ECE-620 Lecture on power system modeling and dynamics including wind turbines

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1. Time scale decomposition

2. Synchronous generator and DFIG-based wind turbine models

3. Small-signal stability analysis (SSSA)

4. Cases of study
   4.1. Load parametrization in SSSA
   4.2. Inertia parametrization in SSSA
   4.3. Effect of wind turbines location on oscillation damping
   4.4. Limit-induced bifurcations in wind farms
1. Time scale decomposition
1. Time scale decomposition

Classification of power system dynamics

- Lightning surge
- Switching surge
- Electro-magnetic transient
- Electro-mechanical transient
- Boiler dynamics

Time [s]

$10^{-7}$ $10^{-5}$ $10^{-3}$ $10^{-1}$ $10^{1}$ $10^{3}$ $10^{5}$

Should our model represent completely all dynamics?

No, it is not recommended due to numerical issues such as numerical instability and excessive simulation time.
1. Time scale decomposition

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No, it is not recommended due to numerical issues such as numerical instability and excessive simulation time.

We will focus on the time frame of electromechanical transients.
1. Time scale decomposition

**General concept.** Consider the following generic system,

\[
\begin{align*}
\dot{x} &= f(x, z) & x(0) &= x^o \\
\dot{z} &= g(x, z) & z(0) &= z^o 
\end{align*}
\]

where \(x \in \mathbb{R}^{n \times 1}\), and \(z \in \mathbb{R}^{m \times 1}\). Assume that the dynamics of \(x\) and \(z\) are distinctive.

To decouple these dynamics, we seek for an integral manifold for \(z = h(x)\) which satisfies the differential equation of \(z\). Thus,

\[
\dot{z} = \frac{\partial z}{\partial x} \dot{x} = \frac{\partial h}{\partial x} f(x, h(x)) = g(x, h(x))
\]

If the initial condition belongs to the manifold, \(z^o = h(x^o)\), we say that the integral manifold is an exact solution of \(\dot{z} = g(x, z)\), and the following reduced order model is an exact model:

\[
\dot{x} = f(x, h(x))
\]
1. Time scale decomposition

A complete linear example. Consider the following second order system (assume that $\varepsilon$ is a small parameter)

$$
\dot{x} = -x + z \\
\varepsilon \dot{z} = -x - z
$$

$x(0) = x^o$ 
$z(0) = z^o$

The system eigenvalues are:

$$
\lambda_1, 2 = -\varepsilon - 1 - \sqrt{\varepsilon^2 - 6\varepsilon + 1}/2\varepsilon
$$

In the limit,

$$
\lambda_1 = \lim_{\varepsilon \to 0} -\varepsilon - 1 - \sqrt{\varepsilon^2 - 6\varepsilon + 1}/2\varepsilon = -1 - 1/2 = -\infty
$$

$$
\lambda_2 = \lim_{\varepsilon \to 0} -\varepsilon - 1 + \sqrt{\varepsilon^2 - 6\varepsilon + 1}/2\varepsilon = 0
$$

Applying L'Hopital's rule we can show that $\lambda_2 \to -2$. As a result, this system has a very fast mode $e^{-t/\varepsilon}$ and a slow mode $e^{-2t}$—we have two distinctive dynamics!
1. Time scale decomposition

A complete linear example. Consider the following second order system (assume that $\varepsilon$ is a small parameter)

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\begin{align*}
\dot{x} &= -x + z \\
\varepsilon \dot{z} &= -x - z
\end{align*}
\]

The system eigenvalues are:

\[
\lambda_{1,2} = \frac{-\varepsilon - 1 \pm \sqrt{\varepsilon^2 - 6\varepsilon + 1}}{2\varepsilon}
\]

In the limit,

\[
\begin{align*}
\lambda_1 &= \lim_{\varepsilon \to 0} \frac{-\varepsilon - 1 - \sqrt{\varepsilon^2 - 6\varepsilon + 1}}{2\varepsilon} = \frac{-1 - 1}{2\varepsilon} = \frac{-1}{\varepsilon} = -\infty \\
\lambda_2 &= \lim_{\varepsilon \to 0} \frac{-\varepsilon - 1 + \sqrt{\varepsilon^2 - 6\varepsilon + 1}}{2\varepsilon} = \frac{0}{0} \text{ indeterminate!}
\end{align*}
\]

Applying L’Hopital’s rule we can show that $\lambda_2 \to -2$. As a result, this system has a very fast mode $e^{-t/\varepsilon}$ and a slow mode $e^{-2t}$—we have two distinctive dynamics!
1. Time scale decomposition

We propose a linear manifold of the form $z = hx$, where $h$ is a real constant. This manifold must satisfy the differential equation of $z$, thus

$$\epsilon \dot{z} = -x - z \quad \Rightarrow \quad \epsilon \frac{\partial z}{\partial x} \dot{x} = -x - hx$$

$$\Rightarrow \epsilon h (-x + hx) = -x - hx$$

$$\Rightarrow (-\epsilon h + \epsilon h^2) x = -(1 + h)x$$

$$\Rightarrow (-\epsilon h + \epsilon h^2) = -(1 + h)$$

$$\Rightarrow \epsilon h^2 + (1 - \epsilon)h + 1 = 0$$

If there exists a solution for $h$, then the manifold $z = hx$ exists.
1. Time scale decomposition

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\[
\Rightarrow (-\epsilon h + \epsilon h^2) = -(1 + h)
\]

\[
\Rightarrow \epsilon h^2 + (1 - \epsilon) h + 1 = 0
\]

If there exists a solution for \( h \), then the manifold \( z = hx \) exists.

The solution for \( h \) is given by:

\[
h(\epsilon) = \frac{-(1 - \epsilon) \pm \sqrt{(1 - \epsilon)^2 - 4\epsilon}}{2\epsilon} \quad \epsilon \in \mathbb{R} \iff \phi(\epsilon) = (1 - \epsilon)^2 - 4\epsilon \geq 0
\]
1. Time scale decomposition

Roots of $\phi(\varepsilon)$ are given by:

$\phi(\varepsilon) = (1 - \varepsilon)^2 - 4\varepsilon = 0$

$\Rightarrow \varepsilon_{1,2} = 3 \pm \sqrt{8}$
1. Time scale decomposition

Roots of $\phi(\varepsilon)$ are given by:

$\phi(\varepsilon) = (1 - \varepsilon)^2 - 4\varepsilon = 0$

$\Rightarrow \varepsilon_{1,2} = 3 \pm \sqrt{8}$

If $\varepsilon < \varepsilon_1$, then we have two solutions for $h$ given by:

$h_1(\varepsilon) = \frac{-(1-\varepsilon)+\sqrt{(1+\varepsilon)^2-4\varepsilon}}{2\varepsilon}$

$h_2(\varepsilon) = \frac{-(1-\varepsilon)-\sqrt{(1-\varepsilon)^2-4\varepsilon}}{2\varepsilon}$

In general, manifolds may not exist and may not be unique.
1. Time scale decomposition

With $\varepsilon \ll \varepsilon_1$, $x$ is the variable associated with the slow mode, while $z$ is associated with the fast mode. As we are interested in the slow mode, we choose the manifold $z = h_1 x$.

In the case $\varepsilon = 0$,

$$\lim_{\varepsilon \to 0} h_1(\varepsilon) = \lim_{\varepsilon \to 0} - (1 - \varepsilon) + \frac{\sqrt{(1 + \varepsilon)^2 - 4\varepsilon}}{2\varepsilon} = -1$$

In the case $\varepsilon \neq 0$, we use a Taylor expansion series to represent $h$ around $-1$. Therefore, the manifold becomes:

$$z = h(\varepsilon)x$$

where $h(\varepsilon) = h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + ...$

By using this manifold in the differential equation of $z$, we get:

$$\varepsilon \dot{z} = -x - z$$

$$\Rightarrow \varepsilon \frac{\partial z}{\partial x} \dot{x} = -x - h(\varepsilon)x = -(1 + h(\varepsilon))x$$
1. Time scale decomposition

\[ \varepsilon \left( h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + ... \right) \left( -x + [h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + ...] \ x \right) = \]
\[ - (1 + h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + ...) x \]
\[ \Rightarrow \left( h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + ... \right) \left( (-1 + h_0) \varepsilon + h_1 \varepsilon^2 + h_2 \varepsilon^3 + ... \right) = \]
\[ - (1 + h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + ...) \]

By equating terms based on the powers of \( \varepsilon \), we obtain:

\[ \varepsilon^0 : 1 + h_0 = 0 \Rightarrow h_0 = -1 \]
\[ \varepsilon^1 : h_0(-1 + h_0) = -h_1 \Rightarrow h_1 = -2 \]
\[ \varepsilon^2 : h_0 h_1 + h_1(-1 + h_0) = -h_2 \Rightarrow h_2 = -6 \], and so on

To sum up, we obtain the following different approximations:

\[ z \approx -x \]  
Zero-order manifold

\[ z \approx -(1 + 2 \varepsilon)x \]  
First-order manifold

\[ z \approx -(1 + 2 \varepsilon + 6 \varepsilon^2)x \]  
Second-order manifold
1. Time scale decomposition

The reduced-order model becomes:

\[
\begin{align*}
\dot{x} &= -x + z \\
z &= hx \\
x(0) = x_0, z(0) = z_0
\end{align*}
\]

\[\Rightarrow \dot{x} = -(1 - h)x \Rightarrow x(t) = x_0 e^{-(1-h)t} \]

Considering \(\varepsilon = 0.1\), we have:

Zero-order manifold : \(x(t) = x_0 e^{-2t}\)

First-order manifold : \(x(t) = x_0 e^{-(2+2\varepsilon)t} = x_0 e^{-2.2t}\)

Second-order manifold : \(x(t) = x_0 e^{-(2+2\varepsilon+6\varepsilon^2)t} = x_0 e^{-2.26t}\)

Exact solution : \(x(t) = \left(\frac{7.7}{6.4} x_0 + \frac{z_0}{6.4}\right) e^{-2.3t} - \left(\frac{1.3}{6.4} x_0 + \frac{z_0}{6.4}\right) e^{-8.7t}\)
1. Time scale decomposition

A general linear example. In an n-dimensional linear case we have:

\[
\dot{x} = Ax + Bz \\
\varepsilon \dot{z} = Cx + Dz 
\]

where \( \varepsilon \) is a small constant; \( x \in \mathbb{R}^{n \times 1} \); \( z \in \mathbb{R}^{m \times 1} \); and \( A, B, C, \) and \( D \) are all matrices with proper dimensions.

A zero-order manifold is obtained for \( z \) by letting \( \varepsilon \) be equal to zero and solving in terms of \( x \) as follows:

\[
\varepsilon \dot{z} = 0 = Cx + Dz \Rightarrow z = -D^{-1}Cx \\
\Rightarrow \dot{x} = \left( A - BD^{-1}C \right) x 
\]

We assume that \( D \) is full rank and, therefore, invertible.
1. Time scale decomposition

A graphical nonlinear example. Initial value problem with $\varepsilon = 0.05$

\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_3 \\
\dot{x}_2 &= -x_2 - x_3 \\
\varepsilon \dot{x}_3 &= \tan^{-1}(1 - x_1 + x_2 - x_3)
\end{align*}
\]

\[
\Rightarrow \quad \dot{x}_3 = \frac{\tan^{-1}(1 - x_1 + x_2 - x_3)}{\varepsilon}
\]
1. Time scale decomposition

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\end{align*}
\]

\[\Rightarrow \dot{x}_3 = \frac{\tan^{-1} (1 - x_1 + x_2 - x_3)}{\varepsilon}\]

Note there exists $\delta$, small, such that if $|1 - x_1 + x_2 - x_3| >> \delta$, then $|\dot{x}_3|$ gets large.
1. Time scale decomposition

A graphical nonlinear example. Initial value problem with $\varepsilon = 0.05$

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\dot{x}_1 &= -x_1 + x_3 \\
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\end{align*}
\]

\[
\Rightarrow \dot{x}_3 = \frac{\tan^{-1}(1 - x_1 + x_2 - x_3)}{\varepsilon}
\]

Note there exists $\delta$, small, such that if $|1 - x_1 + x_2 - x_3| \gg \delta$, then $|\dot{x}_3|$ gets large.

Let $M$ be the set defined by $\{x_1, x_2, x_3 \in \mathbb{R} : 1 - x_1 + x_2 - x_3 = 0\}$

In state space, we say:

If the distance between the system trajectory and $M$ is much larger than $\delta$, then $x_3$ will exhibit a fast response.
1. Time scale decomposition

Example (cont.). Dynamic simulation with $\varepsilon = 0.05$

State-space trajectories from 4 different initial points:

Point 1  
$x_1 = 0.8, \quad x_2 = -0.2, \\
x_3 = 0.5$

Point 2  
$x_1 = 0.0, \quad x_2 = 0.0, \\
x_3 = 0.0$

Point 3  
$x_1 = 0.6, \quad x_2 = -0.8, \\
x_3 = 0.0$

Point 4  
$x_1 = 0.0, \quad x_2 = -0.8, \\
x_3 = 1.0$
1. Time scale decomposition

Example (cont.). Considering a zero-order manifold for $z$ ($\varepsilon = 0$)

$$M : \{x_1, x_2, x_3 \in \mathbb{R} : 1 - x_1 + x_2 - x_3 = 0\}$$

Simplified model:

$$\dot{x}_1 = -x_1 + x_3$$
$$\dot{x}_2 = -x_2 - x_3$$
$$0 = 1 - x_1 + x_2 - x_3$$

Outside $M$ Dynamic of $x_3$ infinitely fast, i.e.,
$$\dot{x}_3 = \infty$$

Inside $M$ Dynamic of $x_3$ in the same time scale of $x_1$ and $x_2$
2. Synchronous generator and DFIG-based wind turbine models
2. Synchronous generator and wind turbine models

Power system models. We typically use a zero-order manifold for the fast dynamics related to, for example, stator flux linkages of synchronous generators. Slow dynamics are fully modeled such as those related to the angular motion of synchronous generators. As a result, power systems are described by a set of differential-algebraic equations (DAEs) of the form:

\[
\begin{align*}
\dot{x} &= f(x, y, \mu) \\
0 &= g(x, y, \mu)
\end{align*}
\]

- $x$ Vector of state variables
  - Angular speed of SGs
  - Angular rotor position of SGs
  - Rotor flux linkages of SGs
  - Variables of SG’s exciters
  - Variables of SG’s governors
  - Variables of turbines

- $y$ Vector of algebraic variables
  - Armature current of SGs
  - Bus voltages

- $\mu$ Vector of parameters
  - Electrical demand
  - Wind speed, solar radiation, others
2. Synchronous generator and wind turbine models

Two-axis model of SGs

\[
\begin{align*}
T'_{do} \frac{dE'_q}{dt} &= -E'_q - (X_d - X'_d)I_d + E_{fd} \\
T'_{qo} \frac{dE'_d}{dt} &= -E'_d + (X_q - X'_q)I_q \\
\frac{d\delta}{dt} &= \omega - \omega_s \\
2H \frac{d\omega}{\omega_s} &= T_M - E'_d I_d - E'_q I_q - (X'_q - X'_d)I_dI_q
\end{align*}
\]

Manifold is associated with the following equivalent circuit:

![Equivalent Circuit Diagram]

Color definition:

- **Blue**: State variables
- **Green**: Algebraic variables
- **Red**: Inputs
2. Generator models

\( E_{fd} \): Exciter’s model (IEEE Type 1)

\[
T_E \frac{dE_{fd}}{dt} = - \left( K_E + S_E(E_{fd}) \right) E_{fd} + V_R
\]

\[
T_A \frac{V_R}{dt} = -V_R + + K_A R_f - \frac{K_A K_F}{T_F} E_{fd} + K_A (V_{ref} - V_t)
\]

\[
T_F \frac{dR_f}{dt} = -R_f + \frac{K_F}{T_F} E_{fd} \quad \text{con} \quad V_R^{min} \leq V_R \leq V_R^{max}
\]
2. Synchronous generator and wind turbine models

$T_m$: Governor’s model (IEESGO)

\[
\begin{align*}
T_4 \frac{dT_m}{dt} &= -T_m + PV \\
T_1 \frac{dy_1}{dt} &= -y_1 + \frac{1}{R_D} \left( \frac{\omega}{\omega_s} - 1 \right) \\
T_3 \frac{dy_3}{dt} &= -y_3 + y_1 \\
y_2 &= \left( 1 - \frac{T_2}{T_3} \right) y_3 + \frac{T_2}{T_3} y_1
\end{align*}
\]

DAEs considering the particular case of $K_2 = 0$

\[
\begin{align*}
PV &= P_C - y_2, \text{ if } P_{\text{min}} \leq P_C - y_2 \leq P_{\text{max}} \\
PV &= P_{\text{max}}, \text{ if } P_C - y_2 > P_{\text{max}} \\
PV &= P_{\text{min}}, \text{ if } P_C - y_2 < P_{\text{min}}
\end{align*}
\]
2. Synchronous generator and wind turbine models

Two-axis model of a DFIG

\[
\begin{align*}
\frac{dE'_{qD}}{dt} &= -\frac{1}{T'_0} \left( E'_{qD} + (X_s - X'_s)I_{ds} \right) + \omega_s \frac{X_m}{X_r} V_{dr} \\
&\quad - (\omega_s - \omega_r) E'_{dD}
\end{align*}
\]

\[
\begin{align*}
\frac{dE'_{dD}}{dt} &= -\frac{1}{T'_0} \left( E'_{dD} - (X_s - X'_s)I_{qs} \right) - \omega_s \frac{X_m}{X_r} V_{qr} \\
&\quad + (\omega_s - \omega_r) E'_{qD}
\end{align*}
\]

\[
\frac{d\omega_r}{dt} = \frac{\omega_s}{2H_D} \left[ T_m - E'_{dD}I_{ds} - E'_{qD}I_{qs} \right]
\]

Manifold is associated with the following equivalent circuit:

Color definition:

- **Blue**: State variables
- **Green**: Algebraic variables
- **Red**: Inputs

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2. Synchronous generator and wind turbine models

Zero-axis model of a DFIG

\[
\frac{d\omega_r}{dt} = \frac{\omega_s}{2H_D} [T_m - X_m I_{qs} I_{dr} + X_m I_{ds} I_{qr}] \\
\omega_s = \frac{\omega_s - \omega_r}{\omega_s}
\]

Manifold is associated with the following equivalent circuit:

Color definition:
- Blue: State variables
- Green: Algebraic variables
- Red: Inputs
2. Synchronous generator and wind turbine models

$V_{qr}$, $V_{dr}$, $T_m$: P-Q controllers and pitch angle controller
2. Synchronous generator and wind turbine models

Control schemes to participate in frequency regulation (P1 and P2)

**P1: Inertial response**

- Maximum power tracking
- Supervisory voltage controller
- Washout Filter
- Wind turbine

**P2: Power reserve**

- Pitch-angle controller
- Wind turbine

---

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3. Small-signal stability analysis
   (Electromechanical oscillations)
3. Small-signal stability analysis

Consider the aforementioned generic representation by the DAEs:

\[
\begin{align*}
\dot{x} &= f(x, y, \mu) \\
0 &= g(x, y, \mu)
\end{align*}
\]  

Given \( \mu \), assume that the system is in equilibrium at the point \((x^e, y^e)\):

\[
\begin{align*}
0 &= f(x^e, y^e, \mu) \\
0 &= g(x^e, y^e, \mu)
\end{align*}
\]

In a neighborhood of \((x^e, y^e)\), the system modeled by \((\ast)\) can be approximately represented by:

\[
\begin{align*}
\dot{x} &= f(x^e, y^e, \mu) + \nabla f_x(x^e, y^e, \mu) \Delta x + \nabla f_y(x^e, y^e, \mu) \Delta y + \text{H.O.T.} \\
0 &= g(x^e, y^e, \mu) + \nabla g_x(x^e, y^e, \mu) \Delta x + \nabla g_y(x^e, y^e, \mu) \Delta y + \text{H.O.T.}
\end{align*}
\]

where \( \Delta x = x - x^e \) and \( \Delta y = y - y^e \). Note also that \( \dot{x} = \frac{d(x-x^e)}{dt} = \Delta \dot{x} \)
3. Small-signal stability analysis

By neglecting H.O.T., we obtain:

\[
\begin{bmatrix}
\Delta \dot{x} \\
0
\end{bmatrix}
= \begin{bmatrix}
\nabla f_x(x^e, y^e, \mu) & \nabla f_y(x^e, y^e, \mu) \\
\nabla g_x(x^e, y^e, \mu) & \nabla g_y(x^e, y^e, \mu)
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
\]

To eliminate algebraic variables, we proceed as follows:

\[
0 = \nabla g_x(x^e, y^e, \mu) \Delta x + \nabla g_y(x^e, y^e, \mu) \Delta y
\Rightarrow \Delta y = -\nabla g_y^{-1}(x^e, y^e, \mu) \nabla g_x(x^e, y^e, \mu) \Delta x
\]

Finally,

\[
\Delta \dot{x} = \left(\nabla f_x(x^e, y^e, \mu) - \nabla f_y(x^e, y^e, \mu) \nabla g_y^{-1}(x^e, y^e, \mu) \nabla g_x(x^e, y^e, \mu)\right) \Delta x
\]

\[
A_s(x^e, y^e, \mu)
\]

This is a linear homogenous system, and the spectrum of matrix \(A_s\) gives us information about the system stability around the equilibrium point \(\Delta x = 0 (x = x^e), \Delta y = 0 (y = y^e)\)
3. Small-signal stability analysis

To take into account a small perturbation, the following initial value problem is considered:

\[ \Delta \dot{x} = A_s \Delta x \]
\[ \Delta x(t_0) = x_0 \neq 0 \]

The explicit solution of this problem is given by:

\[ \Delta x(t) = e^{A_s(t-t_0)} \Delta x_0, \ \forall t \geq t_0 \]

In general, \( A_s \) is a full matrix, and obtaining the exponential of \( A_s \) is not an easy task. However, a solution can be easily obtained using a similarity transformation.
3. Small-signal stability analysis

Assume that $A_s \in \mathbb{R}^{n \times n}$ has $n$ distinct eigenvalues $\lambda_i \in \mathbb{C}$ \( \forall i = \{1, 2, ..., n\} \). Then, each $\lambda_i$ has a unique right eigenvector, $v_i \in \mathbb{C}^{n \times 1}$, and left eigenvector, $w_i \in \mathbb{C}^{n \times 1}$, defined by:

$$A_s v_i = \lambda_i v_i$$
$$w_i^T A_s = \lambda_i w_i^T$$

Note that:
- Right (left) eigenvectors are linearly independent. Defining $V = [v_1, v_2, ..., v_n]$ and $W = [w_1, w_2, ..., w_n]$, we say that $V$ and $W$ are full rank matrices.
- The set of right and left eigenvectors are orthogonal. To prove this, consider $\lambda_i \neq \lambda_j$, then

Multiplying ($\ddagger$) from the right by $v_i$, we have:

$$A_s v_i = \lambda_i v_i$$
$$w_j^T A_s = \lambda_j w_j^T$$

\[ w_j^T \lambda_i v_i = \lambda_j w_j^T v_i \]
\[ (\lambda_i - \lambda_j) w_j^T v_i = 0 \Rightarrow w_j^T v_i = 0, \forall i \neq j \]
3. Small-signal stability analysis

As we are free to choose any magnitude for the eigenvectors, we assign a magnitude in such a way that the set of right and left eigenvectors are orthonormal. We define \( w_j^T v_i = \delta_{ji} \) (Kronecher delta), or in matrix form:

\[
\begin{bmatrix}
  w_1 & w_2 & \ldots & w_n \\
\end{bmatrix}^T
\begin{bmatrix}
  v_1 & v_2 & \ldots & v_n \\
\end{bmatrix} = I_n \Rightarrow W^T = V^{-1}
\]

Finally,

\[
A_s v_1 = \lambda_1 v_1 \\
A_s v_2 = \lambda_2 v_2 \\
\vdots \\
A_s v_n = \lambda_n v_n
\]

\[
\Rightarrow A_s \begin{bmatrix}
  v_1 & v_2 & \ldots & v_n \\
\end{bmatrix} = \begin{bmatrix}
  v_1 & v_2 & \ldots & v_n \\
\end{bmatrix} \begin{bmatrix}
  \lambda_1 & 0 & \ldots & 0 \\
  0 & \lambda_2 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & \lambda_n \\
\end{bmatrix}
\]

\[
\Rightarrow V^{-1} A_s V = \Lambda \\
\Rightarrow W^T A_s V = \Lambda : \text{Diagonalization of } A_s!
\]
3. Small-signal stability analysis

We define a transformed vector of state variables as \( q = W^T \Delta x \). Note that \( \Delta x = W^{-T} q = V q \). Then, the transformed state equations become:

\[
\Delta \dot{x} = A_s \Delta x \Rightarrow W^T \Delta \dot{x} = W^T A_s \Delta x \\
\Rightarrow \dot{q} = W^T A_s V q \Rightarrow \dot{q} = \Lambda q
\]

whose solution is given by:

\[
q(t) = e^{\Lambda t} W^T \Delta x(0) = e^{\Lambda t} \begin{bmatrix} w_1^T \Delta x(0) \\ \vdots \\ w_n^T \Delta x(0) \end{bmatrix}, \forall \ t \geq 0
\]

As \( \Lambda \) is diagonal, \( e^{\Lambda t} \) is also diagonal, and the transformed state variables are decoupled. Therefore,

\[
\forall \ i = \{1, 2, \ldots, n\}, \ q_i(t) = e^{\lambda_i t} w_i^T \Delta x(0)
\]

The term \( e^{\lambda_i t} \) is known as **mode** \( i \)
3. Small-signal stability analysis

By transforming back to the original state variables, we obtain:

\[ \Delta x(t) = Vq = [v_1 \ v_2 \ \ldots \ v_n] \begin{bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{bmatrix} = \sum_{i=1}^{n} q_i(t)v_i \]

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\[ \Rightarrow \forall \ k = \{1, 2, ..., n\}, \ \forall \ t \geq 0, \ \Delta x_k(t) = \sum_{i=1}^{n} e^{\lambda_i t} w_i^T \Delta x(0)v_i(k) \]
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Remarks

a. The appearance of mode $i$ in the state variable $\Delta x_k$ depends on the perturbation $\Delta x(0)$ (excitation of the mode)
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Remarks

a. The appearance of mode \( i \) in the state variable \( \Delta x_k \) depends on the perturbation \( \Delta x(0) \) (excitation of the mode)

b. The relative phase of the mode \( i \) in the state variable \( \Delta x_k \) depends on the phase of \( v_i(k) \) (mode shape)
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\[
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b. The relative phase of the mode \(i\) in the state variable \(\Delta x_k\) depends on the phase of \(v_i(k)\) (mode shape)

c. The intensity of the mode \(i\) on the state variable \(\Delta x_k\) depends on the magnitude of \(w_i^T \Delta x(0)v_i(k)\)
3. Small-signal stability analysis

Common practice (power engineering community): We define the matrix of participation factors as the element-wise product between $W$ and $V$:

$$ P = W \circ V = \begin{bmatrix}
  w_1(1)v_1(1) & w_2(1)v_2(1) & \ldots & w_n(1)v_n(1) \\
  w_1(2)v_1(2) & w_2(2)v_2(2) & \ldots & w_n(2)v_n(2) \\
  \vdots & \vdots & \ddots & \vdots \\
  w_1(n)v_1(n) & w_2(n)v_2(n) & \ldots & w_n(n)v_n(n)
\end{bmatrix} = [p_{ki}] $$

In the particular case that $\Delta x(0) = e_k = [0 \ldots 1 \ldots 0]^T$

$$ \Rightarrow w_i^T \Delta x(0) = w_i^T e_k = w_i(k) $$

$$ \Rightarrow \Delta x_k(t) = \sum_{i=1}^{n} e^{\lambda_i t} w_i(k)v_i(k) $$

$$ \Rightarrow \Delta x_k(t) = \sum_{i=1}^{n} e^{\lambda_i t} p_{ki} $$

intensity of mode $i$ on variable $x_k$
3. Small-signal stability analysis

Electromechanical oscillations

- This refers to low frequency oscillations that occur in synchronous generators (typically between 0.1 to 3 Hz)
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  - Intra-plant mode (typically between 2 to 3 Hz)
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- Oscillations must have a damping ratio of at least 10% in normal operation, and at least 5% after a single contingency
3. Small-signal stability analysis

Natural frequency ($\Omega$), damping ratio ($\sigma$), oscillation frequency ($f_o$)

Consider the mode $e^{\lambda_i t}$ with $\lambda_i = \lambda_{i,x} + j \lambda_{i,y} \in \mathbb{C}$

\[
\Omega_i \triangleq \sqrt{\lambda_{i,x}^2 + \lambda_{i,y}^2}
\]

\[
\sigma_i \triangleq \frac{-\lambda_{i,x}}{|\lambda_i|} = -\frac{\lambda_{i,x}}{\Omega_i} = -\cos \theta_i
\]

$\Rightarrow \lambda_{i,x} = -\sigma_i \Omega_i$

$\Rightarrow \lambda_{i,y} = \sqrt{|\lambda_i|^2 - \lambda_{i,x}^2} = \sqrt{1 - \sigma_i^2 \Omega_i}$

$\Rightarrow f_{i,o} \triangleq \frac{\lambda_{i,y}}{2\pi} = \frac{\sqrt{1 - \sigma_i^2 \Omega_i}}{2\pi}$

Define $A_i \triangleq w_i^T \Delta x(0) v_i(k) = |A_i| e^{j\alpha_i}$. We can prove that:

\[
\Delta x_k(t) = \sum_{i=1}^{n} e^{\lambda_i t} w_i^T \Delta x(0) v_i(k) = \sum_{i=1}^{n} |A_i| e^{-\sigma_i \Omega_i t} \cos \left( \sqrt{1 - \sigma_i^2} \Omega_i t + \alpha_i \right)
\]
4. Cases of study
4. Case of study: 15-bus test system

2 SGs: Two-axis model, IEESGO governor, IEEE Type-1 exciter
5 WTGs: Supervisory control, participation in frequency regulation
### 4. Cases of study: Base case

#### Dominant modes - base case

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>$\sigma$ %</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.165 \pm j 6.524$</td>
<td>2.53</td>
<td>Electromechanical</td>
</tr>
<tr>
<td>$-0.468 \pm j 1.575$</td>
<td>28.5</td>
<td>SG1 and SG2’s exciters</td>
</tr>
<tr>
<td>$-0.404 \pm j 0.723$</td>
<td>48.8</td>
<td>SG1 and SG2’s exciters</td>
</tr>
<tr>
<td>$-0.196 \pm j 0.165$</td>
<td>76.5</td>
<td>SG1 and SG2’s governors</td>
</tr>
</tbody>
</table>

### P1: Frequency regulation through inertial response

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>$\sigma$ %</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.188 \pm j 6.485$</td>
<td>2.90</td>
<td>Electromechanical</td>
</tr>
<tr>
<td>$-0.468 \pm j 1.575$</td>
<td>28.5</td>
<td>SG1 and SG2’s exciters</td>
</tr>
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</tr>
<tr>
<td>$-0.075 \pm j 0.128$</td>
<td>50.7</td>
<td>SG1 and SG2’s governors</td>
</tr>
</tbody>
</table>
4. Cases of study: Load parametrization

Power transfer from SG2 to bus 8

The abrupt change in eigenvalues pathway is due to the hitting of the supervisory control limits.

The supervisory control hits its limits when the load at bus 8 is 178 [MW]

Black-line: No participation in freq. control / Red-line: Participation through inertial response
4. Cases of study: Inertia parametrization

$H_1$ and $H_2$ as parameters (WF at bus 4)

The point A corresponds to the base case when WTGs do not participate in frequency regulation.

The point D corresponds to the base case when WTGs participate in frequency regulation through inertial response.

Dashed-line: No participation in freq. control / Solid-line: Participation through inertial response
4. Case of study: Effect of WF’s location

Now, WF connected at bus 7

---

Power system diagram with key details:
- SG 1
  - $H_1 = 10 \, [s]
  - $P_{G2} = 163 \, MW$
  - $|\bar{V}_2| = 1 \, \text{p.u.}$
- SG 2
  - $H_2 = 25 \, [s]$
- Wind farm with buses regulated by supervisory control
- Power base: 100 MVA
- Rated frequency: 60 Hz
- $V_{15} = 1 \angle 0^\circ \, \text{p.u.}$ (slack bus)
4. Case of study: Inertia parametrization

Inertia of SGs as parameters (two cases: WF at bus 4 and bus 7)
4. Case of study: Summary

Remarks

✓ Two schemes to make WTGs participate in frequency regulation are studied: P1 (inertial response) and P2 (power reserve)
4. Case of study: Summary

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  - P2 does not alter the typical relation between system inertia and damping ratio—the larger $H$, the larger $\sigma$
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✓ Results show that locating WTGs closer to SGs with less inertia will increase the system damping ratio the most. In large systems with many electromechanical modes, determining the location of WTGs is still an open question
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✓ Discontinuities due to variable limits require exhaustive evaluation, as these may lead to the sudden loss of system stability. This possible outcome is discussed in the following case, where WF is connected instead at bus 9
4. Case of study: Limit-induced bifurcation

Wind turbines

Qref (reactive power reference)

Supervisory control

Ideal voltage transformer

PCC

Vref

100+j35 MVA

90+j30 MVA

125+j50 MVA

\[ \bar{V}_{15} = 1 \angle 0^\circ \text{ p.u. (slack bus)} \]

\[ V_{15} = 100 \text{ MVA} \]

\[ V_{15} = 60 \text{ Hz} \]

\[ P_{G2} = 163 \text{ MW} \]

\[ |V_2| = 1 \text{ p.u.} \]

\[ P_{G3} = 85 \text{ MW} \]

\[ |V_3| = 1 \text{ p.u.} \]
4. Case of study: Limit-induced bifurcation

Power transfer from SG1 to load at bus 8. Load at bus 8 increases from 200 to 400 MW

Without supervisory control

Electromechanical mode

HB point

Voltage mode

With supervisory control

Electromechanical mode

LIB point

Voltage mode
4. Case of study: Limit-induced bifurcation

Zoom in of previous trajectory with supervisory voltage control

This unstable eigenvalue trajectory abruptly appears when $Q_{\text{ref}} = Q_{\text{max}}$ (voltage mode).

Stable eigenvalue trajectories (electromechanical modes)

LIB point

$Q_{\text{ref}} = Q_{\text{min}}$

$Q_{\text{min}} < Q_{\text{ref}} < Q_{\text{max}}$

$Q_{\text{ref}} = Q_{\text{max}}$
4. Case of study: Limit-induced bifurcation

Trajectories of fixed voltage and fixed reactive power \( (Q_{max}) \)

- Stable trajectory
- Unstable trajectory

\( (a) \) Loading trajectory when \( Q_{ref}=Q_{max} \)

\( (b) \) Loading trajectory when \( V=V_{ref} \)

\( (c) \) Loading trajectory when hitting limit does not cause instability

Supervisory control reaches its limit

PV trajectory when \( Q=Q_{max} \)

HB point

LIB point

Stable trajectory

Unstable trajectory

\( V_{ref} \)

\( V_{ref} \)
4. Case of study: Limit-induced bifurcation

PV curve and limit-induced bifurcation

- Geometric trajectory when $Q_{\text{ref}} = Q_{\text{min}}$ (classical PV curve)
- Geometric trajectory when $Q_{\text{ref}} = Q_{\text{max}}$ (classical PV curve)
- Supervisory control regulates bus voltage
- Lower limit binds
- Upper limit binds
- HB point trajectory if $Q_{\text{ref}}$ is constant such that $Q_{\text{min}} < Q_{\text{ref}} < Q_{\text{max}}$
References

4. H. Pulgar-Painemal, Wind farm model for power system stability analysis, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2010
Questions?