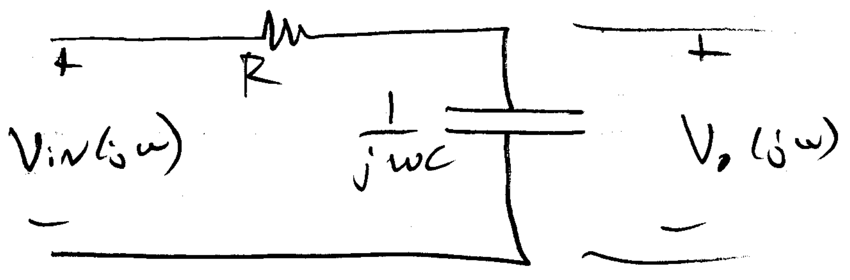
Example 2

You are given the following filter circuit. Determine the transfer function for $V_o(j\omega)/V_{IN}(j\omega)$. What kind of filter is this? Let $R = 10\,\Omega$ 10 Ω and $C = 0.01F$.

If an input signal of

$$V_{IN}(t) = 8\cos(12t)$$

is applied to the filter, what will be $V_o(t)$?



Solution:

$$\frac{V_o(j\omega)}{V_{IN}(j\omega)} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1+j\omega RC}$$

With $R = 10\,\Omega$, $C = 0.01F$

$RC = 10 \times .01 = 0.1$

$$\frac{V_o(j\omega)}{V_{IN}(j\omega)} = \frac{1}{1+j\omega/10} = \frac{1}{1+j0.1\omega}$$

At $\omega = 12$, the input signal frequency,

$$\frac{V_o(j12)}{V_{IN}(j12)} = G(j12) = \frac{1}{1+j1.2}$$

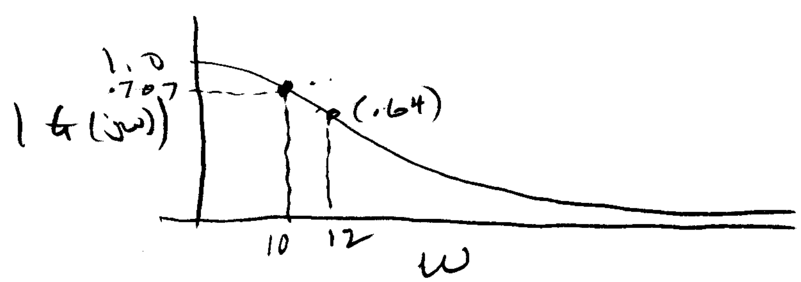
$G(j12) = 0.64\angle -50.2^\circ$

Therefore

$$V_o(t) = |G(j12)|\, 8 \cos(12t + \angle G(j12))$$

$$= (0.64)(8)\cos(12t + (-50.2^\circ))$$

$$V_o(t) = 5.12 \cos(12t - 50.2^\circ)$$



Example 3

This problem shows how to do the previous problem with the aid of MATLAB. We use the same function for $G(j\omega)$,

$$G(s) = \frac{1}{1 + 0.1s} = \frac{1}{0.1s + 1}$$

We write the following program to get the frequency response using Bode in MATLAB.

The output is on the following page.

```
% A simple program to find the frequency of a low
% pass filter using bode of MATLAB.
%  Let the function be
%
%          G(s)  = 1/(0.1s+1)
%
%  call the program bodelow.m;  wlg, Nov, 2002

num = 1;
den = [0.1 1];
bode(num,den)
grid
```

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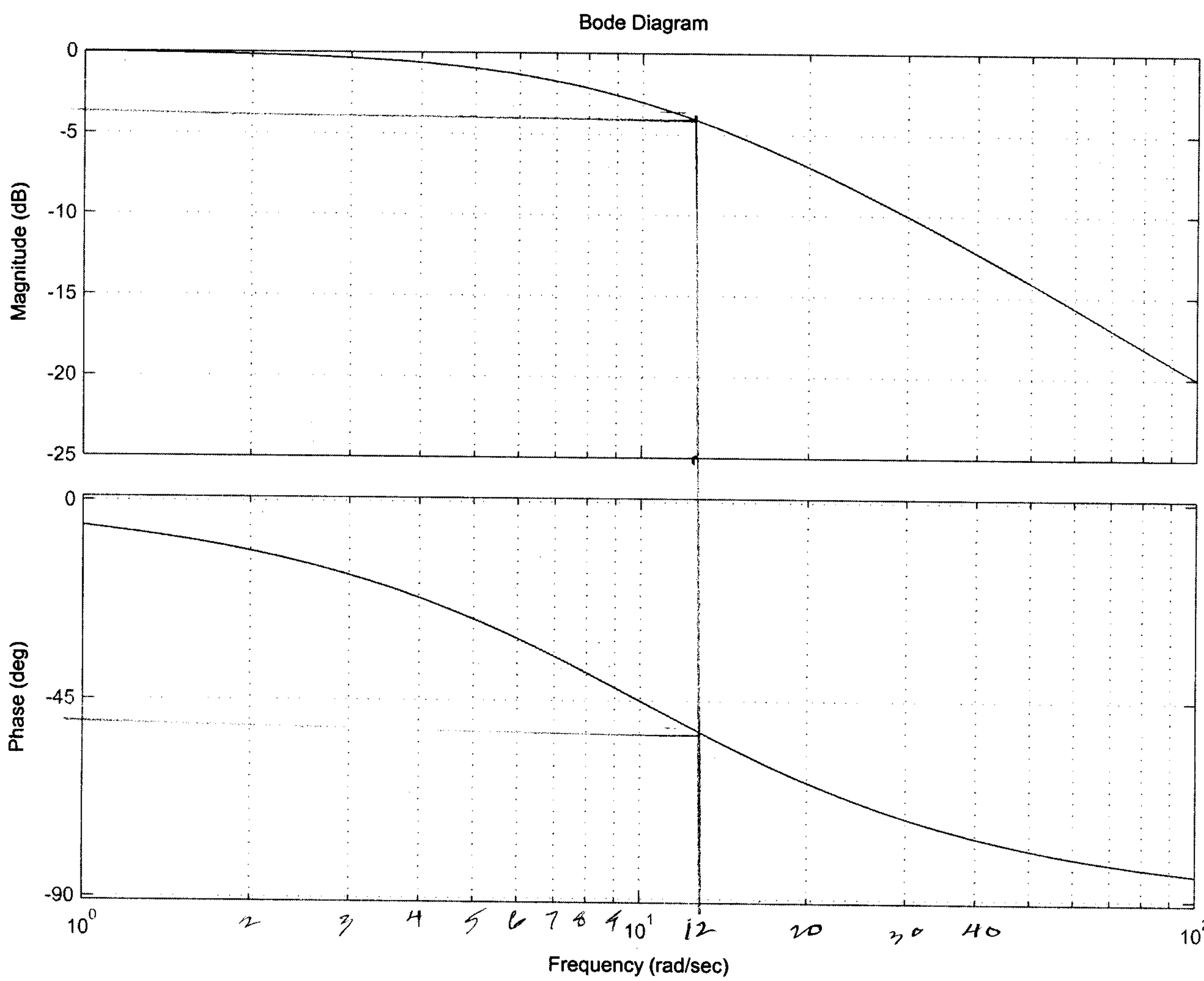

Bode Diagram
Magnitude (dB)
0
-5
-10
-15
-20
-25
Phase (deg)
0
-45
-90
10^0
2
3
4
5
6
7
8
9
10^1
12
20
30
40
10^2
Frequency (rad/sec)

We notice that at $\omega = 12$, the magnitude (in dB) is -4 dB

$$20 \log |G| = -$$

$$|G| = 10^{-4/20} = 0.63$$

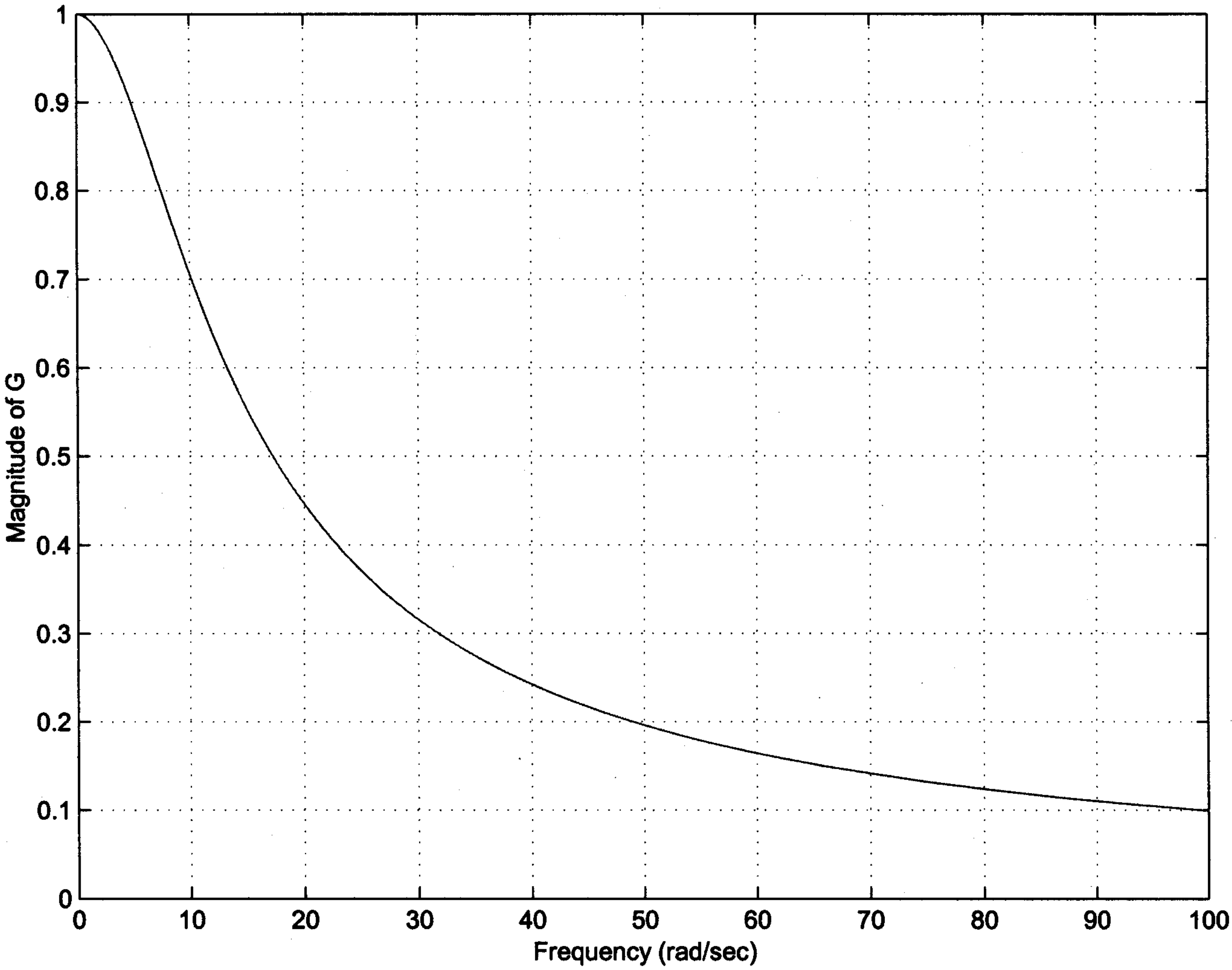
The phase looks like it is ≈ -50. The problem is we are trying to read 12 on a log scale by interpolation. We cannot get a printout directly when we use Bode this way. We can get output if we use freqs. A program for the same function is given below.

The actual program is shown on page 18.23. The plot for $|G|$ is shown on page 18.24 and the angle of G is shown on page 18.25. We still must interpolate for $\omega = 12$. We can make a data dump by using

[w, Gmag, angleG]

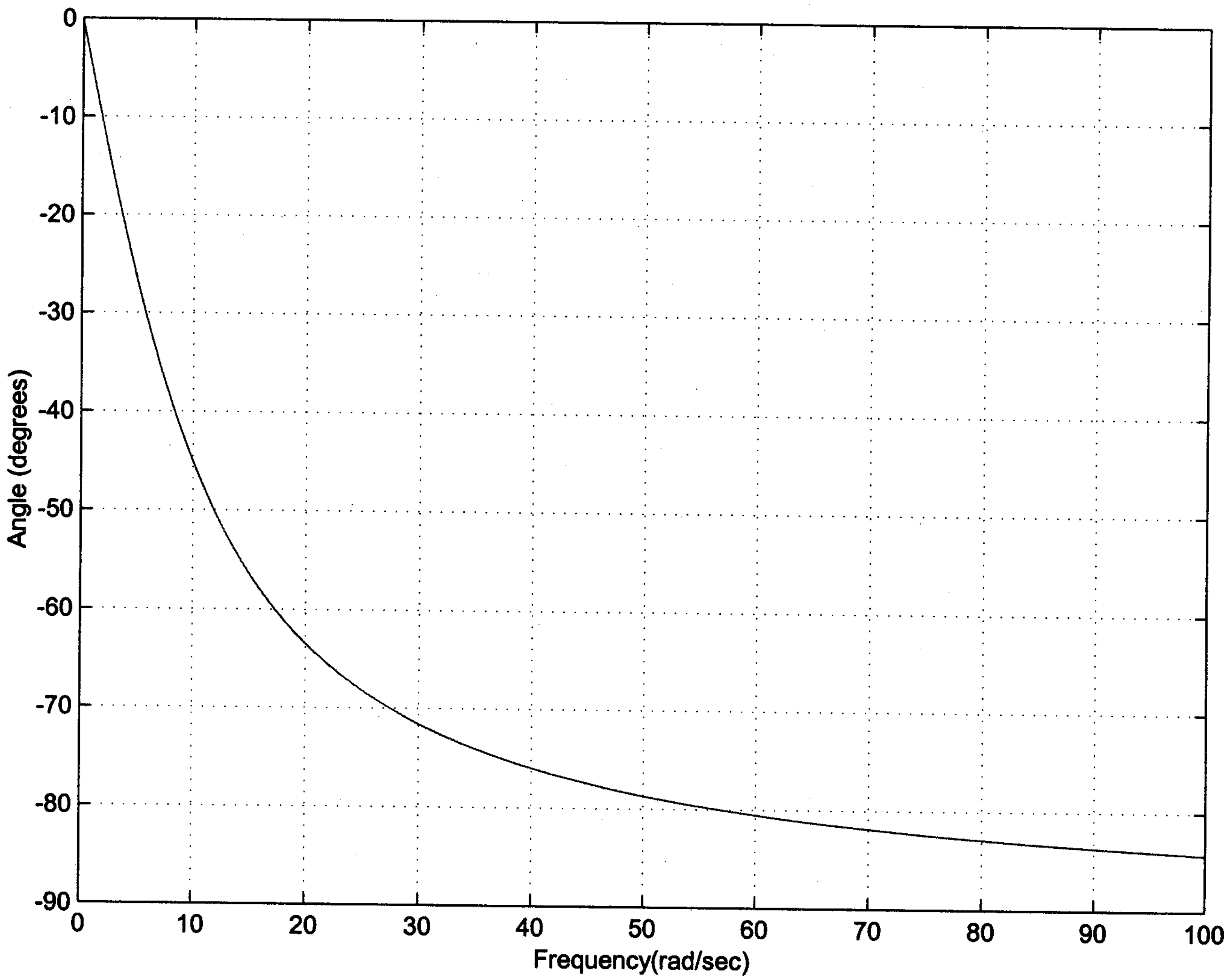
```
%  using freqs to obtain the frequency response
% of a simple low pass filter.
%
%   G(s) = 1/(0.1s+1)
%
%  call the program freqxxs.m
%  wlg: Nov 2002 Office Computer

w = 0:1:20;
num = 1;
den = [0.1 1];

[G,w] = freqs(num,den);

% the above calculates the frequency response at
% 200 values of w.  The w are selected by MATLAB

Gmag = abs(G);
angleG = 57.3*angle(G);

plot(w,Gmag)
grid
ylabel('Magnitude of G')
xlabel('Frequency (rad/sec)')


%plot(w,angleG)
%grid
%ylabel('Angle (degrees)')
%xlabel('Frequency(rad/sec)')
```

18.24



Magnitude of G
Frequency (rad/sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50
60
70
80
90
100

18.25



Angle (degrees)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
0
10
20
30
40
50
60
70
80
90
100
Frequency(rad/sec)

18.25

All we need to do is add this statement to our program, leaving off the ; after the statement. The modified program is shown below.

C:\MATLAB6p1\work\freqxxs.m
November 21, 2002

```
%  using freqs to obtain the frequency response
% of a simple low pass filter.
%
%   G(s) = 1/(0.1s+1)
%
%  call the program freqxxs.m
%  wlg: Nov 2002 Office Computer

w = 0:1:20;
num = 1;
den = [0.1 1];

[G,w] = freqs(num,den);

% the above calculates the frequency response at
% 200 values of w.  The w are selected by MATLAB

Gmag = abs(G);
angleG = 57.3*angle(G);

[w,Gmag,angleG]

%plot(w,Gmag)
%grid
%ylabel('Magnitude of G')
%xlabel('Frequency (rad/sec)')


%plot(w,angleG)
%grid
%ylabel('Angle (degrees)')
%xlabel('Frequency(rad/sec)')
```

Matlab will printout ω, Gmag, angle G in 3 columns for 200 values of ω (log stepping). The page with ω around 12 is shown on page 18.27. We see that at $\omega = 12.03$ we get $|G| = 0.639$ and $\angle G = -50.3$ which are both close to the values we calculated by hand (0.64 and $-50.2°$)

These examples should be helpful in getting you started with using MATLAB to find frequency response.

18.27

4.5529	0.9101	-24.4813
4.7138	0.9045	-25.2399
4.8803	0.8987	-26.0155
5.0526	0.8925	-26.8078
5.2311	0.8861	-27.6165
5.4159	0.8793	-28.4415
5.6072	0.8722	-29.2822
5.8052	0.8648	-30.1384
6.0103	0.8571	-31.0093
6.2226	0.8490	-31.8946
6.4424	0.8407	-32.7935
6.6699	0.8319	-33.7055
6.9055	0.8229	-34.6296
7.1494	0.8135	-35.5652
7.4020	0.8038	-36.5114
7.6634	0.7937	-37.4672
7.9341	0.7834	-38.4317
8.2143	0.7727	-39.4038
8.5045	0.7618	-40.3824
8.8049	0.7505	-41.3666
9.1159	0.7390	-42.3551
9.4379	0.7273	-43.3467
9.7712	0.7152	-44.3404
10.1164	0.7030	-45.3348
10.4737	0.6906	-46.3289
10.8437	0.6779	-47.3213
11.2267	0.6651	-48.3110
11.6232	0.6522	-49.2967
12.0338	0.6391	-50.2773
12.4588	0.6260	-51.2517
12.8989	0.6127	-52.2189
13.3545	0.5994	-53.1777
13.8262	0.5860	-54.1271
14.3146	0.5727	-55.0664
14.8202	0.5593	-55.9944
15.3437	0.5460	-56.9105
15.8857	0.5327	-57.8139
16.4468	0.5195	-58.7038
17.0277	0.5064	-59.5796
17.6291	0.4934	-60.4407
18.2518	0.4805	-61.2866
18.8965	0.4677	-62.1169
19.5640	0.4551	-62.9311
20.2550	0.4427	-63.7289
20.9705	0.4304	-64.5101
21.7112	0.4183	-65.2744
22.4781	0.4065	-66.0216
23.2720	0.3948	-66.7517
24.0940	0.3833	-67.4645
24.9451	0.3721	-68.1601
25.8262	0.3611	-68.8385
26.7384	0.3503	-69.4996
27.6829	0.3397	-70.1437
28.6607	0.3294	-70.7708
29.6730	0.3194	-71.3811
30.7211	0.3095	-71.9748
31.8063	0.2999	-72.5520