ECE472/572 - Lecture 13

Wavelets and Multiresolution Processing

11/15/11

Reference: Wavelet Tutorial
http://users.rowan.edu/~polikar/WAVELETS/WTpart1.html

Roadmap

Preprocessing - low level

- Image Acquisition
- Image Compression
- Morphological Image Processing
- Fourier & Wavelet Analysis

Image Compression
- Image Enhancement
- Image Restoration
- Image Coding

Image Segmentation

Recognition & Interpretation

Representation & Description

Knowledge Base

Questions

- Why wavelet analysis? Isn’t Fourier analysis enough?
- What type of signals needs wavelet analysis?
- What is stationary vs. non-stationary signal?
- What is the Heisenberg uncertainty principle?
- What is MRA?
- Understand the process of DWT
Non-stationary Signals

- Stationary signal
  - All frequency components exist at all time
- Non-stationary signal
  - Frequency components do not exist at all time

\[ x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t) + \cos(2\pi f_4 t) \]
Short-time Fourier Transform (STFT)

- Insert time information in frequency plot

\[
X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt
\]

\[
\text{STFT}_w(t, f) = \int_{-\infty}^{\infty} x(t) w(t-t') \exp(-j2\pi ft) dt
\]

Problem of STFT

- The Heisenberg uncertainty principle
  - One cannot know the exact time-frequency representation of a signal (instance of time)
  - What one can know are the time intervals in which certain band of frequencies exist
  - This is a resolution problem
- Dilemma
  - If we use a window of infinite length, we get the FT, which gives perfect frequency resolution, but no time information.
  - In order to obtain the stationarity, we have to have a short enough window, in which the signal is stationary. The narrower we make the window, the better the time resolution, and better the assumption of stationarity, but poorer the frequency resolution
- Compactly supported
  - The width of the window is called the support of the window
Multi-Resolution Analysis

- MRA is designed to give good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies.
- This approach makes sense especially when the signal at hand has high frequency components for short durations and low frequency components for long durations.
- The signals that are encountered in practical applications are often of this type.

Continuous Wavelet Transform

\[
X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt
\]

\[
STFT_x(t,f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt
\]

\[
CWT_x(r,s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} x(t) \psi^* \left( \frac{t-r}{s} \right) dt
\]

\(\psi(t)\): mother wavelet
Application Examples – Alzheimer’s Disease Diagnosis
Examples of Mother Wavelets

Wavelet Synthesis

\[ CWT_t^\psi (x) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} x(t) \psi^* \left( \frac{t-s}{\tau} \right) ds \]

\[ \int \psi(t) \psi^*(t) dt = 0 \quad \text{for} \ k \neq l \quad \text{and} \quad \int \left| \psi(t) \right|^2 dt = 1 \]

\[ x(t) = \frac{1}{c_0} \int_{-\infty}^{\infty} CWT_t^\psi (\tau, s) \frac{\psi^* \left( \frac{t-s}{\tau} \right)}{\tau} d\tau ds \]

\[ c_0 = \left\{ 2 \pi \int_{-\infty}^{\infty} \left| \psi(t) \right|^2 dt \right\}^{-1/2} \]

The admissibility condition

\[ \int \psi(t) dt = 0 \]

oscillatory

Discrete Wavelet Transform

- The continuous wavelet transform was computed by changing the scale of the analysis window, shifting the window in time, multiplying by the signal, and integrating over all times.
- In the discrete case, filters of different cutoff frequencies are used to analyze the signal at different scales. The signal is passed through a series of highpass filters to analyze the high frequencies, and it is passed through a series of lowpass filters to analyze the low frequencies.
  - The resolution of the signal, which is a measure of the amount of detail information in the signal, is changed by the filtering operations.
  - The scale is changed by upsampling and downsampling operations.
The process halves time resolution, but doubles frequency resolution.

Quadrature Mirror Filters (QMF)

Examples

DWT and IDWT

Perfect reconstruction – needs ideal halfband filters

Daubechies wavelets
DWT and Image Processing

- Image compression
- Image enhancement

2D Wavelet Transforms
Example

Application Example - Denoising

\[ y(t) = x(t) + n(t) \]
\[ W_0(y(t)) = W_0(x(t)) + W_0(n(t)) = W_0(x(t)) + W_0(\varepsilon(t)) \]
\[ x(t) = W_0^{-1}(W_0(y(t)) - w(t)) \]
\[ \hat{y}(t) = W_0^{-1}(W_0(y(t)) - \lambda \cdot \sigma \sqrt{2 \log(N)}), \sigma = \frac{\sqrt{\text{MAE}}}{\sqrt{N}} \]

Application Example - Compression
A Bit of History

- 1976: Croiser, Esteban, Galand devised a technique to decompose discrete time signals
- 1976: Crochiere, Weber, Flanagan did a similar work on coding of speech signals, named subband coding
- 1983: Burt defined pyramidal coding (MRA)
- 1989: Vetterli and Le Gall improves the subband coding scheme

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http://engineering.rowan.edu/~polikar/WAVELETS/WTtutorial.html