Penultimate Polyhedra

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Introduction

These are some notes that I originally hacked up for my sister. They describe how to make polyhedra out of the "penultimate" module. This module is originally described in Jay Ansill's book "Lifestyle Origami," [Ans92] and he attributes the module to Robert Neale. I have omitted how to put the modules together – buy the book, or figure it out for yourself. It's pretty obvious. The pentagon module is pretty much lifted straight from the book (although I've found 3x4 paper easier to work with than 4x4 paper), but the others are my own tweaks.

A note about cutting and glue. The triangle and square modules as pictured have cuts. These are not necessary — you may use inside folds to achieve the same purpose (i.e. the tabs that you are inserting would be too long or wide otherwise). When you do use the inside folds, the tabs become thick, and it takes more patience to get the modules together. Also, the resulting polyhedron is often less stable. However, the choice is yours. If you care more about the purity of the art form (i.e. no cuts or glue), then that is achievable. I'd recommend the dodecahedron and truncated icosahedron as excellent models that are very stable without cuts or glue. However, my personal preference is to cut them and glue them once I'm finished. This is because otherwise, the larger polyhedrons tend to sag after a few months. Gluing has the additional benefit that the polyhedrons are more cat and child proof.

This method of making modules lends itself to many variations besides the ones shown here. All you need is a calculater with trigonometric functions and you can figure them out for yourself. Besides the Platonic and Archimedian solids, I have made various others: rhombic dodecahedron, rhombic triacontahedron, numerous prisms and antiprisms, stella octangula, great and lesser stellated dodecahdra, compound of 5 tetrahedra, compound of 5 octahedra, dual of the snub cube, etc. If you're interested, I can give descriptions of the modules, although perhaps not quickly. I also have pictures of many finished polyhedra online (in gif files) — send me email if you'd like me to send them to you[1].

The polyhedron numbers referenced below are from the pictures of the Archimedean solids in Fuse's book *"Unit Origami"* [Fus90]. Kasahara/Takahama's *"Origami for the Connoisseur"* [KT87] also has pictures of these polyhedra with a different numbering.

I haven't included modules for octagons or decagons. I've made octagonal ones, but they're pretty flimsy, meaning that the resulting polyhedra cannot exist in the same house as cats without the aid of glue or a gun. Of course, that doesn't bother me much. If you can't figure out how to make octagonal or decagonal modules, send me email, and I'll make the diagrams.

If you are interested in polyhedrons, I'd recommend reading Wenninger's *"Polyhedron Models"* [Wen71], Holden's *"Shapes, Space and Symmetry"* [Hol71] and for a more mathematical treatment, Coxeter's *"Regular Polytopes"* [Cox48]. There is a web page with beautiful renderings of the uniform polyhedra at `http://www.inf.ethz.ch/department/TI/rm/unipoly/`.

[1] or see `http://www.cs.utk.edu/~plank/plank/origami/origami.html`

Modular origami is found in many origami books. Notable in these are the Fuse and Kasahara books mentioned above [Fus90, KT87], as well as Gurkewitz's *"3-D Geometric Origami"* [GA95], and Yamaguchi's *"Kusudama"* [Yam90]. Jeannine Mosely has invented a brilliantly simple module for the greater and lesser stellated dodecahedrons. If you are interested in that module, let me know and I'll dig it up for you.

Notes on the 19 Archimedean Solids

These are some of the polyhedra that you can make with the basic modules (triangle, square, pentagon, hexagon). The ones with octagons and decagons can be made with similar modules, but they're pretty flimsy, so I don't include them. Each module is an edge of the polyhedron. The notation is as follows — if it says "sq–tr", then it means to fold a module with a 90-degree angle on one side, and a 60-degree angle on the other. That edge will be used for places on the polyhedron where a triangle meets a square. For example, on the cuboctahedron, all edges are like this.

Coloring is a matter of taste. I have made most of these polyhedra, and some colorings to look much better than others (at least to me). In general, I've found that it's best to make sure that all three edges of any triangle are not the same color. Two are fine. Three tend to blur the fact that it's a triangle.

- **The Tetrahedron (#1)**. 4 triangular faces. 6 modules. All are tr–tr. The last one is usually difficult to get in. I usually color with 2 edges each of three different colors such that each triangular face is composed of edges of three different colors.
- **The Cube (#2)**. 6 square faces. 12 modules. All are sq–sq. Most any coloration works.
- **The Octahedron (#3)**. 8 triangular faces. 12 modules. All are tr–tr. Most any coloration works.
- **The Dodecahedron (#4)**. 12 pentagonal faces. 30 modules. All are pe–pe. This is a great piece of origami – simple to make, and rock solid. It is a good one upon which to learn how to use these modules. There are a couple of neat colorations here. They mostly use ten modules each of three different colors. One way is color such that no two adjacent edges on a pentagonal face are the same. The one I like better is to make the top and bottom pentagons out of color 1. Then have the 10 edges emanating from the top and bottom pentagons be color 2. The remaining ten edges of color 3 form a band around the middle. You can use this same design with two colors by making the band around the middle from color 1. Finally, you can color again with ten modules each of three different colors in the following way: Take five modules of one color, and fit them together as follows:

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  Do that with the remaining pieces so that you have 6 composites like the one above, two of each color. These will fit together to make a dodecahedron with each pair of composites on opposite ends of the dodecahedron. (This is related to inscribing a cube in a dodecahedron).
- **The Icosahedron (#5)**. 20 triangular faces. 30 modules. All are tr–tr. You can color this like the dodecahedron, with ten modules each of three difference colors. It can be colored so that all triangles have edges of each color. Or you can color in a way analgous to the dodecahedron: All triangles meet in groups of five. Take five edges of color 1, and make one vertex of the icosahedron. This will make five incomplete triangles. Complete the triangles with color 2. Repeat this with the remaining five modules of color one, and the remaining five modules of color 2. Now you have made two pentagonal pyramids, which compose the top and the bottom of the icosahedron. Use color 3 for the remaining edges, which make a zig-zag around the middle.
- **Truncated Tetrahedron (#6)**. 4 triangular faces, 4 hexagonal faces. 18 modules: 12 tr–he and 6 he–he. You can color this with three colors as follows: Arrange the he–he modules like they are the edges of a tetrahedron. Then add the tr–he modules so that all triangles have edges of each color. You can do this so that each hexagon has no adjacent edges of the same color, or so that hexagon edges all come in pairs of the same color.
- **Truncated Cube (#7)**. 6 octagonal faces, 8 triangular faces. 36 modules: 12 oc–oc, 24 tr–oc. I haven't included modules for octagons.
- **Truncated Octahedron (#8)**. 6 square faces, 8 hexagonal faces. 36 modules: 12 he–he, 24 sq–he. This works nicely with two colors – all the sq–he modules are one, and all the he–he are another. Or you can use three colors, evenly divided so that opposite squares are the same color (and all edges in a square are the same color), and modules connecting two squares are of the third color.

- **Truncated Dodecahedron (#9)**. 12 decagonal faces, 20 triangular faces. 90 modules: 30 de–de, 60 tr–de. I have not made this one. It requires decagonal faces.
- **Truncated Icosahedron (#10)**. 12 pentagonal faces, 20 hexagonal faces. 90 modules: 60 pe–he, 30 he–he. This makes a beautiful piece of origami. All the ones I've made have been two colors: one for the pe–he modules, and one for the he–he modules. It is surprisingly sturdy.
- **Cuboctahedron (#11)**. 6 square faces, 8 triangular faces. 24 modules, all tr–sq. There are many nice ways to color this one, for example eight modules of each color forming opposite pair of squares. One neat one is to use six modules each of four colors, having each color form a hexagonal band around the middle of the polyhedron.
- **Icosidodecahedron (#12)**. 12 pentagonal faces, 20 triangular faces. 60 modules, all tr–pe. The best coloration I found for this one is to use 20 modules each of three colors. Take color 1 and make the top and bottom pentagons. Take color 2, and complete the triangles around each of these pentagons. This will take all 20 modules of color 2. Take color 3, and form the two edges of the remaining triangles that attach to the triangles of color 2. This will take all 20 modules of color 3. The remaining 10 modules of color 1 form a decagonal band around the middle of the polyhedron, attaching the two halves you have just created.

  It is also possible to divide the edges of this polyhedron into six decagonal bands. Unfortunately, it is hard to get six colors to look nice together.
- **Rhombicuboctahedron (#13)**. 18 squares and 8 triangles. 48 modules: 24 sq–sq and 24 tr–sq. This is another very pretty solid. I have always used 16 modules each of three colors (8 sq–sq and 8 tr–sq), and had each color form two parallel octagons around the middle.
- **Rhombitruncatedcuboctahedron (#14)**. 12 squares, 8 hexagons, 6 octagons. 72 modules: 24 oc–he, 24 oc–sq, 24 sq–he. I made this one with three colors – eight of each module. I made two octagons of each color, and put them at opposite ends of the polyhedron. You figure out the rest. It will hold together well without glue, but if you want to hang it, you had better glue it.
- **Rhombicosidodecahedron (#15)**. 30 squares, 12 pentagons, 20 triangles. 120 modules: 60 sq–pe, 60 tr–sq. This is a difficult polyhedron to make because the creases do not hold together very tightly. It will hold together and look nice if you make sure not to move it or breathe on it. Otherwise, you have to glue it. Unfortunately, you have to be careful how you glue it. If you try to glue the completed model, you will be frustrated by the instability. If you try to glue the model incrementally, it may not fit together very well. My best strategy has been the following: First, make all the tr–sq modules, and from them, make two cuboctahedrons. Now glue just the triangles, let it dry, and take the cuboctahedrons apart. Use four of these triangles and the remaining twelve tr–sq modules, and make another cuboctahedron, and glue together the remaining four triangles. You should now have twenty glued triangles. Make five sq–sq modules, and combine it with ten of the sq–pe modules to make a pentagonal prism. Glue the pentagons together. When it is dry, take it apart, and repeat this five more times, until you have twelve glued pentagons. Now, assemble the rhombicosidodecahedron and glue the squares together. This is a time consuming process, and it will take some thought to ensure that you're doing the colors properly, but you end up with a nice-looking, polyhedron.

  Coloring issues:

  1. I made this one once with two colors, one for each type of module. It was ugly because all the triangles were the same color. I'd recommend one of the following.
  2. Split the modules into three colors (20 sq–pe and 20 tr–sq of each color). Arrange the modules so that each square has edges of the same color. Then arrange the squares so that all trianges have edges of all three colors. This can be done by arranging the squares as if they were edges of a dodecahedron, where each edge of the pentagon is never adjacent to an edge of the same color (this is the first coloration of the dodecahedron suggested above).
  3. Use four colors. Color all of the sq–pe modules with color 1. Then divide the remaining modules into 20 each of colors 2, 3, and 4. Now, all the pentagons will be the same color. Make all the triangles have an edge of each color. If you really want to be studly, you can arrange it so that each square has opposite edges of the same color.

4. I tried a coloring once that is symmetrical from top to bottom. I.e. start by making the top and bottom pentagons color 1. Then make all the edges emanating from them have color 2. Complete the triangles with color 3 (perhaps have color 3 form the decagonal band concentric to the top pentagon). Continue in some similar fasion. It was really ugly, so I converted it into two cuboctahedrons and six pentagonal prisms, and tried a different colering.

- **Rhombitruncatedicosidodecahedron (#16)**. 30 squares, 12 decagons, 20 hexagons. 180 modules: 60 de–sq, 60 de–he, 60 he–sq. I haven't made this one.

- **Snub Cube (#17)**. 6 squares, 32 triangles. 60 modules: 24 tr–sq and 36 tr–tr. I've made two of these, one with 4 colors and one with three. They are both fairly solid and very pretty. In the one with three colors, I colored as follows: Divide both sets of modules into three equal number of colors. With the tr–sq modules, make six squares, two of each color. Take the square of color 1. You'll note from the picture that each vertex of the square has three tr–tr modules incident to it. On one vertex, make these of color 2-1-2. On the next, make them 3-1-3. On the next, make them 2-1-2 again, and on the final vertex, make them 3-1-3 again. This is how it will work with all squares – if a square is of color y, then one pair of opposite vertices will have modules ordered x-y-x, and the other pair will have modules ordered z-y-z. It works out so that each pair of squares is on opposite faces, and the pattern is pleasing.

  To get a snub cube from four colors, simply do the same as above, only make all the tr–sq modules out of color 4. The ordering of the tr–tr modules should be the same.

- **Snub Dodecahedron (#18)**. 12 pentagons and 80 triangles. 150 modules: 60 tr–pe and 90 tr–tr. I tried one of these once, and lost momentum because it was extremely flimsy, much like the rhombicosidodecahedron. I'll try it again soon, using the same glueing strategy as the rhombicosidodecahedron (only using an icosidodecahedron and some octahedrons as the gluing substeps). I will likely using the following coloration. There will be four colors. All the pentagons will be color 1. The other three colors will be devided equally.

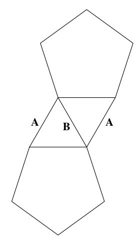


Figure 1: Edges of the snub dodecahedron

There are two types of tr–tr edge — those of type **A** and those of type **B** (see Figure 1). Consider the **A** edges in pairs as in the Figure. There are 30 such pairs that are analagous to edges of a dodecahedron (or icosahedron). Color these pairs in the same was as you color a dodecahedron that has no two adjacent edges of the same color. Color the **B** edges however you want. This should make a symmetrical coloring that is pleasing to view. I'll put the picture on the web if I ever complete it.

References

[Ans92] J. Ansill. *Lifestyle Origami*. HarperCollins Publishers, 10 East 53 Street, New York, NY 10022, 1992.

[Cox48] H. S. M. Coxeter. *Regular Polytopes*. Pitman, New York, 1948.

[Fus90] T. Fusè. *Unit Origami*. Japan Publications Inc., Tokyo and New York, 1990.

[GA95] R. Gurkewitz and B. Arnstein. *3-D Geometric Origami: Modular Polyhedra*. Dover Publications, Inc., New York, 1995.

[Hol71] A. Holden. *Shapes, Space and Symmetry*. Columbia University Press, New York, 1971.

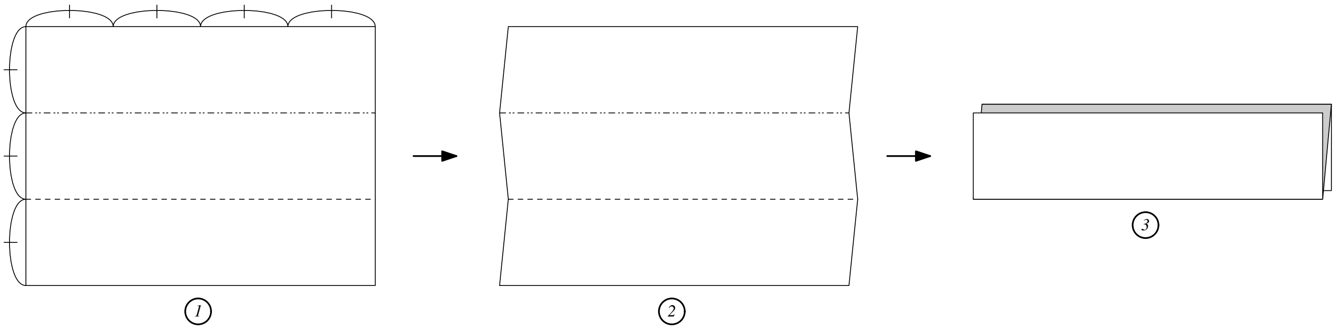
[KT87] K. Kasahara and T. Takahama. *Origami for the Connoisseur*. Japan Publications Inc., Tokyo and New York, 1987.

[Wen71] M. J. Wenninger. *Polyhedron Models*. Cambridge University Press, Cambridge, England, 1971.

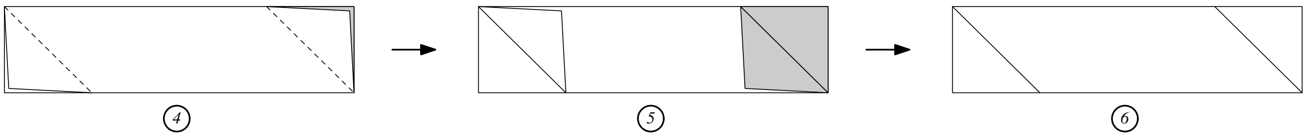
[Yam90] M. Yamaguchi. *Kusudama: Ball Origami*. Shufunotomo/Japan Publications, Tokyo, 1990.

Pentagon Module (108 Degrees)

Start with a 4x3 rectangle, and collapse like an accordian:

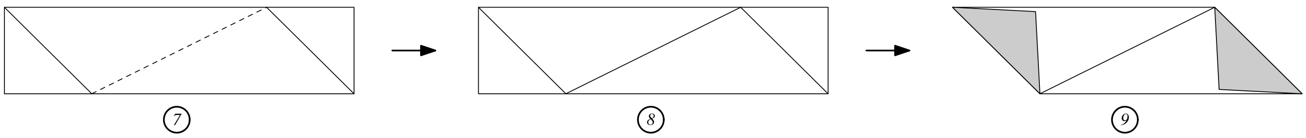


Fold opposite corners in -- use only the top layer -- and unfold

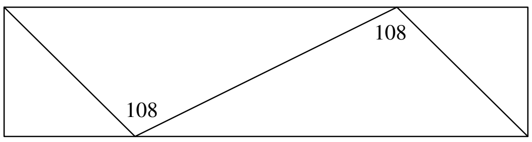


Fold along the dotted line and unfold

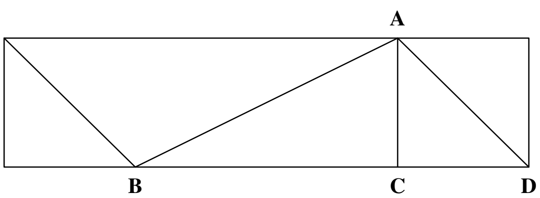
Re-fold the corners, this time folding all layers



The final piece:



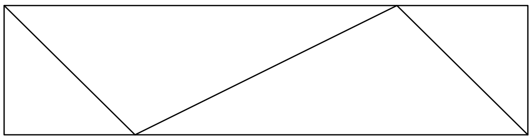
Why?



BC = 2
AC = CD = 1
BAC = atan(BC/AC) = 63.44
CAD = 45
BAD = 63.44+45 = 108.44

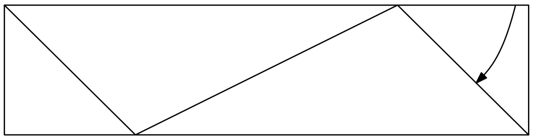
Hexagon Module (120 Degrees)

Get to step 8 of the Pentagon module:

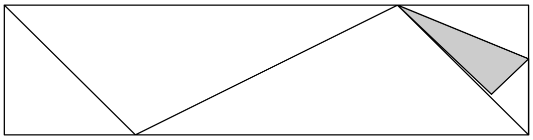


1

Fold the top to the diagonal line (Fold the top layer only):

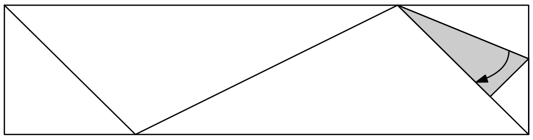


2

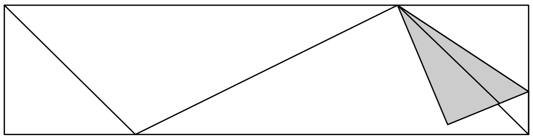


3

Fold this new crease to the diagonal, opening as you fold:

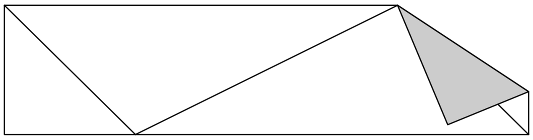


4



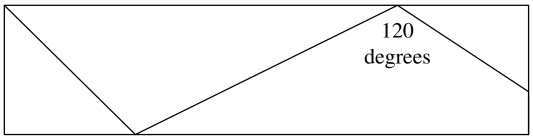
5

Fold all layers along this new crease:



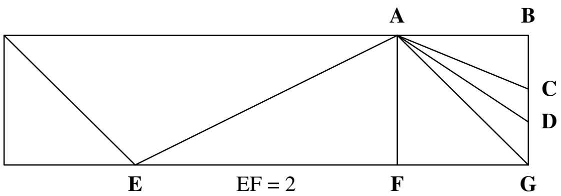
6

Open up. Final fold:



7

Why?



AF = AB = BG = 1

EAF = atan(EF/AF) = 63.44

GAF = 45

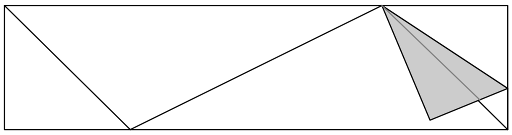
GAC = 45/2 = 22.5

DAC = 22.5/2 = 11.25

EAD = 63.44 + 45 + 11.25 = 119.69 (almost 120)

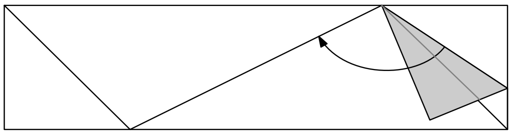
Triangle Module (60 Degrees)

Get to step 5 of the Hexagon module:

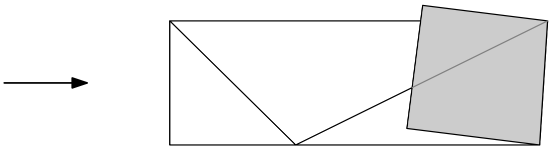


1

Fold the entire module so that this newly created crease matches up with the large crease:

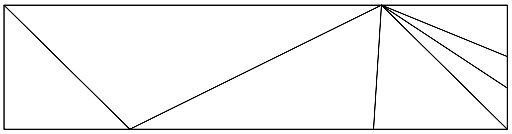


2



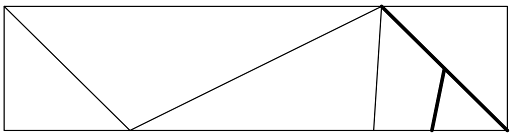
3

Unfold back to the rectangle:



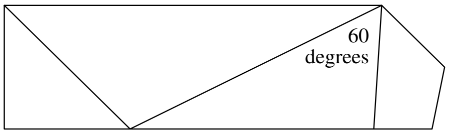
4

Cut along the thick lines (one of them is not along any crease lines)



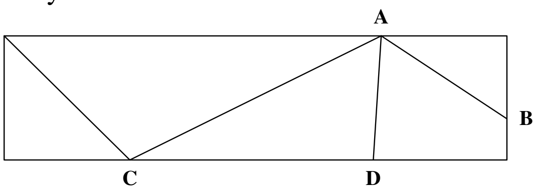
5

The final piece:



6

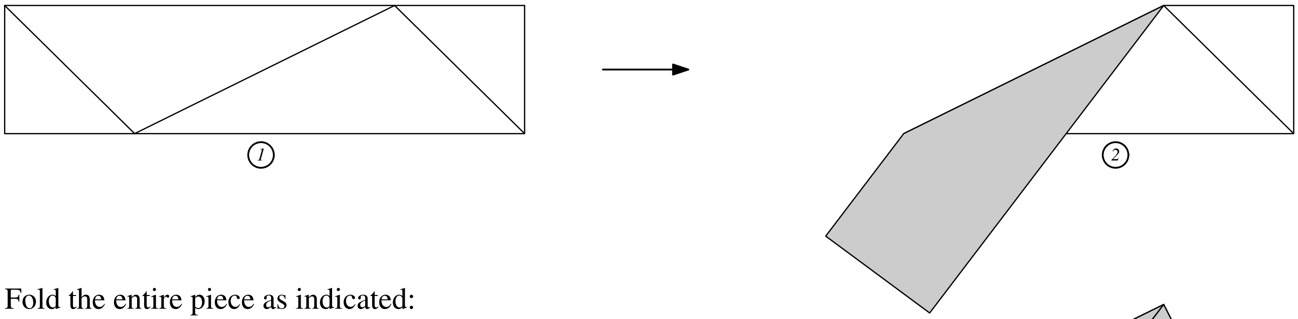
Why?



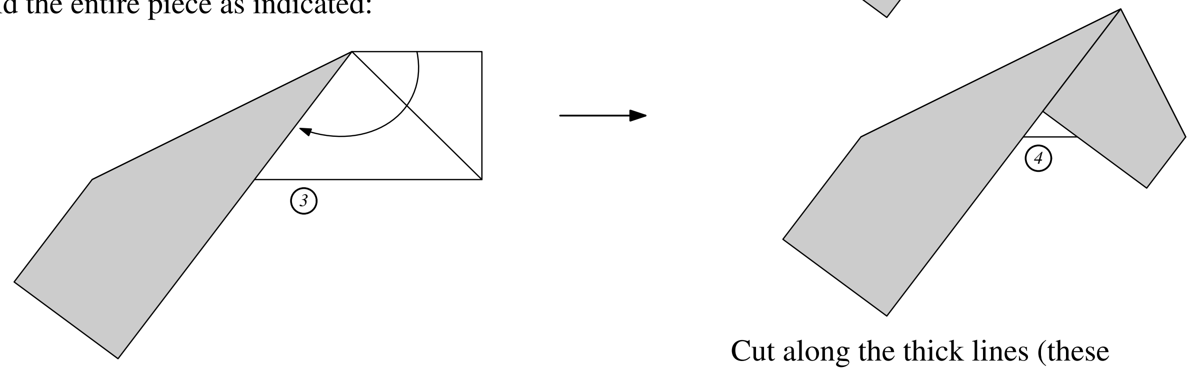
CAB = 119.69 degrees (from the Hexagon module)
CAD = CAB/2 = 59.85 (almost 60)

Square Module (90 Degrees)

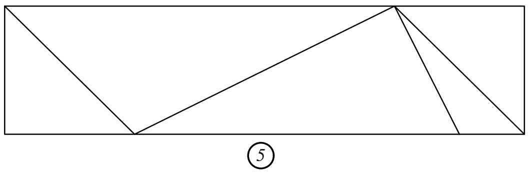
Get to step 8 of the Pentagon module, and fold along the long crease



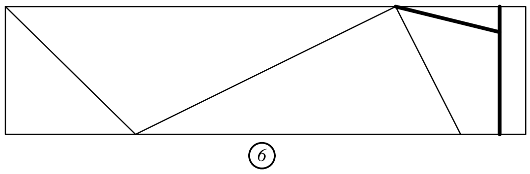
Fold the entire piece as indicated:



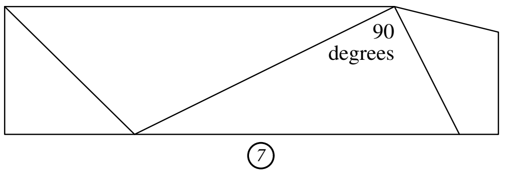
Unfold back to the rectangle:



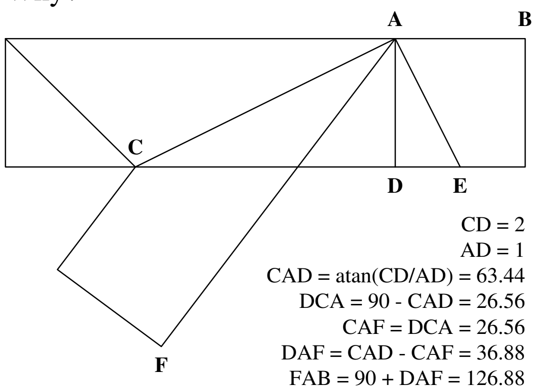
Cut along the thick lines (these are not along any crease lines)



The final piece:



Why?



CD = 2
AD = 1
CAD = atan(CD/AD) = 63.44
DCA = 90 - CAD = 26.56
CAF = DCA = 26.56
DAF = CAD - CAF = 36.88
FAB = 90 + DAF = 126.88
FAE = FAB/2 = 63.44
CAE = FAE + CAF = 90