1. Continuous-time Fourier transform (CTFT) of aperiodic signals. (Aperiodic signals can be interpreted as periodic signals with the period $T \to \infty$, then the frequency $\omega_0 \to 0$, so the harmonics are infinitely close to each other. As a result, the summation in CTFS becomes the integral

(a) The definition:
- The inverse Fourier Transform (the synthesis equation):
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \]
- The Fourier Transform (the analysis equation):
  \[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \]

(b) Relationship between the FS coefficients, $a_k$, and the Fourier transform $X(j\omega)$:
- The FS coefficient of a periodic signal can be expressed in terms of equally spaced samples of the Fourier transform of one period of the periodic signal.
  \[ a_k = \frac{1}{T} X(j\omega)|_{\omega = k\omega_0} \]
- The FT of a periodic signal with FS coefficients \{a_k\} can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the $k$th harmonic frequency $k\omega_0$ is $2\pi$ times the $k$th Fourier series coefficient $a_k$.
  \[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \]

That way, \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \] would be exactly \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} \]
(c) CTFT convergence. (similar to the discussion of convergence of FS)

- Let \( \hat{x}(t) \) be the estimate of \( x(t) \) calculated from the synthesis equation,
  \( \hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \), and \( e(t) = x(t) - \hat{x}(t) \)
- \( x(t) \) is square integrable: if \( \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \), then \( \int_{-\infty}^{\infty} |e(t)|^2 dt \to 0 \). That is, the energy of the error is zero.
- The Dirichlet condition: \( x(t) \) is absolute integrable and well-behaved (i.e., finite number of minima/maxima, finite number of finite discontinuities). If \( \int_{-\infty}^{\infty} |x(t)|dt < \infty \), then \( e(t) \to 0 \) except at discontinuities. That is, \( \hat{x}(t) \) approaches \( x(t) \) at every moment of \( t \) except at discontinuities.
- Gibb’s phenomenon still exists at discontinuities.

(d) Properties of FT and FT pairs (see handout in class): Need to know how to use the tables of properties and transform pairs to solve problems.

(e) System characterization by linear constant-coefficient differential equations (LCDE) - finding the frequency response.

- In the family of LTI systems, a subset of which can be represented using LCDE.
  \[
  \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}
  \]
  where \( N \) is the order of the output and \( M \) is the order of the input.
  We often assume \( N \geq M \).
- Use LCDE to find the frequency response \( H(j\omega) \).
  \[
  H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}
  \]

2. Discrete-time Fourier transform (DTFT) of aperiodic signals

(a) The definition:

- The inverse Fourier Transform (the synthesis equation):
  \[
  x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega
  \]
- The Fourier Transform (the analysis equation):
  \[
  X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}
  \]
(b) Relationship between the DTFS coefficients, \( a_k \), and the DTFT, \( X(e^{j\omega}) \).

- The FS coefficient of a periodic signal can be expressed in terms of equally spaced samples of the Fourier transform of one period of the periodic signal.

\[
a_k = \frac{1}{N} X(e^{j\omega})\big|_{\omega = k\omega_0}
\]

- The FT of a periodic signal with FS coefficients \( \{ a_k \} \) can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the \( k \)th harmonic frequency \( k\omega_0 \) is \( 2\pi \) times the \( k \)th Fourier series coefficient \( a_k \).

\[
X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)
\]

(c) DTFT vs. CTFT

- DTFT is periodic in \( \omega \) with \( 2\pi \) as period, which is why \( x[n] \) uses finite interval of integration in the synthesis equation
- Notation wise, DTFT uses \( X(e^{j\omega}) \) while CTFT uses \( X(j\omega) \).

(d) DTFT convergence.

- Absolute summable: \( \sum_{n=-\infty}^{\infty} |x[n]| < \infty \)
- Square summable (signal has finite energy): \( \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \)

(e) Properties of DTFT and DTFT pairs: Need to know how to use the tables of properties and transform pairs to solve problems.

(f) System characterization by linear constant-coefficient difference equation (LCDE) - finding the frequency response.

- Generalized representation:

\[
\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]
\]

- Use LCDE to find the frequency response \( H(e^{j\omega}) \)

\[
H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}
\]

(g) Duality.
<table>
<thead>
<tr>
<th></th>
<th>Frequency domain (analysis equations)</th>
<th>Dualities</th>
<th>Time domain (synthesis equations)</th>
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</thead>
<tbody>
<tr>
<td><strong>CTFT</strong></td>
<td>( X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt )</td>
<td>( \iff )</td>
<td>( x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega )</td>
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</tr>
<tr>
<td></td>
<td>periodic in frequency (with period ( 2\pi )) continuous in frequency</td>
<td>( \iff )</td>
<td>aperiodic in time discrete in time</td>
</tr>
<tr>
<td><strong>CTFS</strong></td>
<td>( a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt )</td>
<td>( \iff )</td>
<td>( x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} )</td>
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</tr>
<tr>
<td><strong>DTFS</strong></td>
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