ECE 421/521
Electric Energy Systems
Power Systems Analysis I
7 – Optimal Dispatch of Generation

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Background

• In a practical power system, the costs of generating and delivering electricity from power plants are different (due to fuel costs and distances to load centers).
• Under normal conditions, the system generation capacity is more than the total load demand and losses.
• Thus, there is room to schedule generation within capacity limits
  – Minimizing a cost function that represents, e.g.
    • Operating costs
    • Transmission losses
    • System reliability impacts
• This is called Optimal Power Flow (OPF) problem
• A typical problem is the Economic Dispatch (ED) of real power generation
Introduction of Nonlinear Function Optimization

• Unconstrained parameter optimization

• Constrained parameter optimization
  – Equality constraints
  – Inequality constraints
Unconstrained parameter optimization

Minimize cost function $f(x_1, x_2, \cdots, x_n)$

- Necessary conditions for the solution (necessary & sufficient for local minima)
  - Gradient vector \[ \nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right) = 0 \]

  Critical point (f does not change)

- Hessian matrix $H$ is positive definite

\[
H(f) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}
\end{bmatrix}
\]

Local minimum

- All local minima $\rightarrow$ the Global minimum
• Minimize

\[ f(x, y) = x^2 + y^2 \]

\[ \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, 2y) = 0 \]

\[ H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \]

\[ x = 0, \ y = 0 \]
Parameter Optimization with Equality Constraints

Minimize \( f(x_1, x_2, \cdots, x_n) \)

Subject to \( g_k(x_1, x_2, \cdots, x_n) = 0 \quad k = 1, 2, \cdots, K \)

- Introduce Lagrange Multipliers \( \lambda_1 \sim \lambda_K \)

\[
L = f + \sum_{i=1}^{K} \lambda_k g_k
\]

- Necessary conditions for the local minima of \( L \) (also necessary for the original problem)

\[
\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{k=1}^{K} \lambda_k \frac{\partial g_k}{\partial x_i} = 0
\]

\[
\frac{\partial L}{\partial \lambda_k} = g_k = 0
\]
• Minimize $f(x, y) = x^2 + y^2$

Subject to $(x-8)^2 + (y-6)^2 = 25 \implies g(x, y) = (x-8)^2 + (y-6)^2 - 25$

$L = x^2 + y^2 + \lambda[(x-8)^2 + (y-6)^2 - 25]$

$\frac{\partial L}{\partial x} = 2x + \lambda(2x - 16) = 0$

$\frac{\partial L}{\partial y} = 2y + \lambda(2y - 12) = 0$

$\frac{\partial L}{\partial \lambda} = (x-8)^2 + (y-6)^2 - 25 = 0$

Solutions (from the N-R method):
$\lambda=1$, $x=4$ and $y=3$ ($f=25$)
$\lambda=3$, $x=12$ and $y=9$ ($f=225$)
Parameter Optimization with Inequality Constraints

Minimize \( f(x_1, x_2, \cdots, x_n) \)

Subject to:

\[
g_k(x_1, x_2, \cdots, x_n) = 0 \quad k = 1, 2, \cdots, K
\]
\[
u_j(x_1, x_2, \cdots, x_n) \leq 0 \quad j = 1, 2, \cdots, m
\]

- Introduce Lagrange Multipliers \( \lambda_1 \sim \lambda_K \) and \( \mu_1 \sim \mu_m \)

\[
L = f + \sum_{k=1}^{K} \lambda_k g_k + \sum_{j=1}^{m} \mu_j u_j
\]

- Necessary conditions for the local minima of \( L \)

\[
\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{k=1}^{K} \lambda_k \frac{\partial g_k}{\partial x_i} + \sum_{j=1}^{m} \mu_j \frac{\partial u_j}{\partial x_i} = 0 \quad i = 1, \cdots, n
\]
\[
\frac{\partial L}{\partial \lambda_k} = g_k = 0 \quad k = 1, 2, \cdots, K
\]
\[
\frac{\partial L}{\partial \mu_j} = u_j \leq 0
\]
\[
\mu_j u_j = 0 \quad \mu_j \geq 0 \quad j = 1, \cdots, m
\]

Kuhn-Tucker (KKT) necessary condition
• Minimize \( f(x, y) = x^2 + y^2 \)

Subject to \( (x - 8)^2 + (y - 6)^2 = 25 \) \( \rightarrow \) \( g(x, y) = (x - 8)^2 + (y - 6)^2 - 25 \)

\[ 2x + y \geq 12 \quad \Rightarrow \quad u(x, y) = 12 - 2x - y \leq 0 \]

\[ L = x^2 + y^2 + \lambda [(x - 8)^2 + (y - 6)^2 - 25] + \mu (12 - 2x - y) \]

\[ \frac{\partial L}{\partial x} = 2x + \lambda (2x - 16) - 2\mu = 0 \]

\[ \frac{\partial L}{\partial y} = 2y + \lambda (2y - 12) - \mu = 0 \]

\[ \frac{\partial L}{\partial \lambda} = (x - 8)^2 + (y - 6)^2 - 25 = 0 \]

\[ \frac{\partial L}{\partial \mu} = 12 - 2x - y < 0 \quad \mu = 0 \]

or \[ \frac{\partial L}{\partial \mu} = 12 - 2x - y = 0 \quad \mu > 0 \]

Solutions:

\( \mu = 0, \quad \lambda = 3, \quad x = 12 \) and \( y = 9 \) \( (f = 225) \)

\( \mu = 5.6, \quad \lambda = -0.2, \quad x = 5 \) and \( y = 2 \) \( (f = 29) \)

\( \mu = 12, \quad \lambda = -1.8, \quad x = 3 \) and \( y = 6 \) \( (f = 33) \)
**Operating Cost of a Thermal Plant**

- Fuel-cost curve of a generator (represented by a quadratic function of real power)
  \[ C_i = \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]

- Incremental fuel-cost curve:
  \[ \lambda_i = \frac{dC_i}{dP_i} = 2\gamma_i P_i + \beta_i \]
ED Neglecting Losses and No Generator Limits

If transmission line losses are neglected, minimize the total production cost:

\[ C_t = \sum_{i=1}^{n_g} C_i = \sum_{i=1}^{n} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]

subject to

\[ \sum_{i=1}^{n_g} P_i = P_D \]

• Apply the Lagrange multiplier method:

\[ L = C_t + \lambda (P_D - \sum_{i=1}^{n_g} P_i) \]

\[ \frac{\partial L}{\partial P_i} = 0 \quad \frac{\partial C_t}{\partial P_i} - \lambda = 0 \quad \frac{\partial C_t}{\partial P_i} = \lambda = \frac{dC_i}{dP_i} = \beta_i + 2\gamma_i P_i \quad P_i = \frac{\lambda - \beta_i}{2\gamma_i} \quad i = 1, \ldots, n_g \]

\[ \frac{\partial L}{\partial \lambda} = 0 \quad \sum_{i=1}^{n_g} P_i = P_D \quad \sum_{i=1}^{n_g} \frac{\lambda - \beta_i}{2\gamma_i} = P_D \quad \lambda = \frac{P_D + \sum_{i=1}^{n_g} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_g} \frac{1}{2\gamma_i}} \]

All plants must operate at equal incremental cost.
Example 7.4

The fuel-cost functions for three thermal plants are $C_1 \sim C_2$ in $$/h$. $P_1$, $P_2$ and $P_3$ are in MW. $P_D = 800$ MW. Neglecting line losses and generator limits, find the optimal dispatch and the total cost in $$/h

$$C_1 = 500 + 5.3P_1 + 0.004P_1^2$$
$$C_2 = 400 + 5.5P_2 + 0.006P_2^2$$
$$C_3 = 200 + 5.8P_3 + 0.009P_3^2$$

$$\lambda = \frac{P_D + \sum_{i=1}^{n_g} \frac{\beta_i}{2\gamma_i}}{\sum_{i=1}^{n_g} \frac{1}{2\gamma_i}} = \frac{800 + \frac{5.3}{0.008} + \frac{5.5}{0.012} + \frac{5.8}{0.018}}{1 + \frac{1}{0.008} + \frac{1}{0.012} + \frac{1}{0.018}} = \frac{800 + 1443.0555}{263.8889} = 8.5 \$/MWh$$

$$P_i = \frac{\lambda - \beta_i}{2\gamma_i}$$

$$P_1 = \frac{8.5 - 5.3}{2(0.004)} = 400.0000$$

$$P_2 = \frac{8.5 - 5.5}{2(0.006)} = 250.0000$$

$$P_3 = \frac{8.5 - 5.8}{2(0.009)} = 150.0000$$

$$C_i = 6682.5 \$/h$$

The dispatch point is found by equating the incremental cost of each plant to the marginal cost of the total system. This can be represented graphically with the marginal cost of the total system shown as a dashed line and the marginal cost of each individual plant shown as solid lines. The dispatch point is where the dashed line intersects any of the solid lines, indicating the optimal dispatch for each plant.
Solving $\lambda$ by an Iterative Procedure

\[ \sum_{i=1}^{n_g} P_i = \sum_{i=1}^{n_g} \frac{\lambda - \beta_i}{2\gamma_i} = P_D \]

• For a general case:

\[ f(\lambda) = \sum_{i=1}^{n_g} P_i = P_D \]

\[ f(\lambda)^{(k)} + \left( \frac{df(\lambda)}{d\lambda} \right)^{(k)} \Delta \lambda^{(k)} \approx P_D \]

\[ \Delta P^{(k)} = P_D - f(\lambda)^{(k)} = P_D - \sum_{i=1}^{n_g} P_i^{(k)} \]

\[ \Delta \lambda^{(k)} = \frac{P_D - f(\lambda)^{(k)}}{\left( \frac{df(\lambda)}{d\lambda} \right)^{(k)}} = \frac{\Delta P^{(k)}}{\left( \frac{df(\lambda)}{d\lambda} \right)^{(k)}} = \frac{\Delta P^{(k)}}{\sum \left( \frac{dP_i}{d\lambda} \right)^{(k)}} \]

\[ \lambda^{(k+1)} = \lambda^{(k)} + \Delta \lambda^{(k)} \]
Apply the Gradient Method in Example 7.4

\[ P_i^{(k)} = \frac{\lambda^{(k)} - \beta_i}{2\gamma_i} \]

\[ \Delta \lambda^{(k)} = \frac{\Delta P^{(k)}}{\sum (\frac{dP_i}{d\lambda})^{(k)}} = \frac{\Delta P^{(k)}}{\sum \frac{1}{2\gamma_i}} \]

\[ \lambda^{(1)} = 6.0 \]

\[ P_1^{(1)} = \frac{6.0 - 5.3}{2(0.004)} = 87.5000 \]

\[ P_2^{(1)} = \frac{6.0 - 5.5}{2(0.006)} = 41.6667 \]

\[ P_3^{(1)} = \frac{6.0 - 5.8}{2(0.009)} = 11.1111 \]

\[ \Delta P^{(1)} = 800 - (87.5 + 41.6667 + 11.1111) = 659.7222 \]

\[ \Delta \lambda^{(1)} = \frac{659.7222}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} \]

\[ = \frac{659.7222}{263.8888} = 2.5 \]

\[ \lambda^{(2)} = 6.0 + 2.5 = 8.5 \]

\[ P_1^{(2)} = \frac{8.5 - 5.3}{2(0.004)} = 400.0000 \]

\[ P_2^{(2)} = \frac{8.5 - 5.5}{2(0.006)} = 250.0000 \]

\[ P_3^{(2)} = \frac{8.5 - 5.8}{2(0.009)} = 150.0000 \]

\[ \Delta P^{(2)} = 800 - (400 + 250 + 150) = 0.0 \]

\[ C_t = 6682.5 \text{ $ / h} \]
ED Neglecting Losses and Including Generator Limits

• Considering the maximum (rating) and minimum (stability) generation limits, minimize

\[ C_t = \sum_{i=1}^{n_g} C_i = \sum_{i=1}^{n} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]

Subject to

\[ \sum_{i=1}^{n_g} P_i = P_D \]

\[ P_{i(\text{min})} \leq P_i \leq P_{i(\text{max})}, \quad i = 1, 2, \ldots, n_g \]

\[ L = C_i + \lambda (P_D - \sum_{i=1}^{n_g} P_i) + \sum_{i=1}^{n_g} [\mu_i (P_i - P_{i(\text{max})}) + \gamma_i (P_{i(\text{min})} - P_i)] \]

\[ \frac{\partial L}{\partial P_i} = 0 \quad \Rightarrow \quad \frac{\partial C_i}{\partial P_i} - \lambda + \mu_i - \gamma_i = 0 \]

\[ \frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n_g} P_i = P_D \]

\[ \frac{\partial L}{\partial \mu_i} \leq 0 \quad \Rightarrow \quad \mu_i (P_i - P_{i(\text{max})}) = 0, \quad \mu_i \geq 0 \]

\[ \frac{\partial L}{\partial \gamma_i} \leq 0 \quad \Rightarrow \quad \gamma_i (P_{i(\text{min})} - P_i) = 0, \quad \gamma_i \geq 0 \]

\[ P_{i(\text{min})} \leq P_i \leq P_{i(\text{max})}, \quad i = 1, 2, \ldots, n_g \]

\[ \frac{dC_i}{dP_i} = \lambda \]

\[ \frac{dC_i}{dP_i} \leq \lambda \]

\[ \frac{dC_i}{dP_i} \geq \lambda \]

Excluding the plants that reach their limits, the other plants still operate at equal incremental cost
Example 7.6
Consider generator limits (in MW) for Example 7.4, let $P_D=975$ MW

250 $\leq P_1 \leq 450$
150 $\leq P_2 \leq 350$
100 $\leq P_3 \leq 225$

Solution:

$P_1 = 450$ MW
$P_2 = 325$ MW
$P_3 = 200$ MW

$\lambda = 9.4$$/MWh$

$C_t = 8236.25$ $$/h$
\[ P_i^{(k)} = \frac{\lambda^{(k)} - \beta_i}{2\gamma_i} \]

\[ \Delta \lambda^{(k)} = \frac{\Delta P^{(k)}}{\sum (\frac{dP_i^{(k)}}{d\lambda})} = \frac{\Delta P^{(k)}}{\sum \frac{1}{2\gamma_i}} \]

\[ 250 \leq P_1 \leq 450 \]
\[ 150 \leq P_2 \leq 350 \]
\[ 100 \leq P_3 \leq 225 \]

\[ \lambda^{(1)} = 6.0 \]
\[ P_1^{(1)} = \frac{6.0 - 5.3}{2(0.004)} = 87.5000 \]
\[ P_2^{(1)} = \frac{6.0 - 5.5}{2(0.006)} = 41.6667 \]
\[ P_3^{(1)} = \frac{6.0 - 5.8}{2(0.009)} = 11.1111 \]
\[ \Delta P^{(1)} = 975 - (87.5 + 41.6667 + 11.1111) = 834.7222 \]
\[ \Delta \lambda^{(1)} = \frac{834.7222}{\frac{1}{2(0.004)} + \frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{834.7222}{263.8888} = 3.1632 \]

\[ \lambda^{(2)} = 6.0 + 3.1632 = 9.1632 \]
\[ P_1^{(2)} = \frac{9.1632 - 5.3}{2(0.004)} = 482.8947 > P_{1(\text{max})}, \text{so let } P_1 = 450 \]
\[ P_2^{(2)} = \frac{9.1632 - 5.5}{2(0.006)} = 305.2632 \]
\[ P_3^{(2)} = \frac{9.1632 - 5.8}{2(0.009)} = 186.8421 \]
\[ \Delta P^{(2)} = 975 - (450 + 305.2632 + 186.8421) = 32.8947 \]
\[ \Delta \lambda^{(2)} = \frac{1}{\frac{1}{2(0.006)} + \frac{1}{2(0.009)}} = \frac{32.8947}{132.8889} = 0.2368 \]

\[ \lambda^{(3)} = 9.1632 + 0.2368 = 9.4 \]
\[ P_1^{(3)} = 450 \]
\[ P_2^{(3)} = \frac{9.4 - 5.5}{2(0.006)} = 325 \]
\[ P_3^{(3)} = \frac{9.4 - 5.8}{2(0.009)} = 200 \]
\[ \Delta P^{(3)} = 975 - (450 + 325 + 200) = 0.0 \]
Solve the optimal dispatch for $P_D = 550$ MW

\[
C_1 = 500 + 5.3P_1 + 0.004P_1^2 \quad 250 \leq P_1 \leq 450 \quad \text{or} \quad P_1 = 0
\]

\[
C_2 = 400 + 5.5P_2 + 0.006P_2^2 \quad 150 \leq P_2 \leq 350 \quad \text{or} \quad P_2 = 0
\]

\[
C_3 = 200 + 5.8P_3 + 0.009P_3^2 \quad 100 \leq P_3 \leq 225 \quad \text{or} \quad P_3 = 0
\]

Unit Commitment Problem (mixed-integer optimization)

Solution 1:
- $P_1 = 280$ MW
- $P_2 = 170$ MW
- $P_3 = 100$ MW
- $\lambda = 7.54$ $/\text{MWh}$
- $C_t = 4676$ $/\text{h}$

Solution 2:
- $P_1 = 340$ MW
- $P_2 = 210$ MW
- $P_3 = 0$ MW
- $\lambda = 8.02$ $/\text{MWh}$
- $C_t = 4584$ $/\text{h}$

Solution 3:
- $P_1 = 0$ MW
- $P_2 = 340$ MW
- $P_3 = 210$ MW
- $\lambda = 9.58$ $/\text{MWh}$
- $C_t = 47785$ $/\text{h}$

Solution 4:
- $P_1 = 400$ MW
- $P_2 = 0$ MW
- $P_3 = 150$ MW
- $\lambda = 8.5$ $/\text{MWh}$
- $C_t = 4532$ $/\text{h}$
Transmission Loss

• When transmission distances are very small and load density in the system is very high
  – Transmission losses may be neglected
  – All plants operate at equal incremental production cost to achieve optimal dispatch of generation

• However, in a large interconnected network
  – Power is transmitted over long distances with low load density areas
  – Transmission losses are a major factor affecting the optimal dispatch.
Calculation of Transmission Losses

\[ |I| = \frac{P}{|V| \cos \phi} \]

\[ P_L = |I|^2 R = \left( \frac{P}{|V| \cos \phi} \right)^2 R \]

\[ = \frac{R}{|V|^2 \cos^2 \phi} P^2 \]

\[ P_L = BP^2 \]

\[ P_L = |I_1|^2 R_1 + |I_2|^2 R_2 + |I_1 + I_2|^2 R_3 \]

\[ = \left( \frac{P_1}{|V_1| \cos \phi_1} \right)^2 R_1 + \left( \frac{P_2}{|V_2| \cos \phi_2} \right)^2 R_2 \]

\[ + \left( \frac{\cos \alpha \cdot P_1}{|V_1| \cos \phi_1} + \frac{\cos \beta \cdot P_2}{|V_2| \cos \phi_2} \right)^2 R_3 \]

\[ P_L = B_{11} P_1^2 + B_{12} P_1 P_2 + B_{22} P_2^2 \]
More general loss formulas:

• Quadratic form

\[ P_L = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} P_i B_{ij} P_j \]

• Kron’s loss formula

\[ P_L = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} P_i B_{ij} P_j + \sum_{i=1}^{n_g} B_{0i} P_i + B_{00} \]

- \( B_{ij} \), \( B_{0i} \) and \( B_{00} \) are Loss coefficients or B-coefficients.
  
  • See Chapter 7.6 for details of calculating B-coefficients
  
  • Changes with power flows but usually assumed constant and estimated for a power-flow base case
Economic Dispatch Including Losses

- Cost function

\[ C_t = \sum_{i=1}^{n_g} C_i = \sum_{i=1}^{n_g} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]

\[ \frac{dC_i}{dP_i} = \beta_i + 2\gamma_i P_i \]

Incremental fuel cost

Subject to

\[ \sum_{i=1}^{n_g} P_i = P_D + P_L \]

\[ P_L = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} P_i B_{ij} P_j + \sum_{i=1}^{n_g} B_{0i} P_i + B_{00} \]

\[ \frac{\partial P_L}{\partial P_i} = 2 \sum_{j=1}^{n_g} B_{ij} P_j + B_{0i} \]

Incremental transmission loss

\[ P_{i(min)} \leq P_i \leq P_{i(max)} \quad i = 1, \ldots, n_g \]
• Using Lagrange Multipliers

\[ L = \sum_{i=1}^{n_g} C_i + \lambda (P_D + P_L - \sum_{i=1}^{n_g} P_i) + \sum_{i=1}^{n_g} \mu_{i(max)} (P_i - P_{i(max)}) + \sum_{i=1}^{n_g} \mu_{i(min)} (P_i - P_{i(min)}) \]

where

\[ P_L = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} P_i B_{ij} P_j + \sum_{i=1}^{n_g} B_{0i} P_i + B_{00} \]

\[ C_i = \sum_{i=1}^{n_g} C_i = \sum_{i=1}^{n_g} \alpha_i + \beta_i P_i + \gamma_i P_i^2 \]

\[ \frac{\partial L}{\partial \lambda} = 0 \quad \sum_{i=1}^{n_g} P_i = P_D + P_L \]

\[ \frac{\partial L}{\partial P_i} = 0 \quad \frac{\partial C_i}{\partial P_i} + \lambda (0 + \frac{\partial P_L}{\partial P_i} - 1) = 0 \quad \frac{dC_i}{dp_i} + \lambda \frac{\partial P_L}{\partial P_i} = \lambda \quad L_i \frac{dC_i}{dp_i} = \lambda, \quad i = 1, \ldots, n_g \]

\[ \frac{dC_i}{dp_i} = \beta_i + 2\gamma_i P_i \]

\[ \frac{\partial P_L}{\partial P_i} = 2 \sum_{j=1}^{n_g} B_{ij} P_j + B_{0i} \]

\[ \beta_i + 2\gamma_i P_i + 2\lambda \sum_{j=1}^{n_g} B_{ij} P_j + B_{0i}\lambda = \lambda \]
\[ \beta_i + 2\gamma_i P_i + 2\lambda \sum_{j=1}^{n_g} B_{ij} P_j + B_{0i} \lambda = \lambda \quad \iff \quad \left( \frac{\gamma_i}{\lambda} + B_{ii} \right) P_i + \sum_{j=1 \atop j \neq i}^{n_g} B_{ij} P_j = \frac{1}{2} \left( 1 - B_{0i} - \frac{\beta_i}{\lambda} \right) \]

**Solve \( \lambda \) and \( P_i \) for the optimal dispatch:**

\[
\begin{bmatrix}
\frac{\gamma_1}{\lambda} + B_{11} & B_{12} & \cdots & B_{1n_g} \\
B_{21} & \frac{\gamma_2}{\lambda} + B_{22} & \cdots & B_{12} \\
\vdots & \vdots & \ddots & \vdots \\
B_{n_g1} & B_{n_g2} & \cdots & \frac{\gamma_{n_g}}{\lambda} + B_{n_g n_g}
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
\vdots \\
P_{n_g}
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
1 - B_{01} - \frac{\beta_1}{\lambda} \\
1 - B_{02} - \frac{\beta_2}{\lambda} \\
\vdots \\
1 - B_{0n_g} - \frac{\beta_{n_g}}{\lambda}
\end{bmatrix}
\]

\[
\sum_{i=1}^{n_g} P_i = P_D + P_L
\]

**Check inequality constraints:**

If \( P_i < P_{i(\min)} \), let \( P_i \equiv P_{i(\min)} \)

If \( P_i > P_{i(\max)} \), let \( P_i \equiv P_{i(\max)} \)
• Using the Gradient Method: \( f(\lambda) = \sum_{i=1}^{n_g} P_i = P_D + P_L \)

\[
\left(\frac{\gamma_i}{\lambda} + B_{ii}\right)P_i + \sum_{j=1, j \neq i}^{n_g} B_{ij} P_j = \frac{1}{2}(1 - B_{0i} - \beta_i) \hspace{1cm} \Rightarrow \hspace{1cm} P_i = \frac{\lambda(1 - B_{0i}) - \beta_i - 2\lambda \sum_{j \neq i} B_{ij} P_j}{2(\gamma_i + \lambda B_{ii})}
\]

\[
P^{(k)}_i = \frac{\lambda^{(k)}(1 - B_{0i}) - \beta_i - 2\lambda^{(k)} \sum_{j \neq i} B_{ij} P^{(k)}_j}{2(\gamma_i + \lambda^{(k)} B_{ii})}
\]

\[
f(\lambda)^{(k)} = \sum_{i=1}^{n_g} P^{(k)}_i = \sum_{i=1}^{n_g} \frac{\lambda^{(k)}(1 - B_{0i}) - \beta_i - 2\lambda^{(k)} \sum_{j \neq i} B_{ij} P^{(k)}_j}{2(\gamma_i + \lambda^{(k)} B_{ii})}
\]

\[
f(\lambda)^{(k)} + \left(\frac{df(\lambda)}{d\lambda}\right)^{(k)} \Delta\lambda^{(k)} \approx P_D + P_L^{(k)}
\]

1. \( \Delta P^{(k)} = P_D + P_L^{(k)} - \sum_{i=1}^{n_g} P^{(k)}_i \) where \( P_L^{(k)} = \sum_{i=1}^{n_g} \sum_{j=1}^{n_g} P^{(k)}_i B_{ij} P^{(k)}_j + \sum_{i=1}^{n_g} B_{0i} P^{(k)}_i + B_{00} \)

2. \( \Delta\lambda^{(k)} = \frac{\Delta P^{(k)}}{(df(\lambda))^{(k)}} = \frac{\Delta P^{(k)}}{\sum (\frac{\partial P^{(k)}}{\partial \lambda})^{(k)}} \) where \( \sum_{i=1}^{n_g} (\frac{\partial P^{(k)}}{\partial \lambda})^{(k)} = \sum_{i=1}^{n_g} \frac{\gamma_i(1 - B_{0i}) - B_{ii} \beta_i - 2\gamma_i \sum_{j \neq i} B_{ij} P^{(k)}_j}{2(\gamma_i + \lambda^{(k)} B_{ii})^2} \)

3. \( \lambda^{(k+1)} = \lambda^{(k)} + \Delta\lambda^{(k)} \)
Example 7.7

Solve the optimal dispatch of three thermal plants in a power system. The total system load is 150MW. The base for per unit values is 100MVA.

\[ C_1 = 200 + 7.0P_1 + 0.008P_1^2 \ \text{\$/h} \]
\[ C_2 = 180 + 6.3P_2 + 0.009P_2^2 \ \text{\$/h} \]
\[ C_3 = 140 + 6.8P_3 + 0.007P_3^2 \ \text{\$/h} \]

\[ 10 \text{ MW} \leq P_1 \leq 85 \text{ MW} \]
\[ 10 \text{ MW} \leq P_2 \leq 80 \text{ MW} \]
\[ 10 \text{ MW} \leq P_3 \leq 70 \text{ MW} \]

\[ P_{L(pu)} = 0.0218P_{1(pu)}^2 + 0.0228P_{2(pu)}^2 + 0.0179P_{3(pu)}^2 \]

(only \( B_{ii} \neq 0 \))

\[ P_L = \left[ 0.0218\left(\frac{P_1}{100}\right)^2 + 0.0228\left(\frac{P_2}{100}\right)^2 + 0.0179\left(\frac{P_3}{100}\right)^2 \right] \times 100 \text{ MW} \]
\[ = 0.000218P_1^2 + 0.000228P_2^2 + 0.000179P_3^2 \text{ MW} \]
\( P_{i}^{(k)} = \frac{\lambda^{(k)} - \beta_{i}}{2(\gamma_{i} + \lambda^{(k)}B_{ii})} \)

\[
\sum_{i=1}^{n_{g}} \left( \frac{\partial P}{\partial \lambda} \right)^{(k)} = \sum_{i=1}^{n_{g}} \frac{\gamma_{i} + B_{ii} \beta_{i}}{2(\gamma_{i} + \lambda^{(k)}B_{ii})^{2}}
\]

\( \Delta \lambda^{(k)} = \frac{\Delta P^{(k)}}{\sum \left( \frac{\partial P_{L}}{\partial \lambda} \right)^{(k)}} \)

10 MW \( \leq P_{1} \leq 85 \) MW
10 MW \( \leq P_{2} \leq 80 \) MW
10 MW \( \leq P_{3} \leq 70 \) MW

\( \lambda^{(1)} = 8.0 \)

\[
P_{1}^{(1)} = \frac{8.0 - 7.0}{2(0.008 + 0.8 \times 0.000218)} = 51.3136 \text{ MW}
\]

\[
P_{2}^{(1)} = \frac{8.0 - 6.3}{2(0.008 + 0.8 \times 0.000228)} = 78.5292 \text{ MW}
\]

\[
P_{3}^{(1)} = \frac{8.0 - 6.8}{2(0.007 + 0.8 \times 0.000179)} = 71.1575 \text{ MW}
\]

\[
P_{L}^{(1)} = 0.000218(51.3136)^{2} + 0.000228(78.5292)^{2} + 0.000179(71.1575)^{2} = 2.886
\]

\[
\Delta P^{(1)} = 150 + 2.8864 - (51.3136 + 78.5292 + 71.1575) = -48.1139
\]

\[
\sum_{i=1}^{3} \left( \frac{\partial P}{\partial \lambda} \right)^{(1)} = 152.4924
\]

\[
\Delta \lambda^{(1)} = \frac{-48.1139}{152.4924} = -0.31552
\]

\( \lambda^{(4)} = 7.6789 \)

\[
P_{1}^{(4)} = \frac{7.6789 - 7.0}{2(0.008 + 7.679 \times 0.000218)} = 35.0907 \text{ MW}
\]

\[
P_{2}^{(4)} = \frac{7.6789 - 6.3}{2(0.009 + 7.679 \times 0.000228)} = 64.1317 \text{ MW}
\]

\[
P_{3}^{(4)} = \frac{7.6789 - 6.8}{2(0.007 + 7.679 \times 0.000179)} = 52.4767 \text{ MW}
\]

\[
P_{L}^{(4)} = 0.000218(35.0907)^{2} + 0.000228(64.1317)^{2} + 0.000179(52.4767)^{2} = 1.699
\]

\( \Delta P^{(4)} < 0.0001 \)

\[
C_{t} = 1592.65 \text{ $/h}
\]
Homework 8

• ECE521: 6.12, 6.13, 7.9, 7.11
• ECE421: 6.12(a)-(b), 6.13(a), 7.9, 7.11
• Due date: Dec 9 (Monday)