Chapter 5

Thermofluid Engineering and Microsystems

Microfluidic Dynamics

Navier-Stokes equation

1. The momentum equation
2. Interpretation of the NS-equation
3. Characteristics of flows in microfluidics (Re-number)
4. Examples of laminar flows
5. Summary
History of the Navier-Stokes Equation

Navier-Stokes equation is the central relationship of fluid dynamics

• Basic assumptions
  – continuous media
  – continuum mechanics

• In case of liquids:
  – Assumptions fulfilled in macrofluidics as well as microfluidics (down to 10-100 nm) for liquids

How to describe the motion of a fluid?

\[ F = m \cdot a \]

\[ \frac{F}{V} = f_a = m \cdot a = \rho \cdot \frac{dV}{dt} \]

• Velocity depends on space and time \( v = v(x(t), t) \)
• Consider Newton's law for a infinitesimal volume \( V \)
Momentum equation (acceleration) in x-direction:

\[ f_{a,x} = \rho \cdot \frac{dv_x}{dt} = \rho \cdot \frac{d}{dt} v_x(x(t), y(t), z(t), t) = \]

\[ = \rho \cdot \left( \frac{\partial v_x}{\partial x} \frac{dx}{dt} + \frac{\partial v_x}{\partial y} \frac{dy}{dt} + \frac{\partial v_x}{\partial z} \frac{dz}{dt} + \frac{\partial v_x}{\partial t} \right) \]

Momentum equation in 3D (vector notation):

\[ \mathbf{f}_a = \rho \cdot \frac{d\mathbf{v}}{dt} = \rho \cdot (\mathbf{v} \cdot \nabla) + \frac{\partial \mathbf{v}}{\partial t} \]

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \hat{e}_1 \frac{\partial}{\partial x} + \hat{e}_2 \frac{\partial}{\partial y} + \hat{e}_3 \frac{\partial}{\partial z} \]

Nabla operator

Acceleration over Time

\[ \mathbf{f}_a = \rho \cdot \frac{\partial \mathbf{v}}{\partial t} \]
Acceleration along a Stream Line

- Increase of the fluid velocity due to mass conservation
- Fluid has to be accelerated along the streamline

\[ \mathbf{f}_a = \rho \mathbf{v} \nabla \mathbf{v} \]

The Navier-Stokes (NS) Equation

- ... for incompressible Newtonian fluids

\[ \mathbf{f}_a = \rho \left[ \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{friction}} + \mathbf{f}_{\text{volume}} \]

**left hand side**
- Change in momentum (Newton)
  - due to change of velocity over time at a given location
  - due to acceleration of fluid e.g. when moving into smaller flow channel cross sections (also in stationary cases)

**right hand side**
- Forces acting on fluid
- Pressure gradient
- Friction forces
- Volume forces
Interpretation of N-S Equation

\[ \mathbf{f}_a = \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{friction}} + \mathbf{f}_{\text{volume}} \]

Pressure gradient

\[ f_{\text{pressure}} = \frac{dF_{\text{pressure}}}{dV} = -\nabla p \]

Body force (= volume force)

\[ \mathbf{f}_a = \mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{friction}} + \mathbf{f}_{\text{volume}} \]

Actuation of fluid by body force:

- Body force \( f_{\text{volume}} \) acts in the volume itself
Different types of body forces (= volume forces)

- Centrifugal forces (ω rotational speed, r radius, ρ density)
  \[ f_{\text{volume},c} = \rho \omega^2 \mathbf{r} \]

- Gravity
  \[ f_{\text{volume},g} = \rho \mathbf{g} \]

- Electrostatic forces (ρq charge density, E electric field strength)
  \[ f_{\text{volume},q}(\mathbf{x}) = \rho_q(\mathbf{x}) \mathbf{E}(\mathbf{x}) \]

Example: static pressure under gravity

- Only gravity is considered in NS-equation
  \[ f_{\text{volume}} = f_g = \rho \mathbf{g} \]

- Stationary flow (v=const.):
  - friction is zero (no motion)
  - acceleration is zero (\( \frac{dv}{dt} = 0 \))
  \[ f_{\alpha} = f_{\text{pressure}} + f_{\text{friction}} + f_{\text{volume}} \implies -f_{\text{pressure}} = f_{\text{volume}} \]

- Results in:
  \[ \nabla p = \rho \alpha \mathbf{g}, \quad \frac{dp}{dy} = \rho \alpha \mathbf{g} \implies p_{\text{max}} = \rho \alpha \mathbf{g} h \]
Example for water in a micro channel:

- $\rho = 1000\text{kg/m}^3$
- $g = 9.81\text{m/ s}^2$
- $h = 100\ \mu\text{m}$

\[ p_{\text{max}} = 0.981\text{Pa} = 9.81 \times 10^{-6}\text{bar} \]

gravitational effects negligible for microfluidic devices.

Friction

- Friction is a phenomena related to the motion (velocity) of the fluid

\[ F_{\text{friction}} = -\eta \cdot A \cdot \frac{dv_z}{dx} \]

\[ f_{\text{friction}} = \frac{dF}{dV} = \eta \frac{\partial^2 v}{\partial x^2} \]

negative sign signifies that motion is damped by the friction
Simplifications in Microfluidics

\[ \rho_a \left[ \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f}_{\text{Volume},g} \]

- Gravity is neglected
- Influence of convection small, \( \rho (\mathbf{v} \cdot \nabla) \mathbf{v} \to 0 \), i.e. we assume that there is no convection
- If additionally a stationary flow is considered ...

\[ \rho_s \left[ \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f}_{g} \]

Poisson equation (driving pressure and friction are balanced in a stationary laminar flow):

\[ \nabla p = \eta \nabla^2 \mathbf{v} \]

\[ \mathbf{f}_{\text{pressure}} = -\mathbf{f}_{\text{friction}} \]

Characteristics of flows in microfluidics (Re-number)

- Behaviour of flow ...
  - laminar = predictable
  - turbulent = chaotic
- ... is dominated by the ratio between
  - inertial effects (kinetic energy) and
  - frictional effects (damping)
- Friction consumes kinetic energy and converts it into heat
- Motion is slowed/damped down

- **Turbulences** ...
  - are possible only when friction is small compared to the kinetic energy
Reynolds Number

Approximating friction energy

\[ E_{\text{friction}} \propto \| F_{\text{friction}} \| \cdot l = \eta \frac{V}{l} A \cdot l = \eta \frac{V}{l} V \]

Approximating kinetic energy

\[ E_{\text{kin}} \propto mV^2 \]

Ratio of work spent on acceleration to energy dissipated by friction

\[ \frac{E_{\text{kin}}}{E_{\text{friction}}} = \frac{mV^2 l}{\eta V V} = \frac{\rho l V}{\eta} = Re \]

- the Re-number is the most important dimensionless number in microfluidics
- low Re-numbers, i.e. viscous forces dominate, are typical for microfluidics

Critical Reynolds Number

- Critical \( Re^* \) number corresponds to a critical velocity \( V^* \)

\[ V^* = Re^* \frac{\eta}{\rho l} \]

- For a micro device \( V^* \) is hardly reached (\( l = 100 \mu m \rightarrow V^* = 25 \text{ m/s} \))
- Typically \( Re^* \) is in the range of 2300
- As \( Re \) increases further, turbulent character of flow increases
Flow Regimes

Creeping/Stokes flow ($Re < 1$):
- No lateral convection
- Adjacent layers (lamellae) do not "interfere"

Intermediate ($1 < Re < Re^*$)
- Lateral convection becomes increasingly important

Turbulent ($Re^* < Re$)
- Perturbations are amplified
- Curling of field lines
- "Unpredictable" development of field of velocity vectors over time

$Re^* \sim 2300$

Governing Equations

- Fluid flows are determined by knowledge of velocities, pressure, density, viscosity, specific heat and temperature.

1. Mass conservation equation (continuity equation)
   \[
   \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0
   \]
   For incompressible fluids
   \[
   \nabla \cdot \vec{V} = 0
   \]

2. Momentum equation (Navier-Stokes equations)
   \[
   \rho \left( \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right) = -\nabla P + \mu \nabla^2 \vec{V} + \vec{F}
   \]
Newton's Law of Viscosity

- **Dynamic viscosity** \( \eta \) [Pa s = kg m\(^{-1}\) s\(^{-1}\)]

Dynamic viscosities \( \eta \) @ 20°C from CRC Handbook of Chemistry and physics

<table>
<thead>
<tr>
<th>( \eta )</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>air</td>
<td>0.019 mPa s</td>
</tr>
<tr>
<td>oxygen</td>
<td>0.020 mPa s</td>
</tr>
<tr>
<td>benzene</td>
<td>0.652 mPa s</td>
</tr>
<tr>
<td>water</td>
<td>1.002 mPa s</td>
</tr>
<tr>
<td>ethyl alcohol</td>
<td>1.200 mPa s</td>
</tr>
<tr>
<td>oil, olive</td>
<td>84 mPa s</td>
</tr>
<tr>
<td>glycerin</td>
<td>1490 mPa s</td>
</tr>
<tr>
<td>honey</td>
<td>10 000 mPa s</td>
</tr>
</tbody>
</table>
\[
\text{Re} = \frac{\rho lv}{\eta}
\]

- For \( \text{Re} < \text{Re}^* \) the flow is laminar
- For \( \text{Re} > \text{Re}^* \) the flow is turbulent

In microfluidics we (usually) assume
- No gravity
- Incompressibility
- Dominance of viscous forces

Laminar flow, low Re

High degree of laminarity implies that the streamlines are locally parallel.
Turbulent flows

Assumptions
- friction negligible
- no gravity
- → Euler Equation

\[ \rho \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f}_{\text{force}} \]

\[ \rho \left( \frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p = 0 \]

- General solution of Euler equation are wave equations

Velocity vectors unpredictably oscillating in time

Couette Flow

How does the flow look between two plates when
- One plate is at rest
- The other plate is moving at velocity \( v \)

Situation is called “Couette Flow” …
- Flow is driven by viscous drag force acting on the fluid
- Linear velocity gradient
Flow imposed on fluid by shear forces
- No pressure gradient applied $\nabla p=0$
- Stationary flow $\to$ Poisson equation

$$\nabla p = \eta \nabla^2 \mathbf{v} \quad \Rightarrow \quad \nabla p = \eta \frac{\partial^2 v_z}{\partial x^2} \quad \Rightarrow \quad 0 = \eta \frac{\partial^2 v_z}{\partial x^2}$$

Boundary conditions
- Wall at $x=0$ at rest, i.e. $v(x=0)=0$
- Wall at $x=d$ moving at speed $v(x=d)=v_0 = \text{const.}$ in $z$-direction
- Linear flow profile

$$0 = \eta \frac{\partial^2 v_z}{\partial x^2} \quad \Rightarrow \quad v_z(x=0) = \frac{x}{d} v_0$$

Viscosity $\eta$
- internal friction of fluid
- transfer of momentum from one plane sliding parallel to another
- plane thickness is infinite

1. interlocked molecule layer
2. velocity gradient
Laminar pressure driven flow (PDF) through slit

- Solution (no-slip) condition:

\[ v_z(x) = c \cdot (d - x)(d + x) \]

- Peak velocity from Poisson equation (@ x=0):

\[ \frac{\nabla p}{\eta} = \frac{d^2 v_z}{dx^2} = 2c \implies v_z(x) = \frac{\nabla p}{2\eta} \cdot (d - x)(d + x) \]

- Maximum velocity in a slit

\[ v_{z,max} = \frac{\Delta p}{2\eta l} d^2 \]

- Parameters (water):

  \[ \Delta p = 10000 Pa = 0.1 bar \]
  \[ d = 100 \mu m \]
  \[ l = 10 cm \]
  \[ \eta = 0.001 Pas \]

- Maximum velocity:

\[ v_{z,max} = 0.5 \frac{m}{s} \]
Taylor Dispersion

- Parabolic flow profile
- Residence time near walls is high
- Residence time in the middle of the channel minimized
- Liquid plug is diluted very fast (dispersion) → relevant for injection of (biochemical) samples into tubings

Laminar PDF through tube

Pressure-driven flow in capillaries
- Important phenomenon in nature
- e.g., transport of nutrients in plants and animals by heart

Symmetry
- Parabolic flow profile
- Cylindrical symmetry

No slip conditions at the boundary

\[ v_z(r) = \frac{\Delta P}{4\eta l} \left( R_0^2 - r^2 \right) \]
Laminar PDF through tube: example

- Maximum velocity in a tube
  \[ v_{z,\text{max}} = \frac{\Delta p}{4\eta l} R_0^2 \]
- Parameters
  \[ \Delta p = 10000 Pa = 0.1 bar \]
  \[ R_0 = 100 \mu m \]
  \[ l = 10 cm \]
  \[ \eta = 0.001 Pas \]
- Maximum velocity
  \[ v_{z,\text{max}} = 0.25 \frac{m}{s} \]

Hagen-Poiseuille: analogy to electrical circuits

- How to obtain a relation between pressure and flow through a tube?
- Flow:
  \[ I_v = \int_A \mathbf{v} dA = \int_r v_z(r) 2\pi r dr \]
- Circular tube:
  \[ I_v = \frac{\pi R_0^4}{8\eta} \frac{\Delta p}{l} \]
**Laminar Fluid Flow in Circular Conduits**

- **The Hagen-Poiseulle Equation**

This equation relates the volumetric flow, \( Q \), and the corresponding pressure drop, \( \Delta P \).

\[
Q = \frac{\pi a^4}{8\mu} \left[ -\frac{d}{dx}(P + \rho g y) \right]
\]

where \( y \) = elevation of the tube from a reference plane.

The pressure drop in the fluid over the tube length, \( L \) is:

\[
\Delta P = \frac{8\mu L Q}{\pi a^4}
\]

(5.16)

**NOTE:** The pressure drop, \( \Delta P \), meaning a reduction in half in the radius \( \rightarrow 2 \times 16 \) times increase in pressure drop (pumping power)!

The equivalent head loss in relation to \( Q \) is:

\[
h_{f,t} = \frac{128\mu L Q}{\pi \rho g a^4}
\]

(5.17)

**Laminar Fluid Flow in Circular Conduits**

- **The Hagen-Poiseulle Equation**

For conduits with non-circular cross-sections.

In such cases, the hydraulic diameter, \( d_h \), is used in the Hagen-Poiseulle equations.

This diameter is defined as:

\[
d_h = \frac{4A}{p}
\]

(5.19)

where \( A \) = cross-sectional area of fluid flow

\( p \) = wet perimeter.

![Rectangular conduit filled with fluid](image)

\[
d_h = \frac{4A}{p} = \frac{4(wh)}{2(w+h)} = \frac{2wh}{w+h}
\]

![Rectangular conduit filled with fluid up to \( h_i \)](image)

\[
d_h = \frac{4wh_i}{w+2h_i}
\]
Hagen-Poiseuille

\[ \Delta p = \frac{8\eta l}{\pi R_0^4} \cdot I_f \]

Analogy to electrical circuits

\[ U = R \cdot I \]

\[ R_f = C_{\text{geometry}} \cdot \frac{\eta l}{A^2} \]

- the fluidic resistance scales like \( R_0^4 \)
- decreasing \( R_0 \) by a factor 10 \( \Rightarrow \) \( R_f \) increases by \( 10^4 \)

Hagen-Poiseuille: significant role of cross section

- Divide channel into for segments
- Resistance of …
  - every channel increased by factor \( 2^4 = 16 \)
  - all channels together by factor 4

Although same cross section area is used, the fluidic resistance increases dramatically
Summary

- At the boundaries usually no slip conditions are assumed
- Pressure driven flow (PDF) in micro channels typically shows a parabolic profile
  - Zero velocity at the boundary
  - Maximum velocity in the centre of the channel
  - Shear force $F \sim \eta \frac{dv}{dx}$
- Consider Taylor Dispersion when injecting fluid samples
Slit (PDF):
\[ v_z(x) = \frac{V_{\text{p}}}{2\eta} (d-x)(d+x) = v_{z,\text{max}} - \frac{\Delta p}{2\eta}x^2 \]

Tube (PDF): \[ v_z(r) = \frac{\Delta p}{4\eta l} (R_0^2 - r^2) = v_{z,\text{max}} - \frac{\Delta p}{4\eta l}r^2 \]

Volume flow (throughput) scales with \( \sim R^4 \)

Hagen-Poiseuille (tube):
\[ \Delta p = \frac{8\pi l}{\pi R_0^4} \cdot I_z \]

In analogy to electrical circuits, \( \Delta p/l_\theta \) is called fluidic resistance \( R_\theta \)

Fluidic resistance is proportional to:
- Fluid viscosity \( \eta \)
- Length \( l \) of the channel
- Inversely proportional to \( A^2 \)

**Example: capillary filling**

\[ \rho \left[ \frac{\partial}{\partial t} v + (v \cdot \nabla) v \right] = -\nabla p + \eta \nabla^2 v + f_{\text{volume}} \]

**Assumptions**
- Gravity negligible
- Inertia term negligible
- Poisson equation \( \nabla p = \eta \nabla^2 v \)

**How fast does the meniscus travel?**
- Calculate mean velocity in tube cross section (from Hagen-Poiseuille)
\[ v_z(r,t) = \frac{\Delta p}{4\eta z(t)} (R_0^2 - r^2) \]

\[ v_z(t) = \frac{\Delta p}{8\eta z(t)} R^2 \]
Example: capillary filling

\[
v_z(t) = \frac{\Delta p}{8\eta z(t)} R^2
\]

\[
v_z(t) z(t) = \frac{\partial z}{\partial t} z(t) = \frac{\Delta p R^2}{8\eta}
\]

\[
\Delta p = \frac{2\sigma \cos \theta}{R}
\]

\[
z(0) = z_0 \quad \text{boundary condition}
\]

\[
z(t) = \sqrt{\frac{\sigma \cos \theta R}{2\eta}} t + z_0^2
\]

- Meniscus travels very fast when entering a microchannel
- Meniscus gets much slower, with channel length
Self Priming of Microfluidic Chambers

- flat Hydrophilic chamber $\rightarrow$ self-priming via capillary forces
- smooth / droplet shaped design prevents bubble-trapping
  $\rightarrow$ reproducible self-priming without entrapped bubbles

Self Priming of Microfluidic Blind-Channels

- blind channel = channel with just one opening
- channel is hydrophilic
  $\rightarrow$ self-priming via capillary forces
- problem:
  bubble gets trapped at dead end of channel (gas cannot escape)
- solution:
  T-shaped channel cross-section for complete filling without bubble trapping
**Bubble-free Priming Sequence**

1. capillary wicking along edges between lid and sloped walls
2. transition of liquid to bottom layer at dead end of capillary
3. reverse filling of capillary → complete evacuation of gas

![Diagram of priming sequence](image)

**Stokes Drag and Relative Velocity**

- Particles at rest are accelerated by the Stokes Drag
  \[ F_{D,\text{Stokes}} = 6\pi \eta v r_0 = m\ddot{x} \]

- Particles in equilibrium with the flow are not accelerated (stationary condition: relative velocity \( v \) of fluid and particle = zero)
  \[ F_{D,\text{Stokes}} = 6\pi \eta v r_0 \xrightarrow{v \rightarrow 0} 0 \]

→ particles move with the flow

![Diagram of Stokes Drag](image)
Particle Motion in Fluid

1. Stokes force (drag) on particles
   \[ u_{\text{particle}} = u_{\text{fluid}} + \frac{F_{\text{particle}}}{\gamma} \]

2. Buoyancy \( u_g \)

3. DEP
   \[ \tau_a = \frac{m}{\gamma} = \frac{2 \rho a^2}{9 \eta} \leq 10^{-6} \text{ sec for microparticles} \]

spherical particles:
\[ 0.12 \mu m/s \text{ for } 1 \mu m \text{ latex particles} \]
\[ 0.9 \mu m/s \text{ for } 1 \mu m \text{ particles, } 10 \mu m \text{ from electrode, } @SVrms \]

Hydrodynamic Focussing Based on Sheath Flow
Hydrodynamic Focusing

Experimental setup
- Microchannel open to larger vessel
- Liquid from outside is sucked into the channel
- Laminar regime (streamlines do not intersect)

Full solid angle projected onto tiny orifice cross section

Sheath Flow Arrangement

- Stream in the middle of a channel is guided by two side flows
- Increasing $\Phi_1$ and $\Phi_2$
  - allow to focus the sample flow $\Phi_{\text{sample}}$
  - reduce width of lamellae $W_{\text{sample}}$

$\Phi_1 = \Phi_2 > \Phi_{\text{sample}}$
Flow Cytometry

Line-up of particles / cells
- Variation of the width $w_i$ of the sample stream
- Change of position of main stream by adjusting flow in side channels

Application
- Particle (e.g. cell) counting

Hydrodynamic Separation of Particles

- Smaller particles are able to reach regions near the wall
- Larger particles are excluded from the area near the wall
- In the average, larger particles are more centred and travel faster with faster streamlines
Pinched-Flow Fractionation (PFF)

- Y-channel structure
- asymmetric flow-rates of two channels ($Q_1 > Q_2$)
  → particle-containing phase $Q_2$ is pinched at sidewall (particles are aligned on upper wall (a))
- channel-widening (b) leads to spreading stream-lines
  → particles follow the streamline of their center position

M. Nakashima, M. Yamada, M. Seki
Proc. of IEEE MEMS 2014, pp. 33-36

- all particles aligned on the sidewall
- the bigger particles “experience” another part of the parabolic velocity-profile (their centre is located farer in the middle of the channel)
- at the widening, all particles follow “their” stream-line (according to the position of their centre)
  → the smaller particles end-up at another position (in y) compared to the bigger ones

successful separation of two particle sizes (diameter: 15 µm, 30 µm)
**Working Principle:**

- Fluid stream containing particles is focused to a wall until all particles are in contact to the wall ("pinched")
- Small particles get closer to the wall than large particles
- All particles follow the stream lines of their "center of gravity" after the pinched section they leave towards different angles
Hydrodynamic Rectification

- T-junction with adjustable flow-rate ratio \( Q_0/Q_x \)
- inlet-flow \( Q_0 \) = particle suspension with two different sized particles (radius: \( r_1, r_2 \))
- tuning of outlet-flow with different distribution-ratios \( a_x \) \( (Q_1, Q_2, Q_3) \)
- \( \text{fraction } w_y \text{ of main flow exits through vertical outlet} \)


Hydrodynamic Rectification: Performance

- different outlet flow rates in a row (adjusted by different flow resistance of outlets)
- rectified particle position (on lower sidewall)
- two different particles sizes: \( r_1 = 2.1 \mu m \rightarrow \text{Outlet 4} \)
  \( r_2 = 3.0 \mu m \rightarrow \text{Outlet 3} \)
- successful separation

applications:
- sorting of cells (e.g. blood cells)