Digital Signal Processing
Lecture 3 - Discrete-Time Systems

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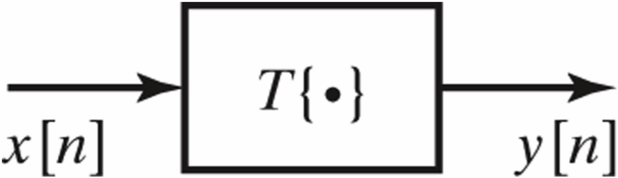
Overview

Recap

- Introduction
  - Three components of DSP
  - Clarifications: Discrete-time vs. Digital
- Discrete-time signals
  - unit sample (impulse), unit step, exponential sequences
  - relationships
  - periodicity of sinusoidal sequences

A general system and an LTI system



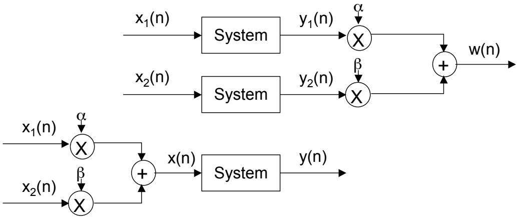
$$y[n] = T\{x[n]\}$$

Linearity

- A system is linear iff

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

  - Homogeneity or scaling or multiplicative property
  - Additivity property

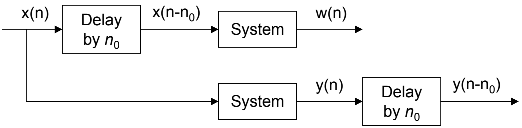


- $w[n]$ needs to be equal to $y[n]$

Time-invariance (or shift-invariance)

- For time-invariant systems, the input-output characteristics do not change over time, i.e.,

$$x[n] \longrightarrow y[n] \text{ implies } x[n-n_0] \longrightarrow y[n-n_0]$$



- $w[n]$ needs to be equal to $y[n-n_0]$

The convolution sum

- The unit sample response uniquely characterizes the system.
- If $\delta[n] \rightarrow h[n]$, then $\delta[n-k] \rightarrow h[n-k]$.

$$x[n] = \sum x[k]\delta[n-k]$$

$$y[n] = \sum x[k]h[n-k] = x[n] * h[n]$$

- Convolution sum is commutative.

$$y[n] = \sum_r x[n-r]h[r] = h[n] * x[n]$$

Stability

- A system is said to be bounded-input bounded-output (BIBO) stable iff every bounded input produces a bounded output. i.e., there exist some finite numbers, B_x and B_y, s.t.

$$|x[n]| \leq B_x < \infty, |y[n]| \leq B_y < \infty$$

  for all n.
- Abosolutely summable: For LTI, if $|x[n]| < \infty$, to have $|y[n]| < \infty$, we need to have

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Causality

- The output of a causal system at any time n depends only on present and past inputs, i.e., $x[n], x[n-1], x[n-2], \cdots$, but not on future inputs, i.e., $x[n+1], x[n+2], \cdots$

Static (or memoryless) vs. Dynamic (or with memory) systems

- For a static system, at any instant n, the output of the system only depends, at most, on the input sample at the same time: $y[n] = T(x[n])$

Exercises

$$y[n] = x[2n]$$

$$y[n] = x[-n]$$

$$y[n] = ax[n] + b$$

$$y[n] = e^{x[n]}$$

$$y[n] = x[n] \cos \omega_0 n,$$

$$h[n] = 2^n u[-n]$$

$$h[n] = 2^n u[n]$$

$$h[n] = (\frac{1}{2})^n u[n]$$

Linear constant-coefficient difference equation (LCDE)

- Nth order LCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- When $N = 0, a_0 = 1$, i.e., 0-th order LCDE

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

That is,

$$h[n] = \begin{cases} b_n, & n = 0, 1, \cdots, M \\ 0, & otherwise \end{cases} = \sum_{k=0}^{M} b_k \delta[n-k]$$

- When $N \neq 0, a_0 = 1$

$$y[n] = \sum_{m=0}^{M} b_m x[n-m] - \sum_{k=1}^{N} a_k y[n-k]$$

Initial conditions

- The homogeneous solution, $y_h[n]$

$$y[n] = y_p[n] + y_h[n]$$

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0 \rightarrow y_h[n] = \sum_{m=1}^{N} A_m z_m^n$$

- If a system is characterized by an LCDE and is further specified to be linear, time-invariant, and causal, then the solution is unique. These auxiliary conditions are referred to as the initial-rest conditions. That is, if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$.
- Note that not for every set of boundary conditions that LCDE would correspond to an LTI system.

Exercises

$$y[n] - ay[n-1] = x[n]$$

Suppose $x[n] = \delta[n]$

- assume $y[n] = 0, n < 0$
- assume $y[n] = 0, n > 0$

Eigenfunctions for LTI systems

- Complex exponentials are eigenfunctions of LTI systems and the frequency response is the eigenvalue of LTI systems.
- Let $x[n] = e^{j\omega n}$, then

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n}\sum_{k} h[k$$

Let $H(e^{j\omega}) = \sum_k h[k]e^{-j\omega k}$, then

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

- Frequency response is the eigenvalue of the system:

$$H(e^{j\omega}) = \sum_k h[k]e^{-j\omega k}$$

- Properties of frequency response:
  - Function of continuous variable ω
  - Periodic function of ω with period of 2π

Exercises

- Find the output to $x[n] = A\cos(\omega_0 n + \phi)$
- Find the frequency response to $y[n] - ay[n-1] = x[n]$
- Find the output to $x[n] = e^{j\omega n}u[n]$ (suddenly applied exponential input)

Fourier-series expansion

- $H(e^{j\omega})$ is a continuous periodic function and would have a Fourier series representation
- $H(e^{j\omega})$ has a Fourier-series expansion in terms of complex exponentials
- The Fourier series coefficients then become the values of the unit sample response

$$H(e^{j\omega}) = \sum_n h[n]e^{-j\omega n}$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Fourier Transform

- Frequency-domain representation of arbitrary sequence $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

The convolution property

- $x[n] * y[n] \leftrightarrow X(e^{j\omega})Y(e^{j\omega})$

$$e^{j\omega_0 n} \rightarrow H(e^{j\omega_0})e^{j\omega_0 n}$$

$$\sum_k A_k e^{j\omega_k n} \rightarrow \sum_k A_k H(e^{j\omega_k})e^{j\omega_k n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

A decomposition of $x[n]$ as a linear combination of complex exponentials

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j\omega})e^{j\omega n}d\omega$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$y[n] = x[n] * h[n]$$

The symmetry property

For real sequences $x[n]$, the symmetry property indicates that the FT of $x[n]$ is a conjugate symmetric function of ω

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

Other useful properties:

- The real part of the FT is an even function: $X_R(e^{j\omega}) = X_R(e^{-j\omega})$
- The imaginary part of the FT is an odd function: $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$
- The amplitude of the FT is an even function: $|X(e^{j\omega})|$
- The phase of the FT is an odd function $X(e^{j\omega})$