ECE442 Communications
Lecture 4. Performance of Modulation

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We consider only additive white Gaussian noise (AWGN), in which the received signal is given by \( r(t) = s(t) + n(t) \).

Signal-to-noise ratio (SNR):

\[
\text{SNR} = \frac{P_r}{(N_0 B)},
\]

where the noise \( n(t) \) has uniform PSD \( N_0 / 2 \) and the bandwidth of the transmitted signal is \( 2B \).

We denote the signal energy per bit by \( E_b \).

We define \( \gamma_s = E_s / N_0 \) (SNR per symbol) and \( \gamma_b = E_b / N_0 \) (SNR per bit).
1. Bit error rate: for both BPSK and QPSK, we have

\[ P_b = Q \left( \sqrt{2\gamma_b} \right). \]

2. Symbol error rate: for QPSK, we have

\[ P_s \approx 2Q \left( \sqrt{\gamma_b} \right). \]
The symbol error probability is given by

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} e^{-\gamma_s \sin^2 \theta} \int_0^\infty z \exp \left[ - \frac{(z - \sqrt{2\gamma_s \cos \theta})^2}{2} \right] dz d\theta.$$  

When $M > 4$, there is no closed-form solution.

We can also obtain an approximation:

$$P_s \approx 2Q \left( \sqrt{2\gamma_s \sin(\pi/M)} \right).$$
The error probability of MPAM is given by

\[ P_s = \frac{2(M - 1)}{M} Q \left( \sqrt{\frac{6\gamma_s}{M^2 - 1}} \right) \]

The error probability of MQAM is given by

\[ P_s = 1 - \left( 1 - \frac{2(\sqrt{M} - 1)}{\sqrt{M}} Q \left( \sqrt{\frac{3\gamma_s}{M - 1}} \right) \right)^2, \]

when \( M = L^2 \).
The error probability of M-FSK is given by

\[ P_s = \sum_{m=1}^{M} (-1)^{m+1} \binom{M-1}{m} \frac{1}{m+1} \exp \left( -\frac{m\gamma_s}{m+1} \right). \]

The error probability of CPFSK is much more complicated.
## Approximations

<table>
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<th>Modulation</th>
<th>$P_s(\gamma_s)$</th>
<th>$P_b(\gamma_b)$</th>
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<tr>
<td>BFSK:</td>
<td>$P_b = Q(\sqrt{\gamma_b})$</td>
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<td>BPSK:</td>
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<tr>
<td>QPSK, 4QAM:</td>
<td>$P_s \approx 2 Q(\sqrt{\gamma_s})$</td>
<td>$P_b \approx Q(\sqrt{2\gamma_b})$</td>
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<td>MPAM:</td>
<td>$P_s \approx \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\gamma_s}{M^2-1}}\right)$</td>
<td>$P_b \approx \frac{2(M-1)}{M \log_2 M} Q\left(\sqrt{\frac{6\gamma_b \log_2 M}{(M^2-1)}}\right)$</td>
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<tr>
<td>MPSK:</td>
<td>$P_s \approx 2 Q\left(\sqrt{2\gamma_s \sin(\pi/M)}\right)$</td>
<td>$P_b \approx \frac{2}{\log_2 M} Q\left(\sqrt{2\gamma_b \log_2 M \sin(\pi/M)}\right)$</td>
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<td>Rectangular MQAM:</td>
<td>$P_s \approx \frac{4(\sqrt{M-1})}{\sqrt{M}} Q\left(\sqrt{\frac{3\gamma_s}{M-1}}\right)$</td>
<td>$P_b \approx \frac{4(\sqrt{M-1})}{\sqrt{M \log_2 M}} Q\left(\sqrt{\frac{3\gamma_b \log_2 M}{(M-1)}}\right)$</td>
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<td>Nonrectangular MQAM:</td>
<td>$P_s \approx 4 Q\left(\sqrt{\frac{3\gamma_s}{M-1}}\right)$</td>
<td>$P_b \approx \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3\gamma_b \log_2 M}{(M-1)}}\right)$</td>
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</table>
Problem 1. Consider BPSK, when the SNR is very high, how fast the bit error rate is decreased (exponentially? linearly?) Justify your answer using either math and simulations.

Problem 2. Consider BPSK. Suppose that the noise power spectral density is -174dBm/Hz and the symbol period is 1 microsecond. Suppose that the transmit power is 0.2W. Consider a free space propagation model. Assume that the antenna gains are 1. Then, what is the maximal transmission distance if the bit error rate needs to be controlled below $10^{-5}$.

Problem 3. Consider the same $E_b/N_0 = 30dB$ and the transmission rate is 1Mbps. Is the bit error rate of QPSK lower than 16QAM? How about $E_b/N_0 = 0dB$?

When fading exists, the received signal is given by 
\[ r(t) = g(t)s(t) + n(t), \]
where \( g(t) \) is the fading channel gain.

There are three performance metrics in fading channels:
- The outage probability \( P_{out} \), defined as the probability that \( \gamma_s \) falls below a given value.
- The average error probability, averaged over the distribution of \( \gamma_s \).
- Combined average error probability and outage, defined as the average error probability that can be achieved some percentage of time or some percentage of spatial locations.
In Rayleigh fading, the outage probability is given by

$$P_{out} = 1 - e^{-\gamma_0 / \bar{\gamma}_s}.$$  

We define the dB fade margin as

$$F_d = -10 \log [-\ln(1 - P_{out})].$$
In Rayleigh fading, the distribution of SNR is given by

\[ P_{\gamma_s}(\gamma) = \frac{1}{\bar{\gamma}_s} e^{-\gamma/\bar{\gamma}_s}. \]

and

\[ P_{\gamma_b}(\gamma) = \frac{1}{\bar{\gamma}_b} e^{-\gamma/\bar{\gamma}_b}. \]
Integrating over $\gamma_s$, we have for BPSK

$$\bar{P}_b \approx \frac{1}{4\bar{\gamma}_b}.$$ 

In AWGN, the error probability decreases exponentially with the SNR. In fading channels, the error probability decreases linearly with SNR.
Comparison of AWGN and Fading Channels
Intersymbol Interference

- ISI is incurred by frequency-selective fading. It gives rise to an irreducible error floor that is independent of the signal power.
- When ISI exists, the SNR becomes

\[ \tilde{\gamma}_s = \frac{P_r}{N_0B + l}, \]

where \( l \) is the power associated with the ISI.