On Uniformly Distributed ON/OFF Arrivals in Virtual Output Queued Switches with Geometric Service Times

Itamar Elhanany[†], Michael Kahane, Dan Sadot
Department of Electrical & Computer Engineering
Ben-Gurion University, Israel
[†]email: itamar@ieee.org

***Abstract*— Virtual output queueing is commonly deployed as a buffering technique in high-performance input-queued switch architectures. This paper presents analysis for discrete-time virtual output queued switches with incoming traffic governed by a uniformly distributed Markov modulated ON/OFF process, and geometrically distributed service times. We utilize the *k*-step first-passage time probability matrix to derive the probability generating function of the inter-arrival times distribution. Based on the latter, closed-form expressions for the queue size distribution and mean delay are obtained. The validity of the analysis is established through computer simulations.**

Keywords- Markov modulated ON/OFF process, Virtual output queueing, Input-queued switches, Traffic modeling, Performance analysis.

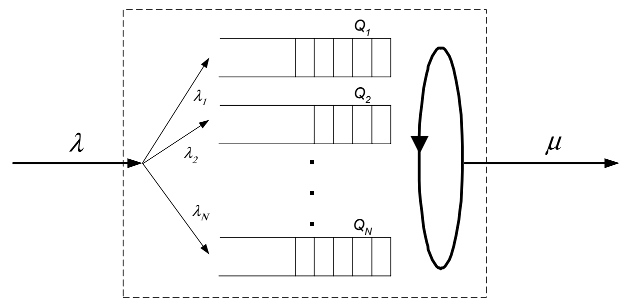

Figure 1. Virtual output queueing system with a mean rate of arrivals λ and a mean service rate μ.

I. Introduction

Input-queued switching architectures, commonly realized using crossbar technologies, are widely deployed in high-performance switches and routers [1], [2], [3] In such systems, each input port may transmit to at most one output port at any given time. A scheduler, whether centralized or distributed, governs the switching process by determining the configuration of the crossbar matrix, thus enabling data cells to traverse the switch fabric. A well-known phenomenon called head-of-line blocking [4] occurs in single queued ingress architectures where contention on a specific egress path prevents non-blocked queued cells/packets from being transmitted, and in doing so limits the overall throughput of the switch.

A common technique for overcoming the head-of-line blocking phenomena is virtual output queueing (VOQ). In VOQ a separate queue is maintained at the ingress port for each of the output destinations, as depicted in figure 1. Since the statistical nature of the traffic arriving at the virtual output queues has significant impact on the scheduler's efficiency, it affects the overall performance, particularly with respect to the latency, throughput and amount of buffer memory required at the ingress modules.

Markov modulated ON-OFF models have been repeatedly incorporated as building blocks for constructing more complex, pragmatic traffic scenarios. Multimedia traffic, which by nature tends to be correlated on several levels, is commonly modeled by a superposition of several ON-OFF sources [5]. The majority of the work presented in the literature, which addresses the analysis of scheduling algorithms under correlated arrivals, focuses on the case of deterministic (constant) service times [6], [7]. For many other applications a more accurate model for the service times is that of a geometrical distribution where in each time slot there is an independent probability of service. This probability of service is typically derived from the scheduling algorithm employed. In most cases, however, evaluation of the switch performance under bursty scenarios is established through simulation results.

In this paper we present analysis for a VOQ system with uniform destination distribution of Markov modulated arrivals and geometrically distributed service times. Based on the probability generating function for the inter-arrival distribution, we obtain closed-form expressions for various performance metrics. Validation of the approach is demonstrated through very good matching between analytical and simulation results. The generic properties of the scheme allows for its application in the context of a wide range of scheduling algorithms.

This paper is organized as follows. Section II provides the queueing notation and model formulations utilized throughout the paper. Section III presents analysis of Markov modulated arrivals for the single-queue case. Section IV focuses on performance analysis of a VOQ system with Markov modulated arrivals, and in section V the conclusions are drawn.

II. QUEUEING MODEL FORMULATION

We begin by assuming a discrete-time queueing system with a single-server and infinite buffer capacity, in which all events occur at fixed time slot intervals. Within each time slot at most a single arrival and a single service event may occur. An early arrival model is considered, for reasons of convenience, such that during a time slot an arrival event will always precede a service event. The service discipline is governed by an i.i.d. Bernoulli process, resulting in geometrically distributed service times. Let μ denote the homogeneous probability of service at any given time slot. The basic stability condition dictates that $\lambda/\mu < 1$, where λ is the arrival rate. Let $Q(n)$ denote the queue occupancy at time slot n, such that

$$Q(n) = \max\langle Q(n-1) + A(n) - D(n), 0\rangle \quad , \qquad (1)$$

where $A(n) \in \{0,1\}$ and $D(n) \in \{0,1\}$ are the number of arrivals and departures during time slot n, respectively. For stability, the arrival rate should converge to the departure rate, such that

$$\lim_{t\to\infty}\left(\frac{\sum_{n=1}^{t} A(n)}{t}\right) = \lim_{t\to\infty}\left(\frac{\sum_{n=1}^{t} D(n)}{t}\right). \qquad (2)$$

It has been shown in the literature [8] that for a *GI/Geo/1* queueing model (general inter-arrival process and geometrically distributed service times), if f_n ($n\geq 1$) is the inter-arrival time distribution, with a probability generating function (p.g.f.),

$$F(z) \equiv \sum_{n=0}^{\infty} f_n z^n = \sum_{n=1}^{\infty} f_n z^n \quad , \qquad (3)$$

then the stationary queue size distribution, $\pi_m = \Pr\{Q = m\}$, will be in the form

$$\pi_m = \begin{cases} 1-\xi & m = 0 \\ \xi(1-\rho)\rho^{m-1} & m \geq 1 \end{cases} \quad , \qquad (4)$$

where ρ is a unique root of the equation

$$z = F(\mu z + (1-\mu)) \qquad (5)$$

that lies in the region (0,1), and ξ is a constant. Taking into consideration the early arrival model, the queue state is examined subsequent to the service phase.

One way of interpreting f_n is the probability of precisely n time slots separating two consecutive arrivals. For the *Geo/Geo/1* model (with parameter λ) the probability of two consecutive arrivals is λ, from which we derive $f_n = \lambda(1-\lambda)^{n-1}$ as the probability of having $n-1$ non-arrival time slots separating two arrivals. The corresponding p.g.f. is $F(z) = \lambda(1-(1-\lambda)z)^{-1}$. When solving for $z=F(z\mu + (1-\mu))$ we find the root

$$\rho = \frac{\lambda(1-\mu)}{\mu(1-\lambda)}, \qquad (6)$$

which is consistent with the well know result for *Geo/Geo/1*, where $\rho = \xi = 1-\pi_\sigma$.

III. THE SINGLE QUEUE WITH ON/OFF ARRIVALS

Consider a discrete-time, two-state Markov chain generating arrivals modeled by an ON-OFF source which alternates between the ON and OFF states. Let the parameters p and q denote the probabilities that the Markov chain remains in states ON and OFF, respectively. An arrival is generated for each time slot that the Markov chain is in state ON. The outcome is a stream of correlated bursts and silent periods both of which are geometrically distributed in length. It can easily be shown that p and q are interchangeable with the mean arrival rate, $\lambda = (1-q)/(2-q-p)$, and mean burst length, $B = 1/(1-p)$. Consequently, the offered load is identical to the steady-state portion of the time the chain spends in state ON.

The probability of two consecutive arrivals, $f_1 = p$, is identical to the probability that following an arrival the Markov chain will remain in state ON. Similarly, f_2 is the probability that following an arrival, the chain transitions to the OFF state and then returns to the ON state. For $n>2$, it is apparent that following a transition from the ON state to the OFF state, there are n-2 time slots during which the chain remains in the OFF state before returning to state ON. As a result we obtain the following general expression for f_n:

$$f_n = \begin{cases} p & n = 1 \\ (1-p)q^{n-2}(1-q) & n > 1 \end{cases}. \qquad (7)$$

The corresponding p.g.f. is

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} f_n z^n = \sum_{n=1}^{\infty} f_n z^n \\ &= pz + \sum_{n=2}^{\infty} q^{n-2}(1-p)(1-q)z^n \\ &= pz + (1-p)(1-q)\frac{z^2}{1-qz} \end{aligned} \qquad (8)$$

The mean inter-arrival time can be found by differentiating $F(z)$ with respect to z and letting $z=1$,

$$\left.\frac{dF(z)}{dz}\right|_{z=1} = \sum_{n=1}^{\infty} n f_n = \frac{2-p-q}{1-q}, \qquad (9)$$

which, as expected, equals to $1/\lambda$.. Next we replace 'z' with '$z\mu$+(1-μ)' in $F(z)$ and solve the equation $z = F(z\mu+(1-\mu))$, to find that the root in the region (0,1) is

$$\rho = \frac{(1-\mu)}{\mu}\left[\frac{1}{\mu(1-p-q)+q}-1\right]. \qquad (10)$$

As with the *Geo/Geo/1* model, the term ρ is a function of both the probability of service, μ, and the parameters of the arrival process. The required condition $\rho<1$ yields that $\mu > (1-q)/(2-p-q) = \lambda$. In order to complete the analysis we need to find ξ or, alternatively, π_0 (given that $\xi = 1-\pi_0$). A departure will occur if, at a given time slot, the queue is granted service and at the same time it is non-empty. Equating the mean probability of arrival to the mean probability of departure, as reflected by (2), we find that

$$\lambda = \mu(1-\pi_0) \;\Rightarrow\; \pi_0 = 1-\frac{\lambda}{\mu}. \qquad (11)$$

Therefore, according to (4), $\xi = \lambda/\mu$ and thus we obtain a close-form expression for the stationary queue size distribution, π_m. Using the latter and Little's theorem, we find two important performance metrics: the mean queue size and mean waiting time,

$$\begin{aligned} E[Q] &= \sum_{n=1}^{\infty} n\pi_n = \frac{\xi}{1-\rho} = \frac{\lambda}{\mu(1-\rho)} \\ E[W] &= E[L]/\lambda = \frac{1}{\mu(1-\rho)} \end{aligned}. \qquad (12)$$

We note that as the probability of service increases, ρ decreases and hence the mean queue size and mean waiting time decrease as well. Figure 2 depicts the mean queue occupancy as a function of the offered load, for a single-queue with two-state Markov modulated arrivals.

IV. THE MULTI-QUEUED MODEL WITH ON/OFF ARRIVALS

We extend the analytical foundations addressed in section III to investigate the case of a switch model deploying virtual output queues, with uniformly distributed Markov modulated arrivals. Letting N denote the number of ports, a burst is defined as a sequence of consecutive arrivals destined to the same output. As with the single queue model, we construct a Markov chain corresponding to the behavior of the examined bursty arrival process (figure 3). The chain consists of N+1 states, N of which represent arrivals going to the N queues, while the last state is the OFF state. We label the ON states as $\kappa_1, \kappa_2, \ldots \kappa_N$, and the OFF state as κ_0. The probability of remaining in the OFF state is q while the probability of remaining in each of the ON states is p. To complement the latter, the probability of returning from any ON state to the OFF state is (1-p) while a transition from the OFF state to any of the ON states is (1-q)/N. Thus, we can represent the

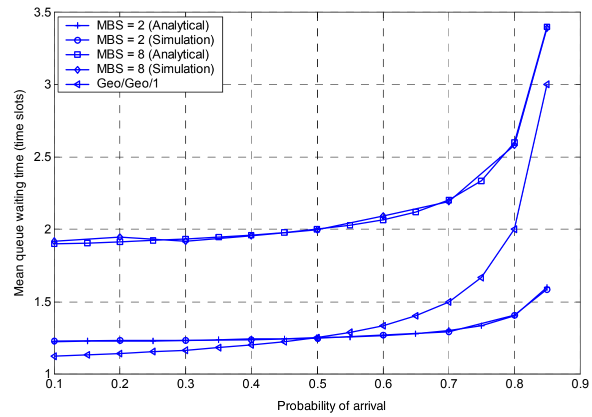

Figure 2. Virtual output queueing system with bursty arrivals at rate λ and geometrically distributed service events with service rate μ.

Markov chain as an $(N+1)\times(N+1)$ transition probability matrix, P, where each element, p_{ij}, denotes the probability of transitioning from the i^{th} state to the j^{th} state among the states of the diagram in figure 3, such that

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots & p_{0N} \\ p_{10} & p_{11} & 0 & \ldots & 0 \\ p_{20} & 0 & p_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ p_{N0} & 0 & 0 & \cdots & p_{NN} \end{pmatrix} \qquad (13)$$

The first row, with the exception of its first element, consists of the probabilities of transitioning from the OFF state to each of the ON states, signifying a beginning of a burst. The first column, with the exception of its first element, contains the probability of returning from each of the ON states to the OFF state (i.e. terminating of a burst). The first element on the diagonal is the probability of remaining in the OFF state while the rest of the diagonal elements are the probabilities of remaining in the various ON states. Accordingly, the examined transition probability matrix is

$$P = \begin{pmatrix} q & \frac{1-q}{N} & \frac{1-q}{N} & \cdots & \frac{1-q}{N} \\ 1-p & p & 0 & \ldots & 0 \\ 1-p & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1-p & 0 & 0 & \cdots & p \end{pmatrix} \qquad (14)$$

We note that P can be represented in terms of the mean burst size, B, and the offered load, λ, by exploiting the relationships described in section II. Moreover, since the distribution of the load is uniform among the queues, each queue receives an average offered load of λ/N. As with the single queue case, we would like to find the p.g.f. of the inter-arrival time distribution from the transition probabilities.

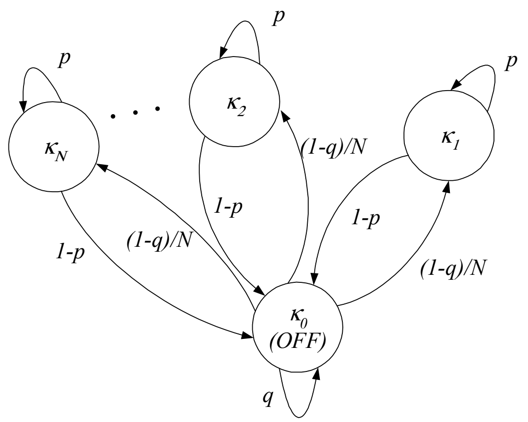

Figure 3. Markov chain characterizing the behavior of a uniformly distributed ON/OFF arrival process to a VOQ with N ports. Each port receives λ/N of the offered load.

The latter is done by utilizing the k-step transition matrix, $P^{(k)}$, in which each element, $p_{ij}^{(k)}$, represents the probability of transitioning from the i^{th} state to j^{th} state in precisely k-steps, with no restrictions made on passing through state j in any of the intermediate steps. In accordance with the Chapman-Kolmogorov equation [9] we have $P^{(k)} = P^k$ ($k \geq 1$), for which the p.f.g. is defined as

$$P(z) = \left[\sum_{n=0}^{\infty} p_{ij}^{(n)} z^n\right] = \left[\sum_{n=0}^{\infty} \left[p_{ij}^{(n)}\right] z^n\right] = \left[\sum_{n=0}^{\infty} P^{(n)} z^n\right] = \left[\sum_{n=0}^{\infty} P^n z^n\right] = \sum_{n=0}^{\infty} (zP)^n = [I - zP]^{-1} \tag{15}$$

where $|z|<1$. The right-hand side of (15) is the matrix form outcome of an infinitely declining geometric series.

We next define the k-step *first passage time* [8] probability matrix, $F^{(k)}$, the elements of which, $f_{ij}^{(k)}$, are the probabilities of transitioning from state i to state j in *precisely* k-steps with the constraint that prior to the k^{th}-step the process has not visited state j. In other words, $f_{ij}^{(k)}$ denotes the probability that the first transitioning instance from state i to state j occurs in precisely k steps. It can be shown [8] that $f_{ij}^{(k)}$ is

$$f_{ij}^{(k)} = \sum_{s_1 \neq j} \sum_{s_2 \neq j} \cdots \sum_{s_{k-1} \neq j} p_{is_1} p_{s_1 s_2} \cdots p_{s_{k-2} s_{k-1}} p_{s_{k-1} j} \tag{16}$$

with $f_{ij}^{(1)} = p_{ij}$ and the s terms are the intermediate states between i and j. Moreover, we observe the clear relationship

$$f_{ij}^{(k)} = \sum_{s \neq j} p_{is} f_{sj}^{(k-1)} \quad . \tag{17}$$

Since the diagonal element, $f_{ii}^{(k)}\big|_{i>1}$, is by definition the probability of k steps separating two consecutive arrivals to queue i, it is identical to the definition of the inter-arrival time distribution for the i^{th} queue. Accordingly, the diagonal elements of the p.g.f. of the first passage time distribution, $F_{ii}(z)\big|_{i>1}$ ($F(z) = [F_{ij}(z)]$), are the p.g.f. of the desired inter-arrival time distribution. It has been shown that the following relationship exists between $P_{ii}(z)$ and $F_{ii}(z)$ [8]:

$$F_{ii}(z) = 1 - \frac{1}{P_{jj}(z)} \quad . \tag{18}$$

Accordingly, as a first step in finding $F_{ii}(z)$, utilizing (15) we are required to find

$$P(z) = (I - zP)^{-1} = \begin{pmatrix} 1 - zq & -z\frac{1-q}{N} & -z\frac{1-q}{N} & \ldots & -z\frac{1-q}{N} \\ -z(1-p) & 1 - zp & 0 & \ldots & 0 \\ -z(1-p) & 0 & 1 - zp & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -z(1-p) & 0 & 0 & \ldots & 1 - zp \end{pmatrix}^{-1} \tag{19}$$

However, we are exclusively interested in finding the diagonal elements of the inverted matrix, $P_{ii}(z)\big|_{i>1} = |M_{ii}| \cdot (|I - zP|)^{-1}$, where $|M_{ij}|$ denotes the determinant of a minor matrix of $[I - zP]$ with i^{th} row and j^{th} column removed. By multiplying each row in $[I - zP]$, except for the first, by $z(1-q)/((1-zp)/N)$ and adding the result to the first row, we reduce $[I - zP]$ to an equivalent triangular matrix in the form

$$\begin{pmatrix} 1 - zq + \frac{z(1-q)}{1-zp} & 0 & 0 & \ldots & 0 \\ -z(1-p) & 1 - zp & 0 & \ldots & 0 \\ -z(1-p) & 0 & 1 - zp & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -z(1-p) & 0 & 0 & \ldots & 1 - zp \end{pmatrix} \tag{20}$$

the determinant of which is simply the multiplication of all its diagonal elements. Applying the same rule to find the determinant of the minor matrix, $|M_{ii}|$, yields

$$P_{ii}(z)\big|_{i>1} = \frac{1 - zq + \dfrac{z(N-1)(1-q)}{N(1-zp)}}{1 - zq + \dfrac{z(1-q)}{1-zp}} \quad . \tag{21}$$

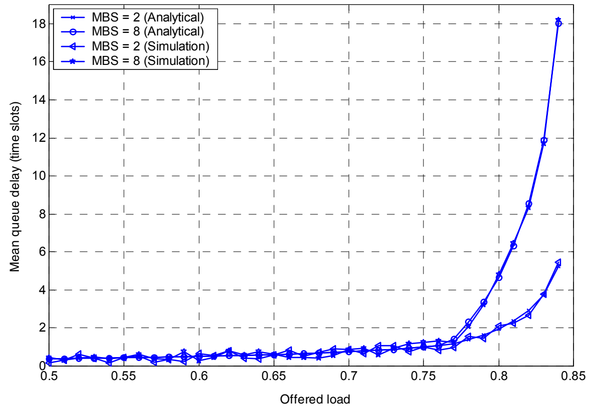

Figure 4. Mean queue waiting time as a function of the offered load for N=32, μ = 0.85 and mean burst sizes of 2 and 8 time slots.

Using (18), we obtain the p.g.f. of the inter-arrival time distribution as

$$F_{ii}(z)\big|_{i>1} = \frac{\left[p^2 q - p(1-p)(1-q)\right]z^3 + \left[(1-p)(1-q)/N - pq - p^2\right]z^2 + pz}{\left[pq - (N-1)(1-p)(1-q)/N\right]z^2 - (p+q)z + 1} \quad (22)$$

To facilitate the completion of the analysis, we are left with solving the equation $F(z\mu' + 1 - \mu') = z$ where μ' denotes the probability of service to each of the queues. To simplify the calculation we may apply the simple variable transform, $z' = z\mu' + 1 - \mu'$, and solve the equivalent equation $F(z') = (z' - 1 + \mu')/\mu'$. Note that since there exists a solution for $0<z<1$ it follows that $0<z'<1$, since $0<\mu'<1$. We can further simplify the resulting third-degree polynomial by factoring out $(z' - 1)$ and solving for z' the following quadratic equation:

$$\left[\mu' p(p+q-1) - pq + \frac{N-1}{N}(1-p)(1-q)\right]z'^2 + \left[\mu'(1-2p-q) + p + q\right]z' + \mu' - 1 = 0 \quad (23)$$

To find μ' we recall that as a byproduct of the stochastic equilibrium that holds for the VOQ we have

$$\lambda = \mu\left(1 - \pi_0^{\,N}(1-\lambda)\right), \quad (24)$$

expressing the balance that must exist between the probability of arrival, λ, and the probability of departure interpreted here as the probability of service multiplied by the probability that at least 1 of the N queues is non-empty. From (24) we extract π_0. Next we find the probability of service to each queue, μ', as the probability of service to the entire VOQ multiplied by the probability that the queue prevails in contending for transmission against the other non-empty queues, such that

$$\mu' = \frac{\mu}{N(1-\pi_0)} = \frac{\mu}{N\left(1 - \left(\frac{\mu - \lambda}{\mu(1-\lambda)}\right)^{-N}\right)}. \quad (25)$$

Substituting (25) in (23) and recalling that $\xi = 1-\pi_0$, we obtain a closed-form expression for the queue size distribution from which the mean queue size and mean delay are directly found. Figure 4 compares the mean queueing delay obtained from simulations results to those obtained using the presented analysis for an ON/OFF process, where N=32, μ=0.85 and the mean burst sizes are 2 and 8 cells. It can be seen that the simulation results clearly validate the accuracy of the analytical deductions.

V. Conclusions

In this paper we present an analytical technique for evaluating the queueing behavior of a virtual output queued switch with non-uniformly distributed ON/OFF arrivals and geometric service times. Utilizing the unique relationship that exists between the transition probabilities and the k-step first passage time distribution, closed-form expressions for the mean queue occupancy and delay are obtained. This distinctive analytical approach can be further exploited to investigate other traffic arrival processes.

References

[1] I. Elhanany, D. Sadot, "A Contention-Free Tbit/sec Packet Switching Architecture for ATM over WDM Networks," *IEICE Trans. on Communications*, Vol. E83-B, No. 2, Feb. 2000.

[2] N. McKeown, "The iSLIP Scheduling Algorithm for Input-Queued Switches," *IEEE/ACM Trans. on Networking*, Vol. 7, No. 2, pp. 188-201, April 1999.

[3] R. Rojas-Cessa, E. Oki, H. J. Chao, **"**CIXOB-k: Combined Input-Crosspoint-Output Buffered Packet Switch," *Proc. IEEE GLOBECOM 2001*, Vol. 4, pp. 2654-2660, Nov. 2001.

[4] M. J. Karol, M. G. Hluchyj, and S. P. Morgan, "Input versus Output Queueing in a Space Division Switch," *IEEE Trans. Communications*, Vol. COM-35, pp. 1347-1356, Dec. 1987.

[5] A. La Corte, A. Lombardo, G. Schembra, "Modeling Superposition of ON-OFF Correlated Traffic Sources in Multimedia Applications," *Proc. INFOCOM 1995*, pp. 993-1000.

[6] H. Bruneel, I. Wuyts, "Analysis of Discrete-Time Multiserver Queueing models with Constant Service Times," *Operations Research Letters*, Vol. 15, pp. 213-236, 1994.

[7] S. Wittevrongel, "Discrete-Time Buffers with Variable-Length Train Arrivals," *Electronics Letters*, Vol. 34, No. 18, pp. 1719-1721, Sep. 1998.

[8] J. J. Hunter, *Mathematical Techniques of Applied Probability: Discrete Time Models*: *Techniques and Applications*, Vol. 2, Academic Press, 1983.

[9] D. Gross, C. M. Harris, *Fundamentals of Queueing Theory*, John Wiley and Sons, 1985.