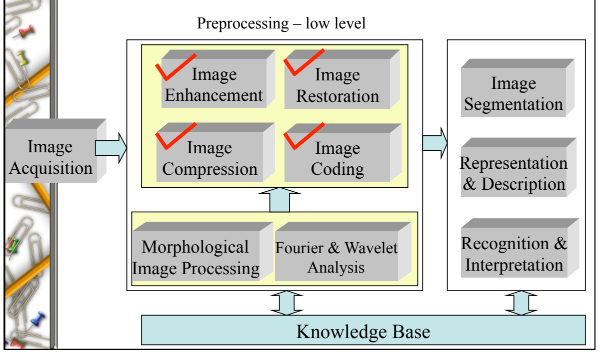
ECE472/572 - Lecture 13

Wavelets and Multiresolution Processing

11/15/11

Reference: Wavelet Tutorial
http://users.rowan.edu/~polikar/WAVELETS/WTpart1.html

Roadmap



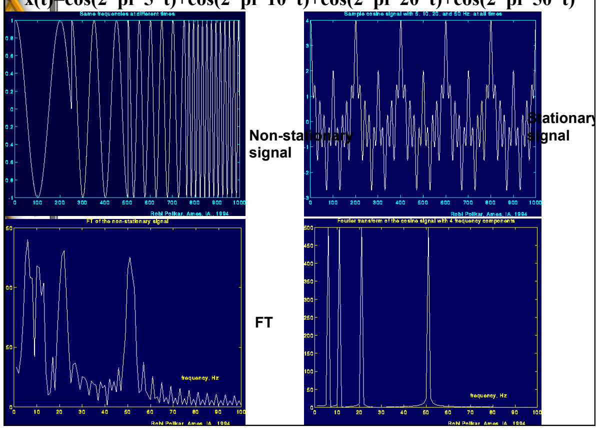
Questions

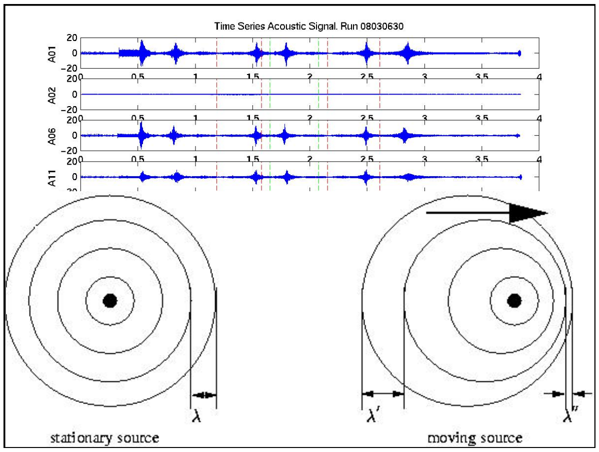
- Why wavelet analysis? Isn't Fourier analysis enough?
- What type of signals needs wavelet analysis?
- What is stationary vs. non-stationary signal?
- What is the Heisenberg uncertainty principle?
- What is MRA?
- Understand the process of DWT

Non-stationary Signals

- Stationary signal
  - All frequency components exist at all time
- Non-stationary signal
  - Frequency components do not exist at all time

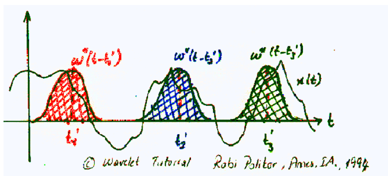
x(t)=cos(2*pi*5*t)+cos(2*pi*10*t)+cos(2*pi*20*t)+cos(2*pi*50*t)





Short-time Fourier Transform (STFT)

- Insert time information in frequency plot

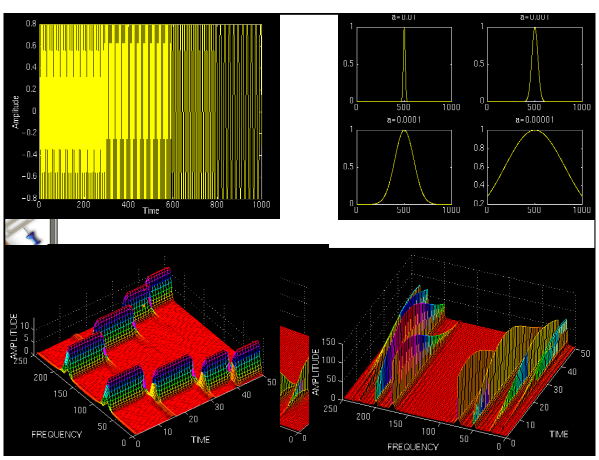


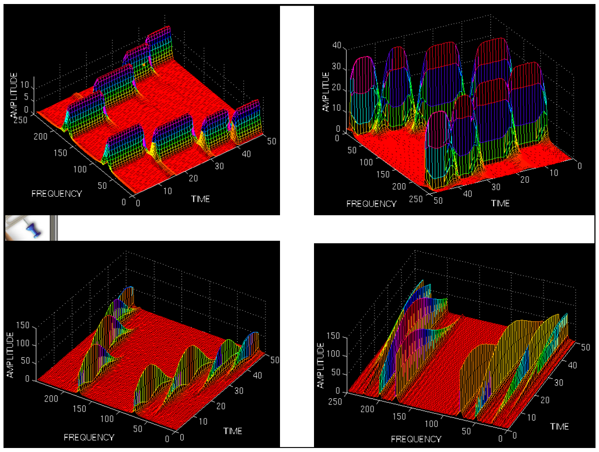
$$X(f)=\int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

$$STFT_X^{w}(t,f)=\int_{-\infty}^{\infty} x(t)w(t-t')\exp(-j2\pi ft)dt$$

Problem of STFT

- The Heisenberg uncertainty principle
  - One cannot know the exact time-frequency representation of a signal (instance of time)
  - What one can know are the time intervals in which certain band of frequencies exist
  - This is a resolution problem
- Dilemma
  - If we use a window of infinite length, we get the FT, which gives perfect frequency resolution, but no time information.
  - in order to obtain the stationarity, we have to have a short enough window, in which the signal is stationary. The narrower we make the window, the better the time resolution, and better the assumption of stationarity, but poorer the frequency resolution
- Compactly supported
  - The width of the window is called the support of the window





Multi-Resolution Analysis

- MRA is designed to give good time resolution and poor frequency resolution at high frequencies and good frequency resolution and poor time resolution at low frequencies.
- This approach makes sense especially when the signal at hand has high frequency components for short durations and low frequency components for long durations.
- The signals that are encountered in practical applications are often of this type.

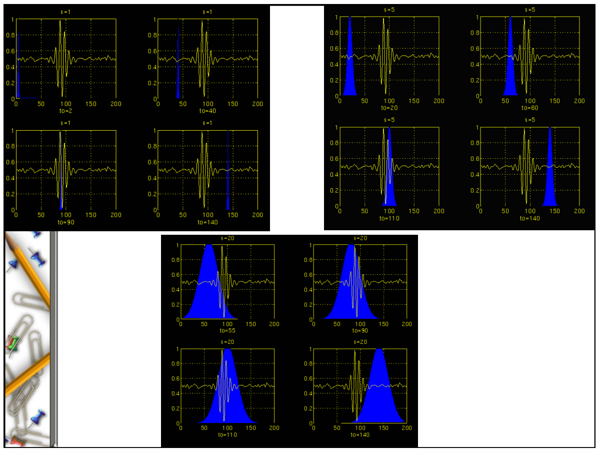
Continuous Wavelet Transform

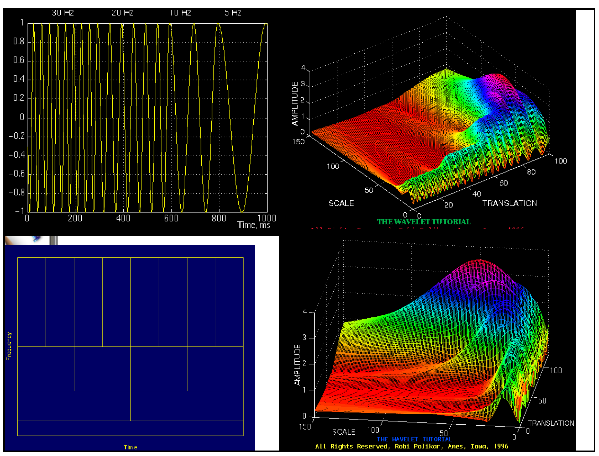
$$X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft)\, dt$$

$$STFT_x^{w}(t, f) = \int_{-\infty}^{\infty} x(t) w(t - t') \exp(-j2\pi ft)\, dt$$

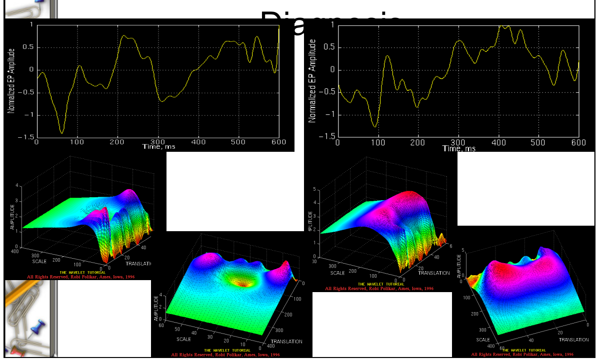
$$CWT_x^{\psi}(\tau, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} x(t) \psi^{*}\left(\frac{t-\tau}{s}\right) dt$$

$\psi(t)$: mother wavelet

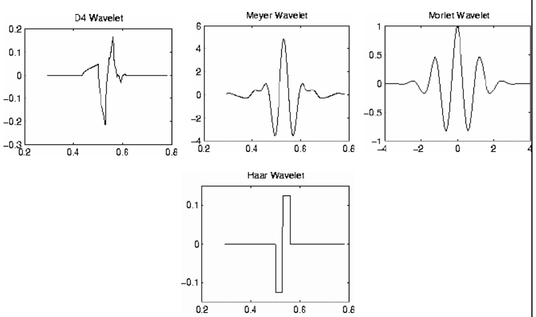




Application Examples – Alzheimer's Disease Diagnosis



Examples of Mother Wavelets



Wavelet Synthesis

$$CWT_x^\psi(\tau,s)=\frac{1}{\sqrt{s}}\int_{-\infty}^{\infty}x(t)\psi\left(\frac{t-\tau}{s}\right)dt$$

orthonormal

$$\int_{-\infty}^{\infty}\psi_k(t)\psi_l^*(t)dt=0 \quad \text{for } k\neq l \quad \text{and} \quad \int_{-\infty}^{\infty}\{\psi_k(t)\}^2dt=1$$

$$x(t)=\frac{1}{c_\psi^2}\int_s\int_\tau CWT_x^\psi(\tau,s)\frac{1}{s^2}\psi\left(\frac{t-\tau}{s}\right)d\tau ds$$

$$c_\psi=\left\{2\pi\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\xi)|^2}{|\xi|}d\xi\right\}^{1/2}<\infty$$

The admissibility condition

$$\int\psi(t)dt=0$$

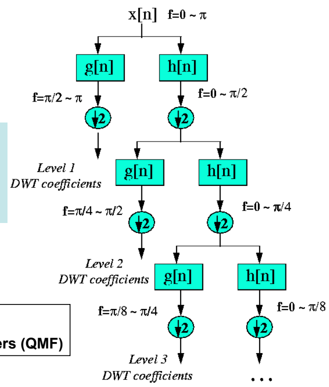
oscillatory

Discrete Wavelet Transform

- The continuous wavelet transform was computed by changing the scale of the analysis window, shifting the window in time, multiplying by the signal, and integrating over all times.
- In the discrete case, filters of different cutoff frequencies are used to analyze the signal at different scales. The signal is passed through a series of highpass filters to analyze the high frequencies, and it is passed through a series of lowpass filters to analyze the low frequencies.
  - The resolution of the signal, which is a measure of the amount of detail information in the signal, is changed by the filtering operations
  - The scale is changed by upsampling and downsampling operations.

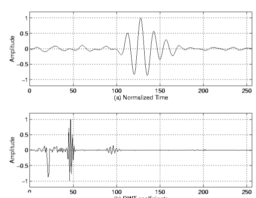
Discrete Wavelet Transform

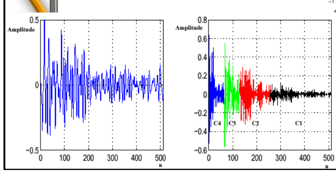
The process **halves** time resolution, but **doubles** frequency resolution



g[L-1-n] = (-1)^n . h[n]
Quadrature Mirror Filters (QMF)

Examples





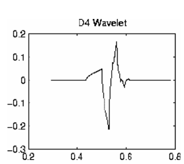
DWT and IDWT

$$g[L-1-n] = (-1)^n . h[n]$$

Quadrature Mirror Filters (QMF)

$$y_{high}[k] = \sum_n x[n].g[-n+2k]$$

$$y_{low}[k] = \sum_n x[n].h[-n+2k]$$



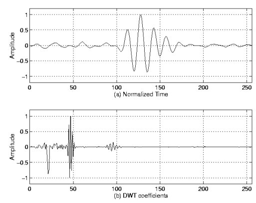
IDWT

$$x[n] = \sum_{k=-\infty}^{\infty} \left(y_{high}[k].g[-n+2k]\right) + \left(y_{low}[k].h[-n+2k]\right)$$

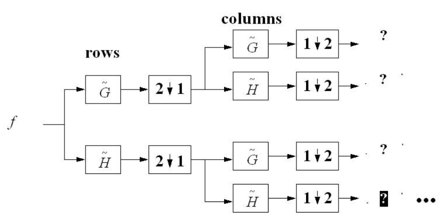
Perfect reconstruction – needs ideal halfband filters
Daubechies wavelets

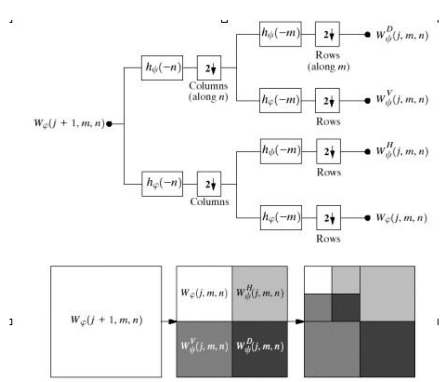
DWT and Image Processing

- Image compression
- Image enhancement

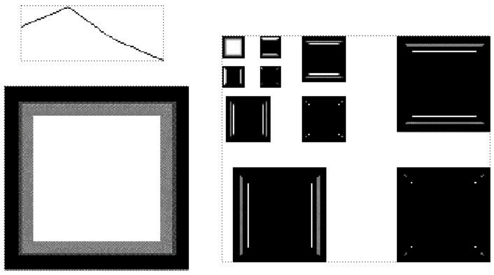


2D Wavelet Transforms





Example



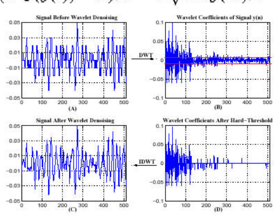
Application Example - Denoising

$$y(t) = x(t) + n(t)$$

$$W_\Psi(y(t)) = W_\Psi(x(t) + n(t)) = W_\Psi(x(t)) + W_\Psi(n(t))$$

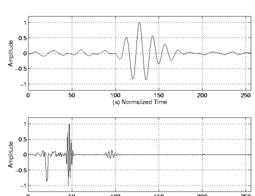
$$x(t) = W_\Psi^{-1}\left(W_\Psi(y(t)) - w(t)\right)$$

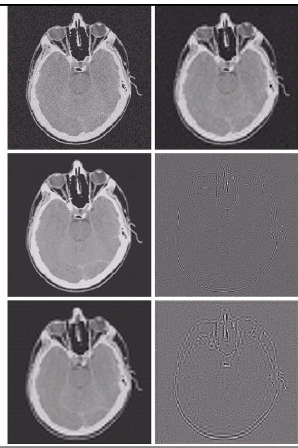
$$\hat{x}(t) = W_\Psi^{-1}\left(W_\Psi(y(t)) - \lambda\right), \lambda = \sigma\sqrt{2log(N)}, \sigma = \frac{MAD}{0.6745}$$



MAD: median of coefficients at the finest decomposition scale

Application Example - Compression





a b
c d
e f

FIGURE 7.26 Modifying a DWT for noise removal: (a) a noisy MRI of a human head; (b), (c) and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbuilt University Medical Center.)

A Bit of History

- 1976: Croiser, Esteban, Galand devised a technique to decompose discrete time signals
- 1976: Crochiere, Weber, Flanagan did a similar work on coding of speech signals, named subband coding
- 1983: Burt defined pyramidal coding (MRA)
- 1989: Vetterli and Le Gall improves the subband coding scheme

Acknowledgement

The instructor thanks the contribution from Dr. Robi Polikar for an excellent tutorial on wavelet analysis, the most readable and intuitive so far.

http://engineering.rowan.edu/~polikar/WAVELETS/WTtutorial.html