Digital Signal Processing
Lecture 4 - z-Transforms

Electrical Engineering and Computer Science
University of Tennessee, Knoxville

Overview

Recap - Discrete-time systems

- Special properties: linearity, TI, stability, causality
- LTI systems: the unit sample response $h[n]$ uniquely characterizes an LTI system

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n]$$

- Linear constant-coefficient difference equation: the solution is unique only with the initial-rest conditions

$$\sum_{k=0}^{N} a_k y[n-k]=\sum_{m=0}^{M} b_m x[n-m]$$

- Frequency response: $H(e^{j\omega})$, complex exponentials are eigenvalues of LTI systems, i.e., if $x[n]=e^{j\omega n}$,

$$y[n]=H(e^{j\omega})x[n]=(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k})e^{j\omega n}$$

- Fourier transform: Generalization of frequency response (a periodic continous function of ω)

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Issue of convergence

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{1}$$

$$|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \leq \sum_{n} |x[n]||e^{-j\omega n}| \tag{2}$$

$$= \sum_{n} |x[n]| \tag{3}$$

- $X(e^{j\omega n})$ converges if $\sum |x[n]| < \infty$, that is, $x[n]$ is absolutely summable.
- Recall: if $h[n]$ is absolutely summable, the system is stable, or $H(e^{j\omega})$ converges
- E.g.: $x[n] = (\frac{1}{2})^n u[n]$, $x[n] = 2^n u[n]$

Definition of the z-transform

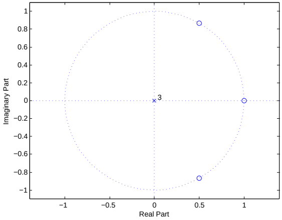
- $x[n] \rightarrow x[n].r^{-n}$, where r^{-n} is a decay function

$$X_r(e^{j\omega}) = \sum_n (x[n]r^{-n})e^{-j\omega n} = \sum_n x[n](re^{j\omega})^{-n}$$

- Define a new complex variable, $z = re^{j\omega}$
- The z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$, $X(z)$ converges if $\sum_{n=\infty}^{\infty} |x[n]r^{-n}| < \infty$
- Relationship with FT: $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$ or $X(e^{j\omega}) = X(z)|_{|z|=1}$
- E.g.: $x[n] = (\frac{1}{2})^n u[n]$, does the z-transform exist? does the FT exist?

The z-plane, the pole-zero plot

- Sum of exponentials of a sequence results in z-transforms that are ratios of polynomials in z
- Zeros of polynomial: roots of the numerator polynomial
- Poles of polynomial: roots of the denominator polynomial
- $|z| = 1$ (or unit circle) is where the Fourier transform equals to the z-transform
- MATLAB function: `zplane`.



Region of convergence

- E.g.:

$$x[n] = -(\frac{1}{2})^n u[-n-1] \tag{4}$$

$$x[n] = (\frac{1}{2})^n u[n] \tag{5}$$

$$x[n] = a^n \text{ for } 0 \leq n \leq N-1 \tag{6}$$

- Region of convergence (R of C): the z-transform exists only for those values of z where $X(z)$ converges.
- Observations:
  - The z-transform is defined by function of z and also the R of C.
  - There won't be any poles in the R of C
  - R of C is bounded by poles or 0 or ∞
  - FT exists only when the R of C includes $|z| = 1$

Different cases

- Finite length sequence: $0 < |z| < \infty$
- Right-sided sequence: $x[n] = 0$ for $n < n_1$

$$R_{x-} < |z| < \infty$$

  where R_{x-} must be the outermost pole in the z-plane
- Left-sided sequence: $x[n] = 0$ for $n > n_1$

$$0 < |z| < R_{x+}$$

  where R_{x+} is the innermost pole
- Two-sided sequence: $R_{x-} < |z| < R_{x+}$ where R_{x-} and R_{x+} are the two poles that are adjacent on the z-plane.

R of C ($\lvert a\rvert < 1, \lvert b\rvert > 1$)	does FT exist	which sided
$\lvert z\rvert < a$		
$a < \lvert z\rvert < b$		
$\lvert z\rvert > b$		

System function

$$y[n] = x[n] * h[n]$$

$$Y(z) = X(z)H(z) \to H(z) = \frac{Y(z)}{X(z)}$$

- $H(z)$ is the system function
- when system is stable?
- when system is causal?
- E.g., What's the system function for $y[n] - \frac{1}{2}y[n-1] = x[n]$?

Properties of the z-transform

- Linearity?
- Time-delay property?
  - What does z^{-1} indicate?
  - Unit delay property of z-transforms

$$x[n-1] \Longleftrightarrow z^{-1}X(z)$$

  - Example: What is $z^{-1}x[n]$

n	n<-1	-1	0	1	2	3	4	5	n>5
x[n]	0	0	2	4	6	4	2	0	0

  - Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n-n_0] \Longleftrightarrow z^{-n_0}X(z)$$

z-transform as an operator

- Unit-delay operator

$$y[n] = D\{x[n]\} = x[n-1]$$

If the input is $x[n] = z^n$,

$$y[n] = D\{x[n]\} = D\{z^n\} = z^{n-1} = z^{-1}x[n] = z^{-1}\{x[n]\}$$

- What is the operator for the first difference?

Convolution and the z-transform

- Convolution in the time domain corresponds to multiplication in the z-domain

$$y[n] = h[n] * x[n] \iff Y(z) = H(z)X(z)$$

- Calculate the output in the z-domain

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$
$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

Cascading systems

- The system function for a cascade of two LTI systems is the product of the individual system functions.

$$h[n] = h_1[n] * h_2[n] \Longleftrightarrow H(z) = H_1(z)H_2(z)$$

- Consider a system described by the difference equations

$$w[n] = 3x[n] - x[n-1], y[n] = 2w[n] - w[n-1]$$

  that represents a cascade of two first-order systems. How to calculate the output?

Factoring the z-polynomials

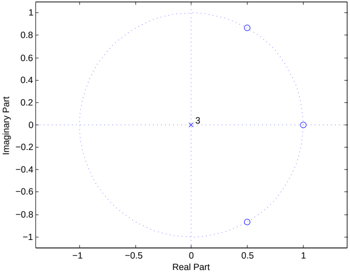
- We can factor z-transform polynomials to break down a large system into smaller modules. The factors of a high-order $H(z)$ would represent component systems that make up $H(z)$ in a cascade connection
- Decompose $H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$ into lower-order cascading systems to help understand the characteristics of the system

Significance of the zeros of $H(z)$

- The zeros of the system function that lie on the unit circle correspond to frequencies at which the gain of the system is zero. Thus, complex sinusoids at those frequencies are blocked or nulled by the system.

Significance of the zeros of $H(z)$ (cont')

- Exercise: $H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$. What does the pole-zero plot indicate? or what kind of input signals would generate a zero output?



- Application example: eliminate jamming signal in a radar or communications system or eliminate the 60 Hz interference from a power line
- Exercise: How to remove signal $x[n] = \cos(\omega n)$?

Nulling filters

- If we want to eliminate a sinusoidal input signal, we would have to remove two signals of the form $z_1^n + z_2^n$

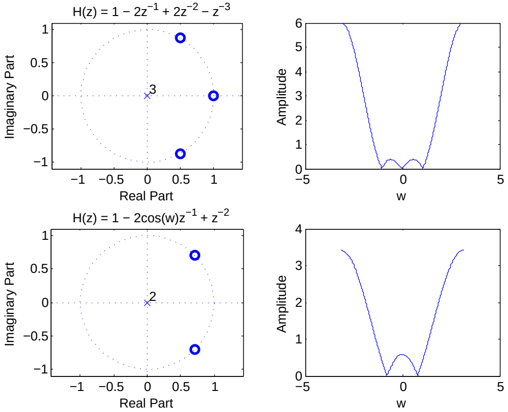
$$x[n] = \cos(\omega n) = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\omega n}$$

with two cascading first-order FIR filters. The second-order FIR filter will have two zeros at $z_1 = e^{j\omega}$ and $z_2 = e^{-j\omega}$.

- To eliminate the first component in $x[n]$, we need a filter with system function $H_1(z) = 1 - z_1 z^{-1}$, and for the second component, a system function of $H_2(z) = 1 - z_2 z^{-1}$, such that

$$\begin{aligned} H(z) &= H_1(z)H_2(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \\ &= 1 - 2\cos\omega z^{-1} + z^{-2} \end{aligned}$$

Revisit - the pole-zero plot vs. the frequency response

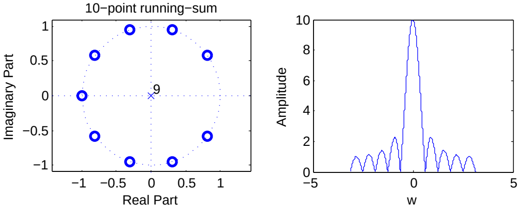


Outline

The L-point running sum filter

$$y[n] = \sum_{k=0}^{L-1} x[n-k], H(z) = \sum_{k=0}^{L-1} z^{-k} = \frac{1-z^{-L}}{1-z^{-1}}$$

- Exercise: What are the roots?
- A 10-point running-sum filter $L = 10$



- Why only 9 poles?
- Why missing a zero at $z = 1$?

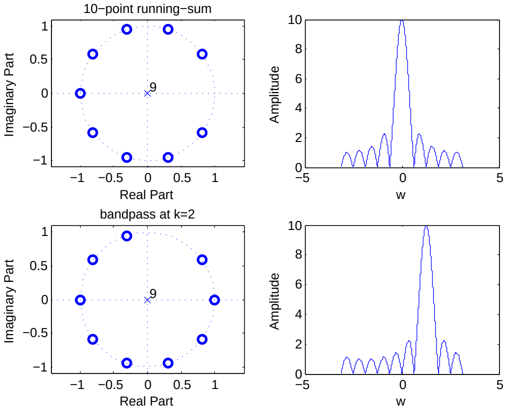
Complex bandpass filters

- We can control the frequency response of an FIR filter by placing its zeros on the unit circle
- Move the passband to a new location with a specified frequency, e.g., $\omega = 2\pi k_0/L$

$$H(z) = \prod_{k=0, k\neq k_0}^{L-1} (1 - e^{j2\pi k/L} z^{-1})$$

  - the index k_0 denotes the one omitted root at $z = e^{j2\pi k_0/L}$
  - What would the pole-zero plot look like?
  - What would the frequency response look like?

Complex bandpass filters (cont')



Complex bandpass filters - the filter coefficient?

- A rotation of the zeros by the angle, $2\pi k_0/L$, is equivalent of shifting the frequency response along the ω-axis by the amount of the rotation.
- Consider $H(z) = G(z/r)$
  - The effect of replacing z in $G(z)$ with z/r is to multiply the roots of $G(z)$ by r and make these the roots of $H(z)$. When r is a complex exponential, this will rotate the complex number through the angle specified.

$$G(z) = \sum_{k=0}^{L-1} z^{-k}, r = e^{j2\pi k_0/L}$$

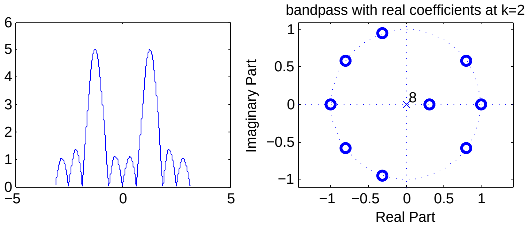
$$H(z) = G(z/r) = G(ze^{-j2\pi k_0/L}) = \sum_{k=0}^{L-1} z^{-k} e^{j2\pi k_0 k/L}$$

  - $b_k = e^{j2\pi k_0 k/L}$ for $k = 0, 1, \cdots, L-1$
  - $H(e^{j\omega}) = \sum_{k=0}^{L-1} e^{j2\pi k_0 k/L} e^{-j\omega k}$

Bandpass filters with real coefficients

- $b_k = \Re\{e^{j2\pi k_0 k/L}\} = \cos(2\pi k_0 k/L)$

$$H(z) = \sum_{k=0}^{L-1} (\cos(2\pi k_0 k/L)) z^{-k} = H_1(z) + H_2(z)$$



Bandpass filters with real coefficients (cont')

$$H(z) = \sum_{k=0}^{L-1} \left(\tfrac{1}{2} e^{j2\pi k_0 k/L} z^{-k} + \tfrac{1}{2} e^{-j2\pi k_0 k/L} z^{-k}\right)$$
$$= \frac{1}{2}\frac{1-z^{-L}}{1-pz^{-1}} + \frac{1}{2}\frac{1-z^{-L}}{1-p^* z^{-1}}$$
$$= \frac{1}{2}\frac{z^L-1}{z^{L-1}(z-p)} + \frac{1}{2}\frac{z^L-1}{z^{L-1}(z-p^*)}$$
$$= \frac{1}{2}\frac{(z^L-1)(z-p^*)+(z^L-1)(z-p)}{z^{L-1}(z-p)(z-p^*)}$$

where $p = e^{j2\pi k_0/L}$

The inverse z-transform

- Formal method - Contour Integration

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C represents a closed contour within the ROC of the z-transform.

- Informal methods
  - Inspection method
  - Power series
  - Partial fraction expansion

Inspection method

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \text{ for } |z| > |a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1 - az^{-1}}, \text{ for } |z| < |a|$$

Power series

- The z-transform is a power series in z.
  $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^n$
- Examples:
  1. $X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})$
  2. $X(z) = \log(1 + az^{-1}), \text{ for } |z| > |a|$
  3. $X(z) = \frac{1}{1-az^{-1}}$
  4. $X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$ for (a) ROC: $|z| > 1$, (b) ROC: $|z| < 0.5$
- Note: If $x[n]$ is a causal sequence, we should seek a power series expansion in negative power of z, then the component of the highest order of z^{-1} should be at the rightmost position of the denominator; If $x[n]$ is not a causal sequence, we should seek a power series expansion in positive power of z, then we should reverse the order of denominator and the the component with the highest order of z^{-1} should be at the leftmost position.
- Drawbacks: No closed-form expression

Partial fraction expansion

- Extension to the inspection method
  $F(x) = \frac{P(x)}{Q(x)} = \sum_{k=1}^{N} \frac{R_k}{x - x_k}$ where R_k is the residue
- The expansion is true with the following two conditions
  - Order of $P(x)$ is less than the order of $Q(x)$
  - No multiple-order roots

$$R_r = F(x)(x - x_r)|_{(x=x_r)}$$

Examples

1. $X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$ for $|z| > \frac{1}{2}$
2. $X(z) = \frac{1+3z^{-1}+\frac{11}{6}z^{-2}+\frac{1}{3}z^{-3}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$. Note that $X(z)$ is an improper rational function where the order the numerator is *larger* than that of the denominator. Use long division with the two polynomials written in "reverse order" to convert it to the sum of a polynomial and a proper rational function.

Examples (cont')

1. $X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$. Note that the two residues are actually complex conjugate pairs. This is a consequence of the fact that the poles are complex conjugate pairs. That is, complex-conjugate poles result in complex-conjugate coefficients in the partial fraction expansion. For example, suppose $X(z) = \frac{A_1}{1-p_1z^{-1}} + \frac{A_2}{1-p_2z^{-1}}$ where $A_1 = A_2^*$ and $p_1 = p_2^*$, then

$$x[n] = A_1(p_1)^n u[n] + A_2(p_2)^n u[n] \tag{7}$$

$$= [|A_1|e^{j\angle A_1}(|p_1|e^{j\angle p_1})^n + |A_2|e^{j\angle A_2}(|p_2|e^{j\angle p_2})^n]u[n] \tag{8}$$

$$= |A_1||p_1|^n \cos(\angle A_1 + n\angle p_1)u[n] \tag{9}$$

Examples (cont')

1. $X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$. Note that $X(z)$ has multiple order poles. So you should find the coefficients for $X(z) = \frac{A_1}{1+z^{-1}} + \frac{A_2}{1-z^{-1}} + \frac{A_3}{(1-z^{-1})^2}$

Sum of geometric series

- Sum of infinite terms in a geometric series

$$\sum_{k=0}^{\infty} A^k = \frac{1}{1-A}, \text{ if } |A| < 1$$

- Sum of the first L terms of a geometric series

$$\sum_{k=0}^{L-1} A^k = \frac{1-A^L}{1-A}$$