Digital Signal Processing
Lecture 8 - Filter Design - IIR

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Overview

Roadmap

- Introduction
- Discrete-time signals and systems - LTI systems
  - Unit sample response $h[n]$: uniquely characterizes an LTI system
  - Linear constant-coefficient difference equation
  - Frequency response: $H(e^{j\omega})$
  - Complex exponential being eigenfunction of an LTI system: $y[n] = H(e^{j\omega})x[n]$ and $H(e^{j\omega})$ as eigenvalue.
- z transform
  - The z-transform, $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
  - Region of convergence - the z-plane
  - System function, $H(z)$
  - Properties of the z-transform
  - The significance of zeros
  - The inverse z-transform, $x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$: inspection, power series, partial fraction expansion
- Sampling and Reconstruction
- Transform domain analysis - nwz

Review - Design structures

- Different representations of causal LTI systems
  - LCDE with initial rest condition
  - $H(z)$ with $|z| > R_+$ and starts at $n = 0$
- Block diagram vs. Signal flow graph and how to determine system function (or unit sample response) from the graphs
- Design structures
  - Direct form I (zeros first)
  - Direct form II (poles first) - Canonic structure
  - Transposed form (zeros first)
- IIR: cascade form, parallel form, feedback in IIR (computable vs. noncomputable)
- FIR: direct form, cascade form, parallel form, linear phase
- Metric: computational resource and precision
- Sources of errors: coefficient quantization error, input quantization error, product quantization error, limit cycles
  - Pole sensitivity of 2nd-order structures: coupled form
  - Coefficient quantization examples: direct form vs. cascade form

Rationale

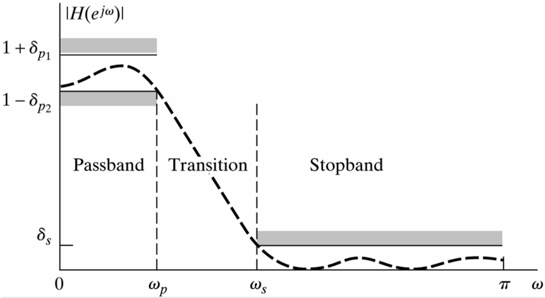
- Review of complex exponentials as eigenfunctions of the LTI
  - $x[n] = e^{j\omega_0 n} \rightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n}$
  - or $x[n] = \cos\omega_0 n \rightarrow y[n] = |H(e^{j\omega_0})|\cos(\omega_0 n + \theta)$
- Separation of signal when they occupate different frequency bands — choose the system function that is unity at selective frequencies
- Given a set of specifications, design a rational transfer function that approximates the ideal filter maintaining specifications of δ_p, δ_s, and the transition band.

Stages of digital filter design

- The specification of the desired properties of the system
  - application-dependent
  - usually done in the frequency domain
- The approximation of the specifications using a causal discrete-time system
- The realization of the system
  - e.g., DSP board

Practical frequency-selective filters

- Approximate ideal filters by a rational function or LCDE



- Factors that affect the filter performance
  - the maximum tolerable passband ripple, $20 \log_{10} \delta_p$
  - the maximum tolerable stopband ripple, $20 \log_{10} \delta_s$
  - the passband edge frequency ω_p
  - the stopband edge frequency ω_s
  - M and N: order of the LCDE

Example

- Design a discrete-time lowpass filter to filter a continuous-time signal with the following specs (with a sampling rate of 10^4 samples/s):
  - The gain should be within ± 0.01 of unity in the frequency band $0 \leq \Omega \leq 2\pi(2000)$
  - The gain should be no greater than 0.001 in the frequency band $2\pi(3000) \leq \Omega$
- Parameter setup
  - $\delta_{p1} = \delta_{p2} = 0.01$, $\delta_s = 0.001$
  - $\omega_p = 2\pi(2000)/10^4$, $\omega_s = 2\pi(3000)/10^4$
  - Ideal passband gain in decibels?
  - maximum passband gain in decibels?
  - maximum stopband gain in decibels?

Design techniques for IIR filters

- Analytical — closed-form solution of transfer function
- Continuous-time $\rightarrow$ Discrete-time
- Algorithmic

General guidelines for CT->DT

$$\begin{array}{ccc} \text{continuous} & \rightarrow & \text{discrete} \\ H_a(s) & \rightarrow & H(z) \\ h_a(t) & \rightarrow & h[n] \end{array}$$

- $j\Omega$-axis (s-plane) $\rightarrow$ unit circle (z-plane)
- if $H_a(s)$ is stable $\rightarrow$ $H(z)$ is stable

Different CT->DT approaches

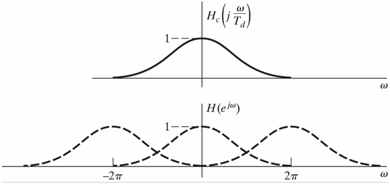
- Mapping differentials to differences
  - $z = 1 + sT$
  - the $j\Omega$-axis is NOT mapped to the unit circle
  - stable poles might not be mapped to inside the unit circle
- Impulse invariance
- Bilinear transformation

Impulse invariance

$$h[n] = T_d h_c(nT_d) \quad (1)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c[\frac{j\omega}{T_d} + \frac{j2\pi k}{T_d}] \quad (2)$$

- Preserve good time-domain characteristics
- Linear scaling of frequency axis, $\omega = \Omega T$
- Existence of aliasing
- Impulse invariance doesn't imply step invariance



Impulse invariance (cont')

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \rightarrow H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

- Mapping poles

$$s = s_k \rightarrow z = e^{s_k T_d}$$

- Preserve residues
- $s = j\Omega \rightarrow z = e^{j\Omega T_d} = e^{j\omega}$, the unit circle
- if s_k is stable, i.e., region of s_k is less than 0, $\rightarrow |z_k| < 1 \rightarrow$ digital filter is stable

Impulse invariance - An example

- Find the system function of the digital filter mapped from the analog filter with a system function $H_c(s) = \frac{s+a}{(s+a)^2+b^2}$. Compare magnitude of the frequency response and pole-zero distributions in the s- and z-plane
- Sol: $H(z) = \frac{1-(e^{-aT}\cos bT)z^{-1}}{(1-e^{-(a+jb)T}z^{-1})(1-e^{-(a-jb)T}z^{-1})}$
- Note that zeros are not mapped. Also note that $|H_s(j\Omega)|$ is not periodic but $|H(e^{j\omega})|$ is.

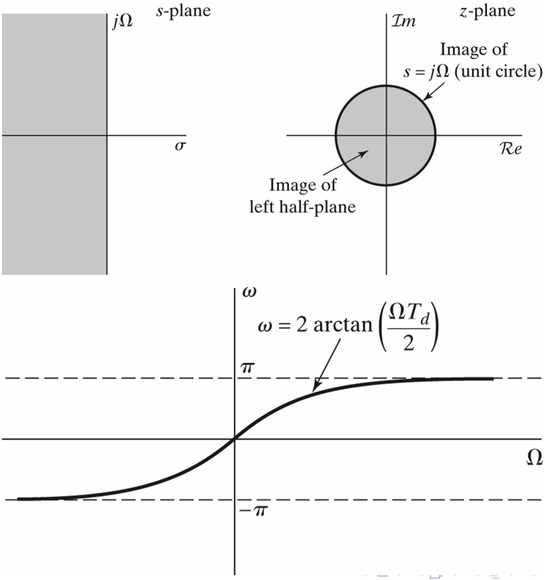
Bilinear transformation

- Mapping from s-plane to z-plane by relating s and z according to a bilinear transformation. $H_c(s) \rightarrow H(z)$

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right), \text{ or } z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

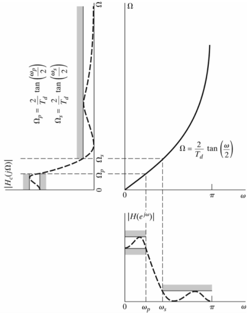
- Two guidelines
  - Preserves the frequency characteristics? I.e., maps the $j\Omega$-axis to the unit circle?
  - Stable analog filter mapped to stable digital filter?
- Important properties of bilinear transformation
  - Left-side of the s-plane $\rightarrow$ interior of the unit circle; Right-side of the s-plane $\rightarrow$ exterior of the unit circle. Therefore, stable analog filters $\rightarrow$ stable digital filters.
  - The $j\Omega$-axis gets mapped exactly once around the unit circle.
    - No aliasing
    - The $j\Omega$-axis is infinitely long but the unit circle isn't $\rightarrow$ nonlinear distortion of the frequency axis

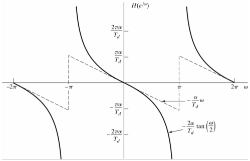
Blinear transformation - Mappings



Bilinear transformation - How to tolerate distortions?

- Prewarp the digital cutoff frequency to an analog cutoff frequency through $\Omega = \frac{2}{T} \tan \frac{\omega}{2}$
- Better used to approximate piecewise constant filters which will be mapped as constant as well
- Can't be used to obtain digital lowpass filter with linear-phase





- Avoid aliasing at the price of distortion of the frequency axis

The class of analog filters

- Butterworth filter
  - $|H_c(j\Omega)|^2 = \frac{1}{1+(\frac{j\Omega}{j\Omega_c})^{2N}}$
  - Note about the butterworth circle with radius Ω_c
  - Ω_c is also called the 3dB-cutoff frequency when $-10log_{10}|H_c(j\Omega)|^2|_{\Omega=\Omega_c} \approx 3$
  - Monotonic function in both passband and stopband
  - Matlab functions: `buttord, butter`
- Chebyshev filter
  - Type I Chebyshev has an equiripple freq response in the passband and varies monotonicaly in the stopband, $|H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_p)}$
  - Type II Chebyshev is monotonic in the passband and equiripple in the stopband, $|H_c(j\Omega)|^2 = \frac{1}{1+\epsilon^2[\frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}]^2}$
  - Matlab functions: `cheb1ord, cheby1, cheb2ord, cheby2`

The class of analog filters (cont'd)

- Elliptic filter
  - $|H_c(j\Omega)^2 = \frac{1}{1+\epsilon^2 R_N^2(\Omega/\Omega_p)}$ where $R_N(\Omega)$ is a rational function of order N satisfying the perperty $R_N(1/\Omega) = 1/R_N(\Omega)$ with the roots of its numerator lying within the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$
  - Equiripple in both the passband and the stopband
  - Matlab functions: `ellipord, ellip`

Example

- Specs of the discrete-time filter: passband gain between 0dB and -1dB, and stopband attenuation of at least -15dB.

$$1 - \delta_p \geq -1dB, \delta_s \leq -15dB$$

$$20\log_{10}|H(e^{j0.2\pi})| \geq -1 \rightarrow |H(e^{j0.2\pi})| \geq 10^{-0.05} = 0.891 \quad (3)$$

$$20\log_{10}|H(e^{j0.3\pi})| \leq -15 \rightarrow |H(e^{j0.3\pi})| \leq 10^{-0.75} = 0.177 \quad (4)$$

Example (cont'd)

- Impulse invariance
  - Round up to the next integer of N
  - Due to aliasing problem, meet the passband exactly with exceeded stopband

$$1 + \left(\frac{j\frac{0.2\pi}{T}}{j\Omega_c}\right)^{2N} = 10^{0.1} \tag{5}$$

$$1 + \left(\frac{j\frac{0.3\pi}{T}}{j\Omega_c}\right)^{2N} = 10^{1.5} \tag{6}$$

- Bilinear transformation
  - Round up to the next integer of N
  - By convention, choose to meet the stopband exactly with exceeded passband

$$1 + \left(\frac{j2\tan(0.1\pi)}{j\Omega_c}\right)^{2N} = 10^{0.1} \tag{7}$$

$$1 + \left(\frac{j2\tan(0.15\pi)}{j\Omega_c}\right)^{2N} = 10^{1.5} \tag{8}$$

Example - Comparison