ECE 522 - Power Systems Analysis II
Spring 2021

Generator Modeling

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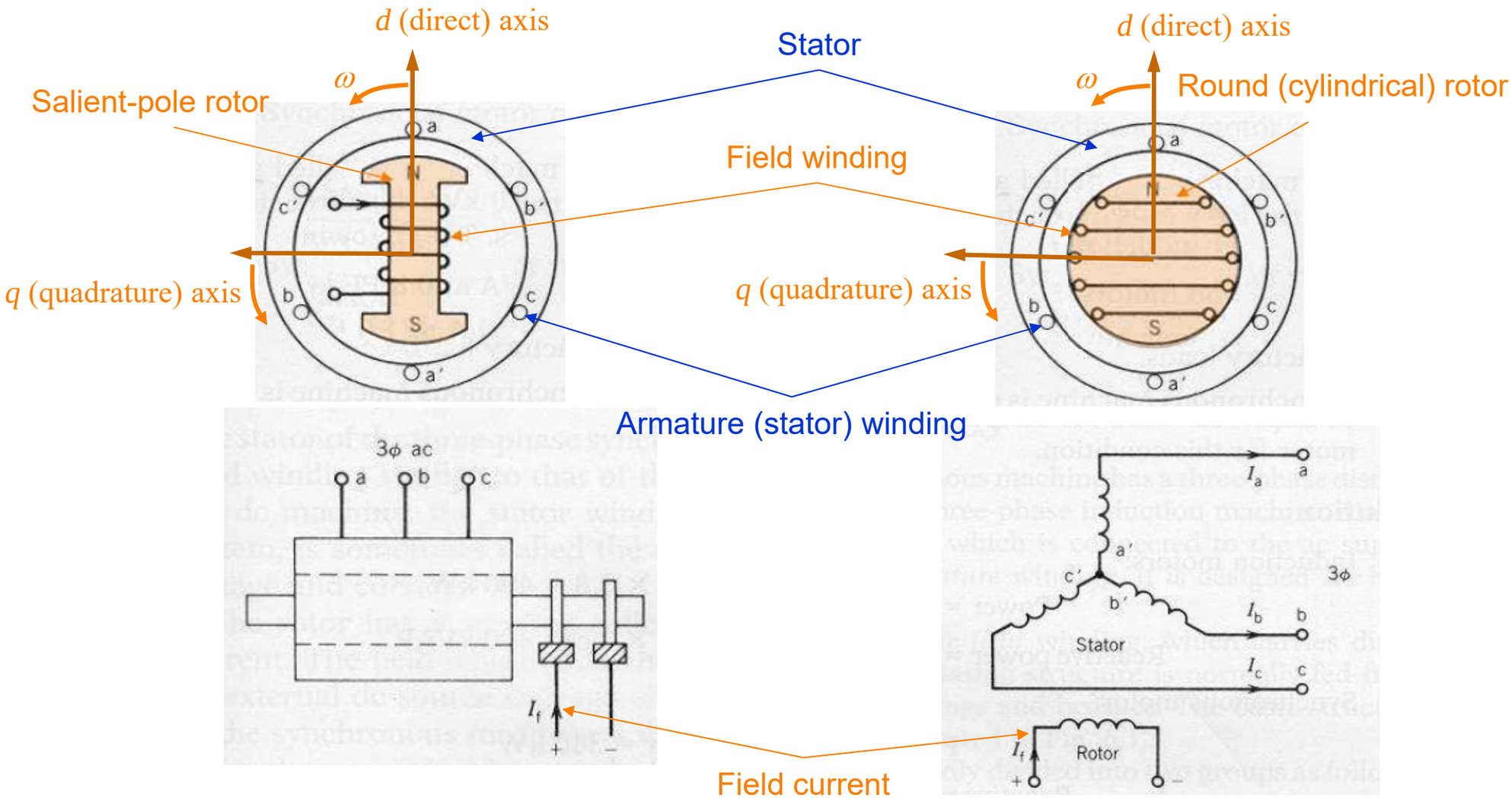
Outline

Modeling of synchronous generators for Stability Studies

- Synchronous Machine Modeling
- Simplified Models for Stability Studies
- Materials:
  - Kundur's Chapters 3-5
  - Saadat's Chapter 8

Synchronous Generators

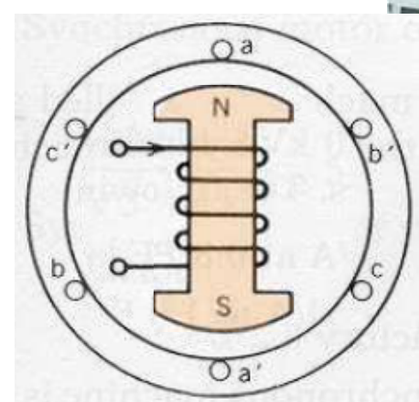


$$f = \frac{P}{2} \times \frac{n}{60}$$

(n: synchronous speed in rpm; P: the number of poles)

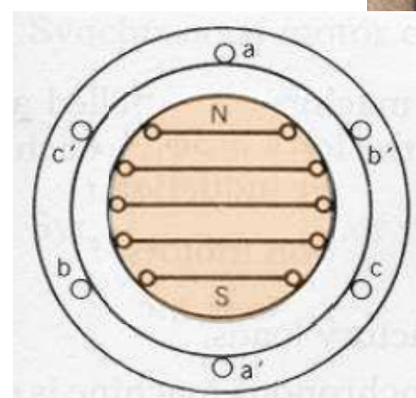
Types of Rotors

- **Salient pole rotors**
  - Have concentrated windings on poles and non-uniform air gap
  - Short axial length and large diameter to extract the maximum power from a waterfall
  - On hydraulic turbines operated at low speeds at 50-300 r/min (having a large number of poles)
  - Have a squirrel-cage windings (damper windings) embedded in the pole-faces to help damp out speed oscillations



12 poles salient-pole rotor (12 MW)

- **Cylindrical/round rotors**
  - Distributed winding and uniform air gap
  - Large axial length and small diameter to limit the centrifugal forces
  - Steam and gas turbines, operated at high speeds, typically 1800-3600r/min (4 or 2-pole)
  - Eddy in the solid steal rotor gives damping effects
  - 70% of large synchronous generators (150-1500MVA)



Round rotor generator under construction

(Source: http://emadrlc.blogspot.com)

Stator and Rotor Windings

Armature windings:

- a-a', b-b' and c-c' windings

Rotor windings:

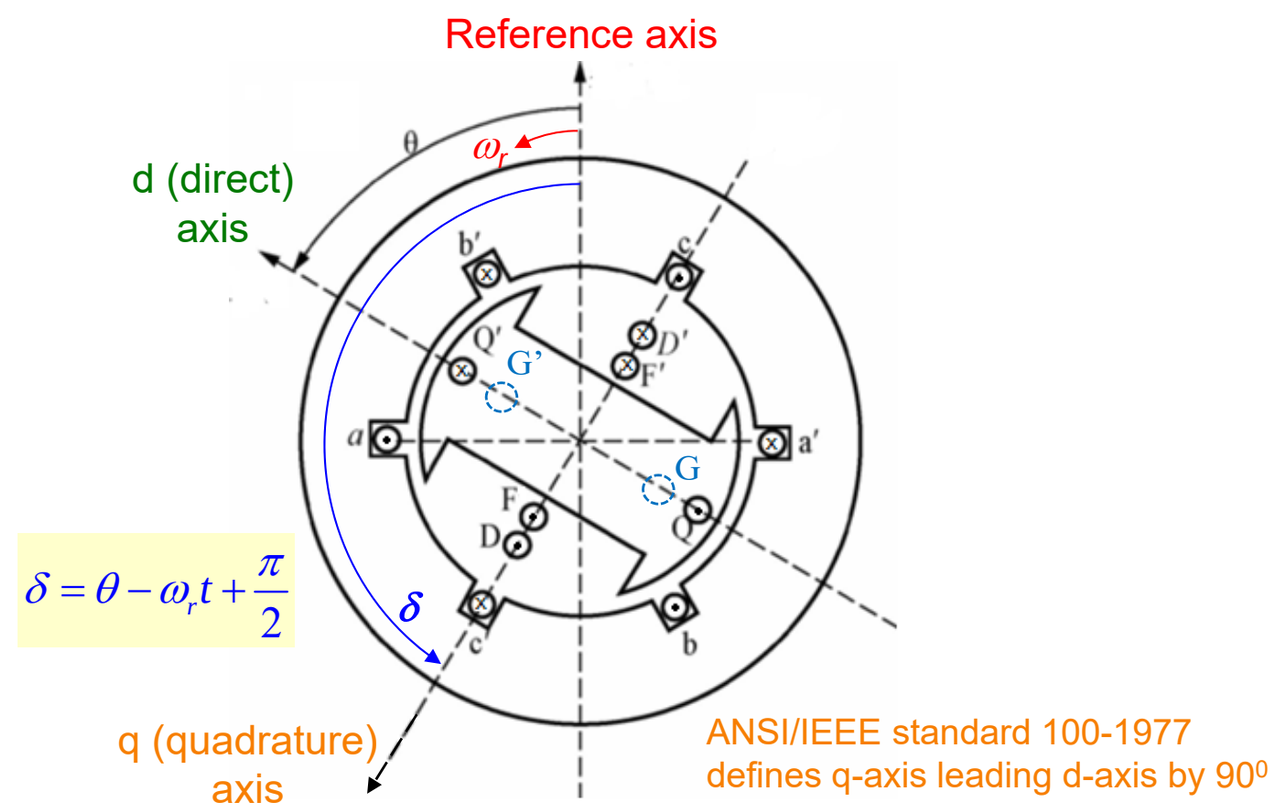
- Field windings
  - Field winding F-F' produces a flux on the d-axis.
- Damper windings
  - Two amortisseur/damper windings D-D' and Q-Q' respectively on d- and q-axes, which could be actual windings or effective parts of the solid steal rotor.
  - For a round-rotor machine, for symmetry, consider two windings on each axis by adding a second damper winding G-G' to the q-axis

Total number of windings:

- Salient pole: 3+3 (a, b, c, F, D, Q)
- Round-rotor: 3+4 (a, b, c, F, D, Q, G)

Rotor position

- θ is the displacement of d-axis from the axis of a-a'
- Consider a reference frame rotating synchronously with d and q-axes at speed ω_r (assumed to be along with the axis of a-a' at t=0 in the figure)
- δ is the displacement of q-axis from the rotating reference axis

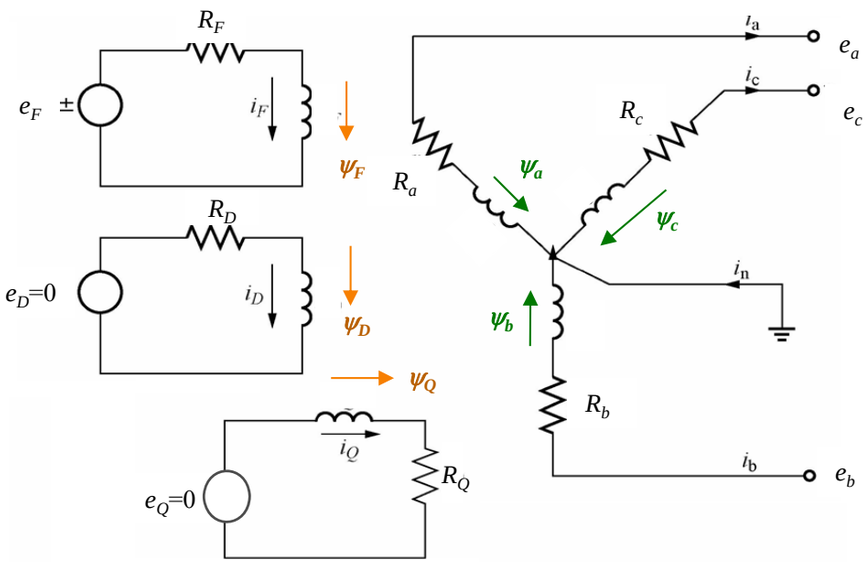


$$\delta = \theta - \omega_r t + \frac{\pi}{2}$$

ANSI/IEEE standard 100-1977 defines q-axis leading d-axis by 90^0

Voltage and Flux Equations (Salient-pole machine)

- The main objective of synchronous machine modeling is to find the minimum set of constant parameters by which voltage and flux equations are in the simplest form for desired accuracy.
- All windings are modeled as magnetically coupled circuits with inductances depending on θ.



- Each rotor winding (F, D, Q) is considered a load in its circuit having ψ and i in the same direction.
- Each stator winding (a, b, c) is considered a source in its circuit having ψ and i in opposite directions.

$$e_F = R_F i_F + \frac{d\psi_F}{dt} \qquad e_a = -R_a i_a + \frac{d\psi_a}{dt}$$

$$\psi_F = -\sum_{k=a,b,c} l_{Fk} i_k + \sum_{j=F,D,Q} l_{Fj} i_j \qquad \psi_a = -\sum_{k=a,b,c} l_{ak} i_k + \sum_{j=F,D,Q} l_{aj} i_j$$

Rotor self-inductances (l_{FF}, l_{DD}, l_{QQ})

Rotor mutual inductances (l_{FD}, l_{DQ}, l_{FQ})

Stator self-inductances (l_{aa}, l_{bb}, l_{cc})

Stator mutual inductances (l_{ab}, l_{bc}, l_{ac})

Stator-to-rotor mutual inductances (l_{aF}, l_{bD}, l_{aQ})

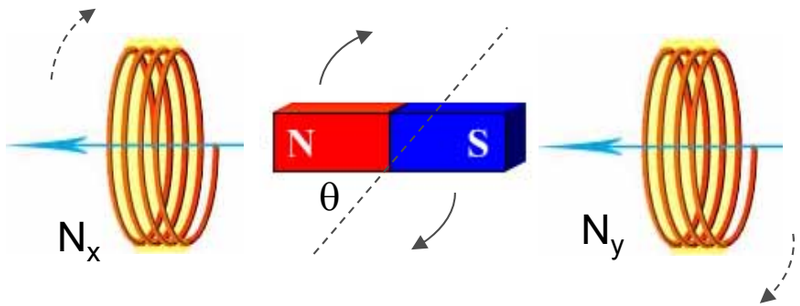
$$\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_F \\ e_D = 0 \\ e_Q = 0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_b & 0 & 0 & 0 & 0 \\ 0 & 0 & R_c & 0 & 0 & 0 \\ 0 & 0 & 0 & R_F & 0 & 0 \\ 0 & 0 & 0 & 0 & R_D & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e}_{abc} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix}$$

$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} l_{aa} & l_{ab} & l_{ac} & l_{aF} & l_{aD} & l_{aQ} \\ l_{ba} & l_{bb} & l_{bc} & l_{bF} & l_{bD} & l_{bQ} \\ l_{ca} & l_{cb} & l_{cc} & l_{cF} & l_{cD} & l_{cQ} \\ l_{Fa} & l_{Fb} & l_{Fc} & l_{FF} & l_{FD} & l_{FQ} \\ l_{Da} & l_{Db} & l_{Dc} & l_{DF} & l_{DD} & l_{DQ} \\ l_{Qa} & l_{Qb} & l_{Qc} & l_{QF} & l_{QD} & l_{QQ} \end{bmatrix} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

Variations of Self- and Mutual Inductances



Each winding is either stationary (on stator) or rotating (on rotor) together with the magnet:
x, y ∈ { a, b, c, F, D, Q }

What is each l_{xy}?

$$\begin{bmatrix} \psi_a \\ \psi_b \\ \psi_c \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \left[\begin{array}{ccc|ccc} l_{aa} & l_{ab} & l_{ac} & l_{aF} & l_{aD} & l_{aQ} \\ l_{ba} & l_{bb} & l_{bc} & l_{bF} & l_{bD} & l_{bQ} \\ l_{ca} & l_{cb} & l_{cc} & l_{cF} & l_{cD} & l_{cQ} \\ \hline l_{Fa} & l_{Fb} & l_{Fc} & l_{FF} & l_{FD} & l_{FQ} \\ l_{Da} & l_{Db} & l_{Dc} & l_{DF} & l_{DD} & l_{DQ} \\ l_{Qa} & l_{Qb} & l_{Qc} & l_{QF} & l_{QD} & l_{QQ} \end{array}\right] \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$\mathbf{L}_{SS}$ $\mathbf{L}_{SR}$ $\mathbf{L}_{RS}$ $\mathbf{L}_{RR}$

$l_{xy} = N_x \times N_y \times P_{xy} = l_{yx}$ (Symmetric)

$N_x \times N_y$: effective coupling

P_{xy} : permeance of the mutual flux path (mainly influenced by air gap)

- **Stator self/mutual inductances (e.g. l_{aa} and l_{ab}):** $N_x \times N_y$ is constant; P_{xy} is a periodic function (assumed to be cosine) of θ with a period 180° due to periodic air gap changes
  $P_{xy} \approx P_0 + P(\theta+\alpha) \rightarrow l_{xy} = l_0 + l_2 \cos 2(\theta+\alpha)$

- **Stator to Rotor Mutual Inductances (e.g. l_{aF}):** P_{xy} between stator and rotor windings is almost constant; however, their effective coupling is a periodic function of θ with period 360°; the flux leakage can be ignored (l_0=0).
  $N_x \times N_y \approx N_0 \cos(\theta+\beta) \rightarrow l_{xy} = l_1 \cos(\theta+\beta)$

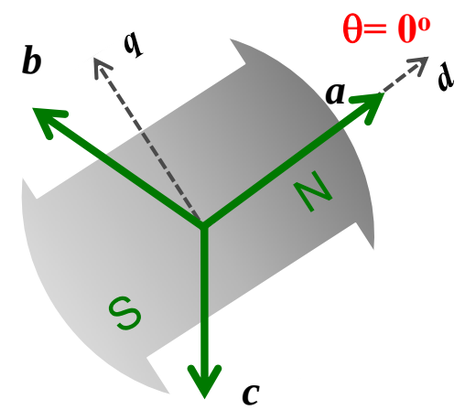
- **Rotor Inductances are all constant** using a reference revolving with the rotor

Stator self-inductances (l_{aa}, l_{bb}, l_{cc}) and Mutual Inductances (l_{ab}, l_{bc}, l_{ac})

- l_{aa} is equal to the ratio of flux linking phase a winding to current i_a, with zero currents in all other circuits (maximum P_{aa} at θ=0° or 180°)

$$l_{aa}=L_s + L_m \cos 2\theta$$
$$l_{bb}=L_s + L_m \cos 2(\theta-120^\circ)$$
$$l_{cc}=L_s + L_m \cos 2(\theta+120^\circ)$$

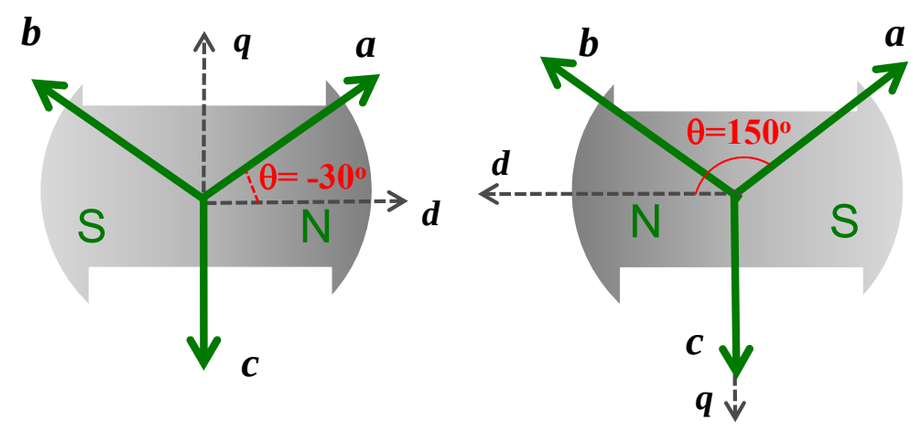
$$L_s > L_m \geq 0$$



- l_{ab} <0 since windings a and b have 120° (>90°) displacement (maximum P_{ab} at θ = -30° or 150°)

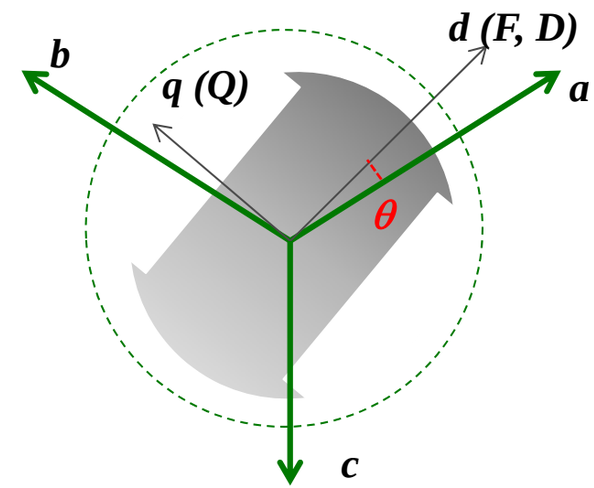
$$l_{ab} = -M_s - L_m\cos 2(\theta+30^\circ) = l_{ba}$$
$$l_{bc} = -M_s - L_m\cos 2(\theta-90^\circ) = l_{bc}$$
$$l_{ca} = -M_s - L_m\cos 2(\theta+150^\circ) = l_{ac}$$

$$M_s \approx L_s/2$$



Stator to Rotor Mutual Inductances

(L_{SR}: l_{aF} l_{bF} l_{cF} l_{aD} l_{bD} l_{cD} l_{aQ} l_{bQ} l_{cQ})



- The rotor sees a constant permeance if neglecting variations in the air gap due to stator slots

- When the flux linking a stator winding and a rotor winding reaches the maximum when they aligns with each other and is 0 when they are displaced by 90°

  (maximum $N_a \times N_F$ at θ=0°)

- d-axis

$$l_{aF} = l_{Fa} = M_F \cos\theta \qquad l_{aD} = l_{Da} = M_D \cos\theta$$

$$l_{bF} = l_{Fb} = M_F \cos(\theta\text{-}120^\circ) \qquad l_{bD} = l_{Db} = M_D \cos(\theta\text{-}120^\circ)$$

$$l_{cF} = l_{Fc} = M_F \cos(\theta+120^\circ) \qquad l_{cD} = l_{Dc} = M_D \cos(\theta+120^\circ)$$

- q-axis

$$l_{aQ} = l_{Qa} = -M_Q \sin\theta$$

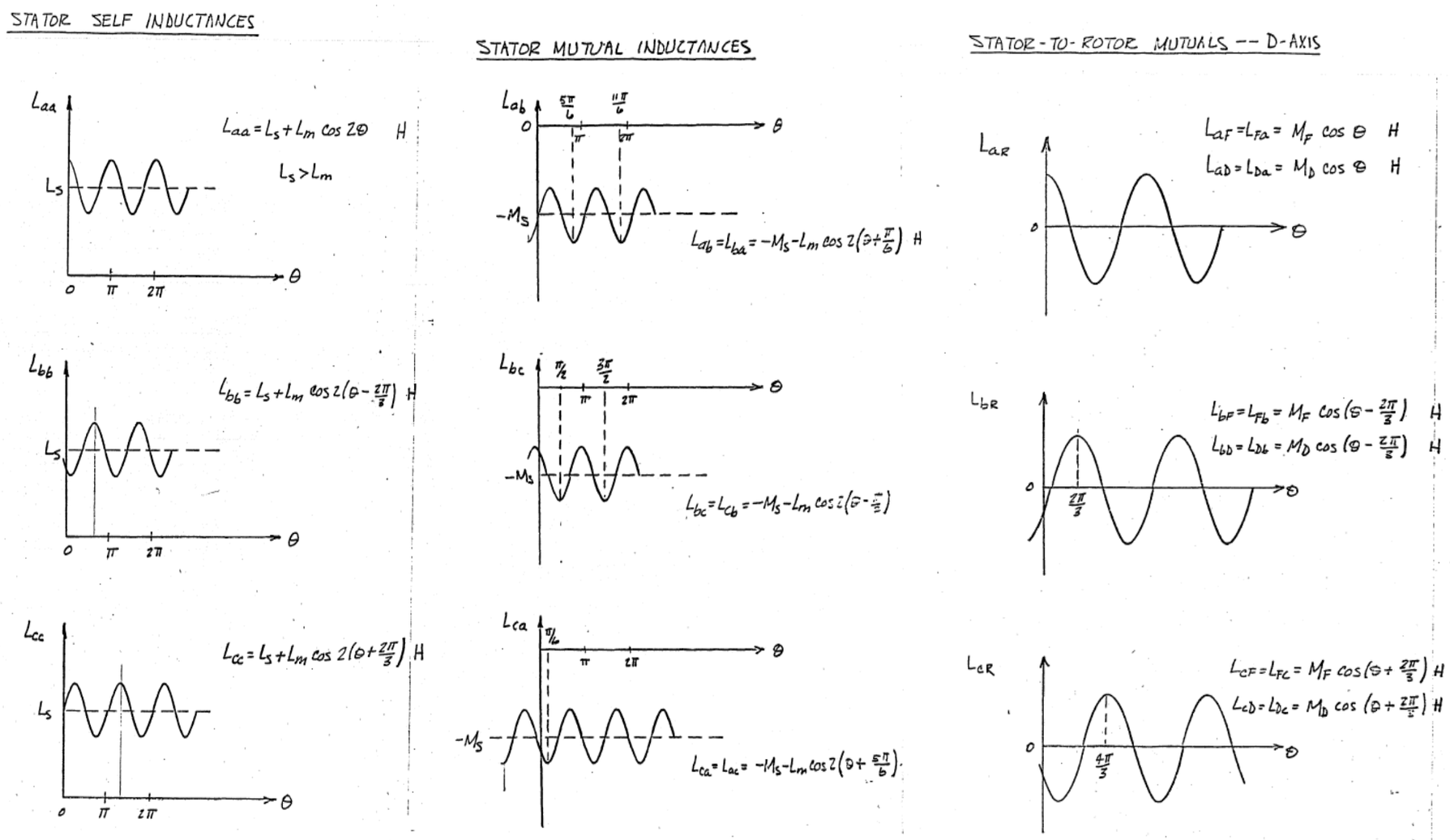
$$l_{bQ} = l_{Qb} = -M_Q \sin(\theta\text{-}120^\circ)$$

$$l_{cQ} = l_{Qc} = -M_Q \sin(\theta+120^\circ)$$

What if we define q-axis lagging d-axis by 90°?

(no negative signs)

For Salient-pole Rotors



For round rotors, which of the curves will be different?

Rotor Inductances (L_{RR}: l_{FF}, l_{DD}, l_{QQ}, l_{FD}, l_{FQ}, l_{DQ})

- They are all constant
  - Rotor self inductances

$$l_{FF} \triangleq L_F$$
$$l_{DD} \triangleq L_D$$
$$l_{QQ} \triangleq L_Q$$

  - Rotor mutual inductances

$$l_{FD} = l_{DF} \triangleq M_R$$
$$l_{FQ} = l_{QF} = 0$$
$$l_{DQ} = l_{QD} = 0$$

Summary

$$\begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

$$\boldsymbol{L}_{SS} = \begin{bmatrix} L_s + L_m \cos 2\theta & -M_s - L_m \cos 2\left(\theta + \frac{\pi}{6}\right) & -M_s - L_m \cos 2\left(\theta + \frac{5\pi}{6}\right) \\ -M_s - L_m \cos 2\left(\theta + \frac{\pi}{6}\right) & L_s + L_m \cos 2(\theta - \frac{2\pi}{3}) & -M_s - L_m \cos 2\left(\theta - \frac{\pi}{2}\right) \\ -M_s - L_m \cos 2\left(\theta + \frac{5\pi}{6}\right) & -M_s - L_m \cos 2\left(\theta - \frac{\pi}{2}\right) & L_s + L_m \cos 2(\theta + \frac{2\pi}{3}) \end{bmatrix}$$

$$\boldsymbol{L}_{SR} = \begin{bmatrix} M_F \cos\theta & M_D \cos\theta & -M_Q \sin\theta \\ M_F \cos\left(\theta - \frac{2\pi}{3}\right) & M_D \cos\left(\theta - \frac{2\pi}{3}\right) & -M_Q \sin\left(\theta - \frac{2\pi}{3}\right) \\ M_F \cos\left(\theta + \frac{2\pi}{3}\right) & M_D \cos\left(\theta + \frac{2\pi}{3}\right) & -M_Q \sin\left(\theta + \frac{2\pi}{3}\right) \end{bmatrix} \qquad \boldsymbol{L}_{RS} = \boldsymbol{L}^T{}_{SR} \qquad \boldsymbol{L}_{RR} = \begin{bmatrix} L_F & M_R & 0 \\ M_R & L_D & 0 \\ 0 & 0 & L_Q \end{bmatrix}$$

Further thinking:

- $\mathbf{L}_{RR}$ is constant because it is in a reference frame rotating with the rotor
- $\mathbf{L}_{SS}$ and $\mathbf{L}_{SR}$ are θ(t) dependent due to variations of $N_x \times N_y$ or P_{xy} caused by the rotation of the rotor relative to the stator

$$\psi_{FDQ} = -\,\mathbf{L}_{RS}(\theta) \times \mathbf{i}_{abc} + \mathbf{L}_{RR} \times \mathbf{i}_{FDQ}$$

Only $\mathbf{L}_{SR}(\theta)$ is explicitly dependent of θ

$$\Leftrightarrow \mathbf{L}_{RS}(\theta) \times \mathbf{i}_{abc} = -\psi_{FDQ} + \mathbf{L}_{RR} \times \mathbf{i}_{FDQ}$$

RHS is in the rotor reference and independent of θ.

$$\text{LHS} = [\mathbf{L}_{RS}(\theta)\,\mathbf{P}^{-1}(\theta)] \times [\mathbf{P}(\theta)\,\mathbf{i}_{abc}] \stackrel{\text{def}}{=} \mathbf{L}'_{RS} \times \mathbf{i}'$$

$$\mathbf{L}_{RS}(\theta)\times\mathbf{i}_{abc}=\begin{bmatrix}M_F & & \\ & M_D & \\ & & M_Q\end{bmatrix}\begin{bmatrix}\cos\theta & \cos(\theta-\frac{2\pi}{3}) & \cos(\theta+\frac{2\pi}{3})\\ \cos\theta & \cos(\theta-\frac{2\pi}{3}) & \cos(\theta+\frac{2\pi}{3})\\ -\sin\theta & -\sin(\theta-\frac{2\pi}{3}) & -\sin(\theta+\frac{2\pi}{3})\end{bmatrix}\begin{bmatrix}i_a\\ i_b\\ i_c\end{bmatrix}$$

$$=\begin{bmatrix}M_F & & \\ & M_D & \\ & & M_Q\end{bmatrix}\begin{bmatrix}i_a\cos\theta+i_b\cos(\theta-\frac{2\pi}{3})+i_c\cos(\theta+\frac{2\pi}{3})\\ i_a\cos\theta+i_b\cos(\theta-\frac{2\pi}{3})+i_c\cos(\theta+\frac{2\pi}{3})\\ -\left[i_a\sin\theta+i_b\sin(\theta-\frac{2\pi}{3})+i_c\sin(\theta+\frac{2\pi}{3})\right]\end{bmatrix}$$

$\rightarrow$ $\boldsymbol{i_d}$

$\rightarrow$ $\boldsymbol{i_q}$

$$\mathbf{L}_{RS}=\begin{bmatrix}M_F\cos\theta & M_F\cos(\theta-\frac{2\pi}{3}) & M_F\cos(\theta+\frac{2\pi}{3})\\ M_D\cos\theta & M_D\cos(\theta-\frac{2\pi}{3}) & M_D\cos(\theta+\frac{2\pi}{3})\\ -M_Q\sin\theta & -M_Q\sin(\theta-\frac{2\pi}{3}) & -M_Q\sin(\theta+\frac{2\pi}{3})\end{bmatrix}$$

$$\sin\alpha\cos\beta+\sin(\alpha-\frac{2\pi}{3})\cos(\beta-\frac{2\pi}{3})+\sin(\alpha+\frac{2\pi}{3})\cos(\beta+\frac{2\pi}{3})$$
$$=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)+\sin(\alpha+\beta-\frac{4\pi}{3})+\sin(\alpha-\beta)++\sin(\alpha+\beta+\frac{4\pi}{3})+\sin(\alpha-\beta)\right]$$
$$=\frac{3}{2}\sin(\alpha-\beta)$$

- For balanced steady-state conditions:
  $i_a = I_m \sin\omega_s t$, $i_b = I_m \sin(\omega_s t - 2\pi/3)$, $i_c = I_m \sin(\omega_s t + 2\pi/3)$, $\theta=\omega_r t+\delta-\pi/2$, $\omega_r\approx\omega_s$

$$i_a\cos\theta+i_b\cos(\theta-\frac{2\pi}{3})+i_c\cos(\theta+\frac{2\pi}{3})=I_m\sin\omega_s t\cos\theta+I_m\sin(\omega_s t-\frac{2\pi}{3})\cos(\theta-\frac{2\pi}{3})+I_m\sin(\omega_s t+\frac{2\pi}{3})\cos(\theta+\frac{2\pi}{3})$$
$$=\frac{3I_m}{2}\sin(\omega_s t-\theta)=\frac{3I_m}{2}\sin(\omega_s t-\omega_r t-\delta+\frac{\pi}{2})\approx\frac{3I_m}{2}\cos\delta \quad \sim \boldsymbol{i_d}$$

$$i_a\sin\theta+i_b\sin(\theta-\frac{2\pi}{3})+i_c\sin(\theta+\frac{2\pi}{3})=I_m\sin\omega_s t\cos(\theta-\frac{\pi}{2})+I_m\sin(\omega_s t-\frac{2\pi}{3})\cos(\theta-\frac{2\pi}{3}-\frac{\pi}{2})+I_m\sin(\omega_s t+\frac{2\pi}{3})\cos(\theta+\frac{2\pi}{3}-\frac{\pi}{2})$$
$$=\frac{3I_m}{2}\sin(\omega_s t-\theta+\frac{\pi}{2})=\frac{3I_m}{2}\sin(\omega_s t-\omega_r t-\delta+\pi)\approx\frac{3I_m}{2}\sin\delta \quad \sim \boldsymbol{i_q}$$

$$i_a+i_b+i_c=0 \quad \sim \boldsymbol{i_0}$$

Park's (dq0) Transformation

$$\mathbf{i}_{0dq} = \mathbf{P}\mathbf{i}_{abc}$$

$$\begin{bmatrix} i_0 \\ i_d \\ i_q \end{bmatrix} = \begin{bmatrix} k_0 & & \\ & k_d & \\ & & k_q \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \cos\theta & \cos(\theta-2\pi/3) & \cos(\theta+2\pi/3) \\ -\sin\theta & -\sin(\theta-2\pi/3) & -\sin(\theta+2\pi/3) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} = \begin{bmatrix} k_0 & k_0 & k_0 \\ k_d\cos\theta & k_d\cos(\theta-2\pi/3) & k_d\cos(\theta+2\pi/3) \\ -k_q\sin\theta & -k_q\sin(\theta-2\pi/3) & -k_q\sin(\theta+2\pi/3) \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

For balanced steady-state conditions:

$i_a = I_m \sin\omega_s t$

$i_b = I_m \sin(\omega_s t - 2\pi/3)$

$i_c = I_m \sin(\omega_s t + 2\pi/3)$

$\theta = \omega_r t + \delta - \pi/2, \quad \omega_r \approx \omega_s$

$i_d = k_d I_m \sin(\omega_s t - \theta) \times 3/2 \quad = k_d I_m \cos\delta \times 3/2$

$i_q = -k_q I_m \cos(\omega_s t - \theta) \times 3/2 = -k_q I_m \sin\delta \times 3/2$

$i_0 = 0$

All constant currents!

- If we expect the transformation to be power invariant:

$$P_{3\phi} = e_a i_a + e_b i_b + e_c i_c = \mathbf{e}_{abc}^T \mathbf{i}_{abc} = (\mathbf{P}^{-1}\mathbf{e}_{0dq})^T \mathbf{P}^{-1}\mathbf{i}_{0dq} = \mathbf{e}_{0dq}^T \mathbf{P}^{-T}\mathbf{P}^{-1}\mathbf{i}_{0dq} = e_0 i_0 + e_d i_d + e_q i_q$$

$\mathbf{P}^{-T}\mathbf{P}^{-1} = \mathbf{I} \rightarrow \mathbf{P}^{-1} = \mathbf{P}^T$ or $\mathbf{P}^T\mathbf{P} = \mathbf{I}$ (orthogonal matrix)

$k_d = k_q = \sqrt{\frac{2}{3}}$ and $k_0 = \frac{1}{\sqrt{3}}$

$$\mathbf{P} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos(\theta-2\pi/3) & \cos(\theta+2\pi/3) \\ -\sin\theta & -\sin(\theta-2\pi/3) & -\sin(\theta+2\pi/3) \end{bmatrix} \quad \mathbf{P}^{-1} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1/\sqrt{2} & \cos\theta & -\sin\theta \\ 1/\sqrt{2} & \cos(\theta-2\pi/3) & -\sin(\theta-2\pi/3) \\ 1/\sqrt{2} & \cos(\theta+2\pi/3) & -\sin(\theta+2\pi/3) \end{bmatrix}$$

What if we define q-axis lagging d-axis by 90°?

Flux Equations

$$\mathbf{i}_{0dq} = \mathbf{P}\mathbf{i}_{abc}$$

$$\boldsymbol{\psi}_{0dq} = \mathbf{P}\boldsymbol{\psi}_{abc}$$

$$\begin{bmatrix} \mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}'_{SS} & \mathbf{L}'_{SR} \\ \mathbf{L}'_{RS} & \mathbf{L}'_{RR} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix}$$

$$\begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_F \\ \psi_D \\ \psi_q \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_q \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & kM_D & 0 \\ 0 & 0 & L_q & 0 & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & M_R & 0 \\ 0 & kM_D & 0 & M_R & L_D & 0 \\ 0 & 0 & kM_Q & 0 & 0 & L_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ -i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$L_0 = L_s - 2M_s, \quad L_d = L_s + M_s + \frac{3}{2}L_m, \quad L_q = L_s + M_s - \frac{3}{2}L_m, \quad k = \sqrt{\frac{3}{2}}$$

Voltage Equations

$$\mathbf{e}_{0dq} = \mathbf{P}\mathbf{e}_{abc} \Rightarrow \begin{bmatrix} \mathbf{e}_{abc} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{0dq} \\ \mathbf{e}_{FDQ} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e}_{abc} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{0dq} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{abc} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\psi}}_{abc} \\ \dot{\boldsymbol{\psi}}_{FDQ} \end{bmatrix} \neq \begin{bmatrix} \dot{\boldsymbol{\psi}}_{0dq} \\ \dot{\boldsymbol{\psi}}_{FDQ} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \mathbf{e}_{0dq} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{abc} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \frac{d}{dt} \left\{ \begin{bmatrix} \mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} \right\}$$

$$\Leftrightarrow \begin{bmatrix} \mathbf{e}_{0dq} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{0dq} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} \frac{d}{dt} \mathbf{P}^{-1} \boldsymbol{\psi}_{0dp} \\ 0 \end{bmatrix}$$

Note: **P** is NOT constant

$$\mathbf{P}\frac{d}{dt}\mathbf{P}^{-1} = \mathbf{P}\frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{P}^{-1} = \omega_r \mathbf{P}\frac{d}{d\theta}\mathbf{P}^{-1} = \frac{2\omega_r}{3} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ -\sin\theta & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \end{bmatrix} \times \begin{bmatrix} 0 & -\sin\theta & -\cos\theta \\ 0 & -\sin(\theta - \frac{2\pi}{3}) & -\cos(\theta - \frac{2\pi}{3}) \\ 0 & -\sin(\theta + \frac{2\pi}{3}) & -\cos(\theta + \frac{2\pi}{3}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_r \\ 0 & \omega_r & 0 \end{bmatrix}$$

$$\mathbf{P}\frac{d}{dt}\mathbf{P}^{-1}\psi_{0dq} = \begin{bmatrix} 0 \\ -\omega_r \psi_q \\ \omega_r \psi_d \end{bmatrix} = \begin{bmatrix} 0 \\ (-\omega_r L_q) \times (-i_q) - \omega_r k M_Q i_Q \\ \omega_r L_d \times (-i_d) + \omega_r k M_F i_F + \omega_r k M_D i_D \end{bmatrix}$$

Voltage Equations (cont'd)

Assume $\mathbf{R}_{abc}=R_a\mathbf{I}$

$$\begin{bmatrix} e_0 \\ e_d \\ e_q \\ e_F \\ e_D=0 \\ e_Q=0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & -\omega_r L_q & 0 & 0 & -\omega_r kM_Q \\ 0 & \omega_r L_d & R_a & \omega_r kM_F & \omega_r kM_D & 0 \\ 0 & 0 & 0 & R_F & 0 & 0 \\ 0 & 0 & 0 & 0 & R_D & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ -i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & kM_D & 0 \\ 0 & 0 & L_q & 0 & 0 & kM_Q \\ 0 & kM_F & 0 & L_F & M_R & 0 \\ 0 & kM_D & 0 & M_R & L_D & 0 \\ 0 & 0 & kM_Q & 0 & 0 & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} -i_0 \\ -i_d \\ -i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$\begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_q \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \left[\begin{array}{ccc|ccc} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & kM_D & 0 \\ 0 & 0 & L_q & 0 & 0 & kM_Q \\ \hline 0 & kM_F & 0 & L_F & M_R & 0 \\ 0 & kM_D & 0 & M_R & L_D & 0 \\ 0 & 0 & kM_Q & 0 & 0 & L_Q \end{array}\right] \begin{bmatrix} -i_0 \\ -i_d \\ -i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix} = \left[\begin{array}{c|ccc|cc} R_a & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & R_a & 0 & 0 & -\omega_r L_q & -\omega_r kM_Q \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ \hline 0 & \omega_r L_d & \omega_r kM_F & \omega_r kM_D & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{array}\right] \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \left[\begin{array}{c|ccc|cc} L_0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{array}\right] \frac{d}{dt} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}$$

$$(-\omega_r L_q)\times(-i_q) - \omega_r kM_Q i_Q = -\omega_r \psi_q \qquad \omega_r L_d \times (-i_d) + \omega_r kM_F i_F + \omega_r kM_D i_D = \omega_r \psi_d$$

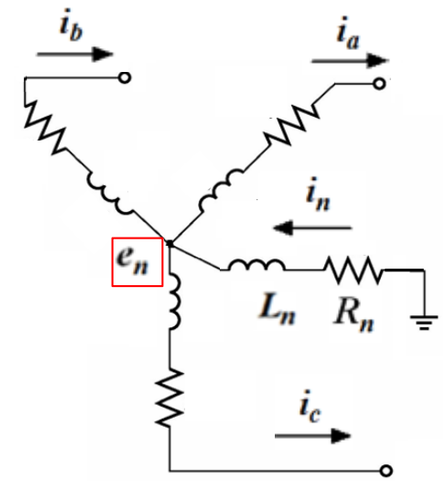
$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix} = \left[\begin{array}{c|ccc|cc} R_a & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{array}\right] \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \left[\begin{array}{c|ccc|cc} L_0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{array}\right] \frac{d}{dt} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_r \psi_q \\ 0 \\ 0 \\ \omega_r \psi_d \\ 0 \end{bmatrix}$$

If the neutral line-to-ground voltage is not zero

$$\begin{bmatrix} \mathbf{e}_{0dq} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{0dq} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P}\frac{d}{dt}\mathbf{P}^{-1} \times \boldsymbol{\psi}_{0dp} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{P} \times \mathbf{e}_n \\ \mathbf{0} \end{bmatrix}$$

$$\mathbf{e}_n \triangleq \begin{bmatrix} e_n \\ e_n \\ e_n \end{bmatrix} = -(R_n i_n + L_n \frac{di_n}{dt}) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} R_n & R_n & R_n \\ R_n & R_n & R_n \\ R_n & R_n & R_n \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} - \begin{bmatrix} L_n & L_n & L_n \\ L_n & L_n & L_n \\ L_n & L_n & L_n \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

$$= -\mathbf{R}_n \mathbf{i}_{abc} - \mathbf{L}_n \frac{d}{dt} \mathbf{i}_{abc}$$



$$\mathbf{P} \times \mathbf{e}_n = \mathbf{P}\ (-\mathbf{R}_n\ \mathbf{i}_{abc} - \mathbf{L}_n\ d\mathbf{i}_{abc}/dt\ )$$

$$= -\ \mathbf{P}\ \mathbf{R}_n\ \mathbf{P}^{-1}\ \mathbf{P}\ \mathbf{i}_{abc} - \mathbf{P}\ \mathbf{L}_n\ \mathbf{P}^{-1}\ \mathbf{P}\ d\mathbf{i}_{abc}/dt$$

$$= -\ \mathbf{P}\ \mathbf{R}_n\ \mathbf{P}^{-1}\ \mathbf{i}_{0dq} - \mathbf{P}\ \mathbf{L}_n\ \mathbf{P}^{-1}\ (d\mathbf{i}_{0dq}/dt - d\mathbf{P}/dt \times \mathbf{i}_{abc})$$

$$= -\begin{bmatrix} 3R_n i_0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3L_n\, di_0/dt \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{P}\mathbf{R}_n\mathbf{P}^{-1} = \begin{bmatrix} 3R_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{P}\mathbf{L}_n\mathbf{P}^{-1} = \begin{bmatrix} 3L_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{e}_{0dq} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{0dq} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{0dq} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \boldsymbol{\psi}_{0dq} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} -3R_n i_0 - 3L_n \frac{d}{dt} i_0 \\ -\omega_r \psi_q \\ \omega_r \psi_d \\ \hline \mathbf{0}_{3\times 1} \end{bmatrix}$$

Note: $\mathbf{P}\ \mathbf{L}_n\ \mathbf{P}^{-1} \times d\mathbf{P}/dt \times \mathbf{i}_{abc} \equiv \mathbf{0}$

Summary: Flux and Voltage Equations

$$\mathbf{i}_{0dq} = \mathbf{P}\mathbf{i}_{abc} \qquad \mathbf{P} = \sqrt{2/3}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ \cos\theta & \cos(\theta - 2\pi/3) & \cos(\theta + 2\pi/3) \\ -\sin\theta & -\sin(\theta - 2\pi/3) & -\sin(\theta + 2\pi/3) \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{SS} & \mathbf{L}_{SR} \\ \mathbf{L}_{RS} & \mathbf{L}_{RR} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} \quad \Longleftrightarrow \quad \boldsymbol{\psi}_{0dq} = \mathbf{P}\boldsymbol{\psi}_{abc}$$

$$\begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_F \\ \psi_D \\ \psi_q \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}$$

$$L_0 = L_s - 2M_s$$

$$L_d = L_s + M_s + \frac{3}{2}L_m$$

$$L_q = L_s + M_s - \frac{3}{2}L_m$$

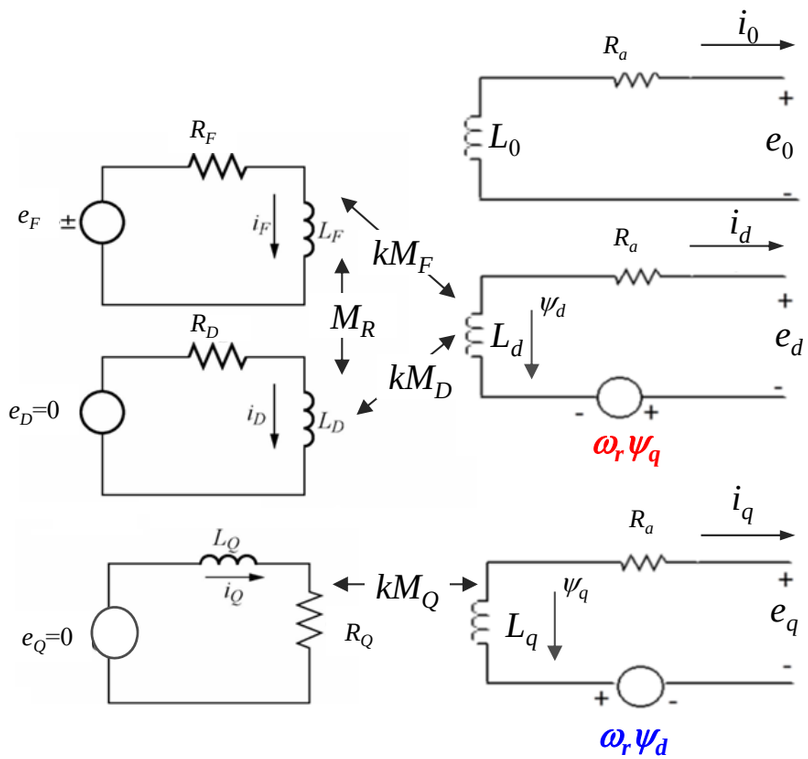
$$k = \sqrt{3/2}$$

$$\begin{bmatrix} \mathbf{e}_{abc} \\ \mathbf{e}_{FDQ} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{abc} & 0 \\ 0 & \mathbf{R}_{FDQ} \end{bmatrix} \begin{bmatrix} -\mathbf{i}_{abc} \\ \mathbf{i}_{FDQ} \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \boldsymbol{\psi}_{abc} \\ \boldsymbol{\psi}_{FDQ} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_n \\ \mathbf{0} \end{bmatrix} \quad \Longleftrightarrow \quad \mathbf{e}_{0dq} = \mathbf{P}\mathbf{e}_{abc} \qquad \mathbf{R}_{abc} = R_a\mathbf{I}$$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ 0 & 0 & 0 & 0 & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{bmatrix} \frac{d}{dt}\begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_r\psi_q \\ 0 \\ 0 \\ \omega_r\psi_d \\ 0 \end{bmatrix}$$

Winding Circuits (Equivalent Transformers) after Park's Transformation

$$
\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix}
=
\begin{bmatrix}
R_a & 0 & 0 & 0 & 0 & 0 \\
0 & R_a & 0 & 0 & 0 & 0 \\
0 & 0 & R_F & 0 & 0 & 0 \\
0 & 0 & 0 & R_D & 0 & 0 \\
0 & 0 & 0 & 0 & R_a & 0 \\
0 & 0 & 0 & 0 & 0 & R_Q
\end{bmatrix}
\begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}
+
\begin{bmatrix}
L_0 & 0 & 0 & 0 & 0 & 0 \\
0 & L_d & kM_F & kM_D & 0 & 0 \\
0 & kM_F & L_F & M_R & 0 & 0 \\
0 & kM_D & M_R & L_D & 0 & 0 \\
0 & 0 & 0 & 0 & L_q & kM_Q \\
0 & 0 & 0 & 0 & kM_Q & L_Q
\end{bmatrix}
\frac{d}{dt}
\begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}
+
\begin{bmatrix} 0 \\ -\omega_r \psi_q \\ 0 \\ 0 \\ \omega_r \psi_d \\ 0 \end{bmatrix}
$$

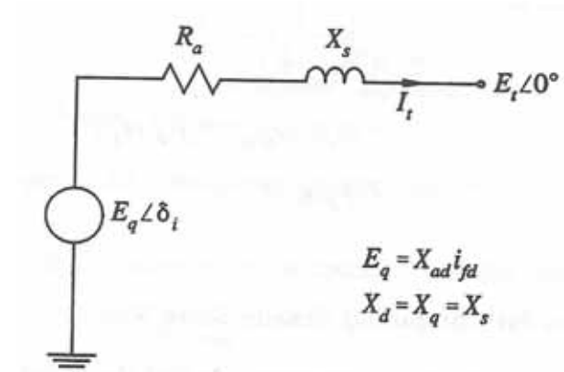


$\omega_r\psi_d$ is a speed voltage caused by d-axis flux in the q-axis winding

$-\omega_r\psi_q$ is a speed voltage caused by q-axis flux in the d-axis winding

Voltage equations follow Faraday's law

$$\mathcal{E} = \frac{1}{q}\oint_{\text{wire}} \mathbf{F}\cdot d\boldsymbol{\ell} = \oint_{\text{wire}} (\mathbf{E} + \boxed{\mathbf{v}\times\mathbf{B}})\cdot d\boldsymbol{\ell}$$



Naming Convention

My Notes	Kundur's book
i_F	i_{fd}
i_D	i_{kd}
i_Q	i_{kq}
L_s	L_{aa0}
M_s	L_{ab0}
L_m	L_{aa2}
M_F	L_{afd}
M_D	L_{akd}
M_Q	L_{akq}
M_R	L_{fkd}
L_F	L_{ffd}
L_D	L_{kkd}
L_Q	L_{kkq}

Alternative Park's Transformation

- If k_d=k_q=2/3 and k_0=1/3, i_a, i_b, i_c, i_d, i_q all have the same amplitude I_m, or in other words, a unit-to-unit relationship holds

$$i_a = I_m \sin\omega_s t$$
$$i_b = I_m \sin(\omega_s t - 2\pi/3)$$
$$i_c = I_m \sin(\omega_s t + 2\pi/3)$$

$$\tilde{\mathbf{P}} = \frac{2}{3}\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ \cos\theta & \cos(\theta-\frac{2\pi}{3}) & \cos(\theta+\frac{2\pi}{3}) \\ -\sin\theta & -\sin(\theta-\frac{2\pi}{3}) & -\sin(\theta+\frac{2\pi}{3}) \end{bmatrix} \qquad \tilde{\mathbf{P}}^{-1} = \begin{bmatrix} 1 & \cos\theta & -\sin\theta \\ 1 & \cos(\theta-\frac{2\pi}{3}) & -\sin(\theta-\frac{2\pi}{3}) \\ 1 & \cos(\theta+\frac{2\pi}{3}) & -\sin(\theta+\frac{2\pi}{3}) \end{bmatrix}$$

$$i_d = k_d I_m \sin(\omega_s t-\theta)\times 3/2 \quad = I_m \sin(\omega_s t-\theta)$$
$$i_q = -k_q I_m \cos(\omega_s t-\theta)\times 3/2 = -I_m \cos(\omega_s t-\theta)$$
$$i_0 = k_0(i_a + i_b + i_c)$$

- However, $\tilde{\mathbf{P}}$ is not isometric. An isometric **P** that preserves distances or inner products is orthogonal: $\mathbf{P}^T\mathbf{P}=\mathbf{I}$.

$$P_{3\phi} = e_a i_a + e_b i_b + e_c i_c = \mathbf{e}_{abc}^T \mathbf{i}_{abc} = \mathbf{e}_{0dq}^T \tilde{\mathbf{P}}^{-T}\tilde{\mathbf{P}}^{-1}\mathbf{i}_{0dq} = 3e_0 i_0 + \frac{3}{2}e_d i_d + \frac{3}{2}e_q i_q \qquad \neq e_0 i_0 + e_d i_d + e_q i_q$$

- L_{ad}-L_{aq} based per unit system: all per unit mutual inductances between the stator and rotor circuits are equal to $\bar{L}_{ad}$ in d-axis or $\bar{L}_{aq}$ in q-axis, where $\bar{L}_{ad} = \bar{L}_d - \bar{L}_l$ and $\bar{L}_{aq} = \bar{L}_q - \bar{L}_l$
- This reduces the number of constant parameters in model without losing its accuracy.

$$\begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_q \\ \psi_F \\ \psi_D \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & M_F & M_D & 0 \\ 0 & 0 & L_q & 0 & 0 & M_Q \\ 0 & \frac{3}{2}M_F & 0 & L_F & M_R & 0 \\ 0 & \frac{3}{2}M_D & 0 & M_R & L_D & 0 \\ 0 & 0 & \frac{3}{2}M_Q & 0 & 0 & L_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ -i_q \\ i_F \\ i_D \\ i_Q \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{\psi}_0 \\ \bar{\psi}_d \\ \bar{\psi}_q \\ \bar{\psi}_F \\ \bar{\psi}_D \\ \bar{\psi}_Q \end{bmatrix} = \begin{bmatrix} \bar{L}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{L}_l + \bar{L}_{ad} & 0 & \bar{L}_{ad} & \bar{L}_{ad} & 0 \\ 0 & 0 & \bar{L}_l + \bar{L}_{aq} & 0 & 0 & \bar{L}_{aq} \\ 0 & \bar{L}_{ad} & 0 & \bar{L}_F & \bar{M}_R & 0 \\ 0 & \bar{L}_{ad} & 0 & \bar{M}_R & \bar{L}_D & 0 \\ 0 & 0 & \bar{L}_{aq} & 0 & 0 & \bar{L}_Q \end{bmatrix} \begin{bmatrix} -\bar{i}_0 \\ -\bar{i}_d \\ -\bar{i}_q \\ \bar{i}_F \\ \bar{i}_D \\ \bar{i}_Q \end{bmatrix}$$

Per Unit Representation

- Each of stator/rotor winding circuit involves quantities, e.g. e, i, S, Z and f, but only three of them are independent.
- Using the machine ratings (stator-side) as the base values

$e_{s\,base}$ (V) peak value of rated line-to-neutral voltage

$i_{s\,base}$ (A) peak value of rated line current

f_{base} (Hz) rated frequency

$$\bar{X}(p.u.) = \frac{X}{Z_{base}} = \frac{2\pi f}{2\pi f_{base}} \times \frac{L}{L_{base}}$$

If $f = f_{base}$ $\bar{X} = \bar{L}$

$S_{3\phi\,base}$ (VA) $= 3\frac{e_{s\,base}}{\sqrt{2}} \times \frac{i_{s\,base}}{\sqrt{2}} = \frac{3}{2} e_{s\,base} \times i_{s\,base}$

ω_{base} (elec. rad/s) $= 2\pi f_{base} = \omega_{mbase} \times (\frac{p}{2})$ e.g. ≈377(rad/s) for 60Hz

t_{base}(s) $= 1/\omega_{base} = 1/(2\pi f_{base})$ ≈0.0027 sec.

$Z_{s\,base}$ (Ω) $= e_{s\,base}/i_{s\,base}$

$L_{s\,base}$ (H) $= Z_{s\,base}/\omega_{base}$

$\psi_{s\,base}$ (Wb·turns) $= L_{s\,base} \times i_{s\,base} = e_{s\,base}/\omega_{base}$

T_{base} (N·m) $= \frac{S_{3\phi\,base}}{\omega_{mbase}} = \frac{3}{2}(\frac{p}{2})\psi_{s\,base} \times i_{s\,base}$

Bases	d	q	0	F	D	Q
1	f_{base}					
2	$e_{s\,base}$			$S_{3\phi\,base}$		
3	$i_{s\,base}$			$i_{F\,base}$	$i_{D\,base}$	$i_{Q\,base}$

i_{Fbase}, i_{Dbase} and i_{Qbase} should enable a symmetric per-unit inductance matrix.

L_{ad}-L_{aq} based Per Unit System

$$\begin{bmatrix}\psi_d\\ \psi_q\\ \psi_0\\ \psi_F\\ \psi_D\\ \psi_Q\end{bmatrix}=\begin{bmatrix}L_d & 0 & 0 & M_F & M_D & 0\\ 0 & L_q & 0 & 0 & 0 & M_Q\\ 0 & 0 & L_0 & 0 & 0 & 0\\ \frac{3}{2}M_F & 0 & 0 & L_F & M_R & 0\\ \frac{3}{2}M_D & 0 & 0 & M_R & L_D & 0\\ 0 & \frac{3}{2}M_Q & 0 & 0 & 0 & L_Q\end{bmatrix}\begin{bmatrix}-i_d\\ -i_q\\ -i_0\\ i_F\\ i_D\\ i_Q\end{bmatrix}$$

Step 1: Symmetrizing

$$\begin{bmatrix}\bar{\psi}_d\\ \bar{\psi}_q\\ \bar{\psi}_0\\ \bar{\psi}_F\\ \bar{\psi}_D\\ \bar{\psi}_Q\end{bmatrix}=\begin{bmatrix}\bar{L}_d & 0 & 0 & \bar{M}_F & \bar{M}_D & 0\\ 0 & \bar{L}_q & 0 & 0 & 0 & \bar{M}_Q\\ 0 & 0 & \bar{L}_0 & 0 & 0 & 0\\ \bar{M}'_F & 0 & 0 & \bar{L}_F & \bar{M}_R & 0\\ \bar{M}'_D & 0 & 0 & \bar{M}_R & \bar{L}_D & 0\\ 0 & \bar{M}'_Q & 0 & 0 & 0 & \bar{L}_Q\end{bmatrix}\begin{bmatrix}-\bar{i}_d\\ -\bar{i}_q\\ -\bar{i}_0\\ \bar{i}_F\\ \bar{i}_D\\ \bar{i}_Q\end{bmatrix}$$

1. Divide both sides of equation of ψ_d by $\psi_{s\,base}=L_{s\,base}\times i_{s\,base}$

$$\frac{\psi_d}{\psi_{s\,base}}=-\frac{L_d\times i_d}{L_{s\,base}\times i_{s\,base}}+\frac{M_F\times i_F}{L_{s\,base}\times i_{s\,base}}+\frac{M_D\times i_D}{L_{s\,base}\times i_{s\,base}}$$

$$=-\frac{L_d\times i_d}{L_{s\,base}\times i_{s\,base}}+\frac{M_F\times i_{F\,base}}{L_{s\,base}\times i_{s\,base}}\times\frac{i_F}{i_{F\,base}}+\frac{M_D\times i_{D\,base}}{L_{s\,base}\times i_{s\,base}}\times\frac{i_D}{i_{D\,base}}$$

$$\bar{\psi}_d=-\bar{L}_d\times\bar{i}_d+\bar{M}_F\times\bar{i}_F+\bar{M}_D\times\bar{i}_D$$

2. Divide both sides of equation of ψ_F by $\psi_{F\,base}=L_{F\,base}\times i_{F\,base}$

$$\frac{\psi_F}{\psi_{F\,base}}=-\frac{3}{2}\frac{M_F\times i_d}{L_{F\,base}\times i_{F\,base}}+\frac{L_F\times i_F}{L_{F\,base}\times i_{F\,base}}+\frac{M_R\times i_D}{L_{F\,base}\times i_{F\,base}}$$

$$=-\frac{3}{2}\frac{M_F\times i_{S\,base}}{L_{F\,base}\times i_{F\,base}}\times\frac{i_d}{i_{s\,base}}+\frac{L_F\times i_F}{L_{F\,base}\times i_{F\,base}}+\frac{M_R\times i_{D\,base}}{L_{F\,base}\times i_{F\,base}}\times\frac{i_D}{i_{D\,base}}$$

$$\bar{\psi}_F=-\bar{M}'_F\times\bar{i}_d+\bar{L}_F\times\bar{i}_F+\bar{M}_R\times\bar{i}_D$$

3. Equate two per unit values: $\bar{M}_F=\bar{M}'_F$

$$\frac{M_F\times i_{F\,base}}{L_{s\,base}\times i_{s\,base}}=\frac{3}{2}\frac{M_F\times i_{S\,base}}{L_{F\,base}\times i_{F\,base}}$$

$$L_{F\,base}\,i^2_{F\,base}=\frac{3}{2}L_{S\,base}\,i^2_{S\,base}$$

$$\omega_{base}L_{F\,base}\,i^2_{F\,base}=\frac{3}{2}\omega_{base}\,L_{S\,base}\,i^2_{S\,base}$$

$$e_{F\,base}\,i_{F\,base}=\frac{3}{2}e_{S\,base}\,i_{S\,base}=S_{3\phi\,base}$$

Similarly,

$$e_{D\,base}\,i_{D\,base}=S_{3\phi\,base}\qquad e_{Q\,base}\,i_{Q\,base}=S_{3\phi\,base}$$

All rotor circuits have the same VA_{base} that equals the stator 3-phase VA_{base}

L_{ad}-L_{aq} based Per Unit System (cont'd)

$$\begin{bmatrix} \bar{\psi}_d \\ \bar{\psi}_q \\ \bar{\psi}_0 \\ \bar{\psi}_F \\ \bar{\psi}_D \\ \bar{\psi}_Q \end{bmatrix} = \begin{bmatrix} \bar{L}_d & 0 & 0 & \bar{M}_F & \bar{M}_D & 0 \\ 0 & \bar{L}_q & 0 & 0 & 0 & \bar{M}_Q \\ 0 & 0 & \bar{L}_0 & 0 & 0 & 0 \\ \bar{M}_F & 0 & 0 & \bar{L}_F & \bar{M}_R & 0 \\ \bar{M}_D & 0 & 0 & \bar{M}_R & \bar{L}_D & 0 \\ 0 & \bar{M}_Q & 0 & 0 & 0 & \bar{L}_Q \end{bmatrix} \begin{bmatrix} -\bar{i}_d \\ -\bar{i}_q \\ -\bar{i}_0 \\ \bar{i}_F \\ \bar{i}_D \\ \bar{i}_Q \end{bmatrix}$$

Step 2: Finding base values

$$\begin{bmatrix} \bar{\psi}_0 \\ \bar{\psi}_d \\ \bar{\psi}_q \\ \bar{\psi}_F \\ \bar{\psi}_D \\ \bar{\psi}_Q \end{bmatrix} = \begin{bmatrix} \bar{L}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{L}_l + \bar{L}_{ad} & 0 & \bar{L}_{ad} & \bar{L}_{ad} & 0 \\ 0 & 0 & \bar{L}_l + \bar{L}_{aq} & 0 & 0 & \bar{L}_{aq} \\ 0 & \bar{L}_{ad} & 0 & \bar{L}_F & \bar{M}_R & 0 \\ 0 & \bar{L}_{ad} & 0 & \bar{M}_R & \bar{L}_D & 0 \\ 0 & 0 & \bar{L}_{aq} & 0 & 0 & \bar{L}_Q \end{bmatrix} \begin{bmatrix} -\bar{i}_0 \\ -\bar{i}_d \\ -\bar{i}_q \\ \bar{i}_F \\ \bar{i}_D \\ \bar{i}_Q \end{bmatrix}$$

$$\bar{L}_d = \bar{L}_l + \bar{L}_{ad} \qquad \bar{L}_q = \bar{L}_l + \bar{L}_{aq}$$

$$L_{ad} = L_d - L_l \qquad \bar{L}_{ad} = \bar{M}_F = \bar{M}_D$$

$$\bar{L}_{ad} = \frac{L_{ad}}{L_{s\,base}} = \frac{M_F \times i_{F\,base}}{L_{s\,base} \times i_{s\,base}} = \frac{M_D \times i_{D\,base}}{L_{s\,base} \times i_{s\,base}}$$

$$L_{aq} = L_q - L_l \qquad \bar{L}_{aq} = \bar{M}_Q$$

$$\bar{L}_{aq} = \frac{L_{aq}}{L_{s\,base}} = \frac{M_Q \times i_{Q\,base}}{L_{s\,base} \times i_{s\,base}}$$

$i_{F\,base} \triangleq \frac{L_{ad}}{M_F} \times i_{S\,base}$, A

$e_{F\,base} \triangleq S_{3\phi\,base} / i_{F\,base}$, V

$Z_{F\,base} \triangleq e_{F\,base} / i_{F\,base}$

$= S_{3\phi\,base} / i^2_{F\,base}$, Ω

$L_{F\,base} \triangleq Z_{F\,base} / \omega_{base}$, H

$\psi_{F\,base} \triangleq L_{F\,base} \times i_{F\,base}$, Wb·turns

$M_{F\,base} = L_{S\,base} \times i_{S\,base} / i_{F\,base}$

$i_{D\,base} \triangleq \frac{L_{ad}}{M_D} \times i_{S\,base}$, A

$e_{D\,base} \triangleq S_{3\phi\,base} / i_{D\,base}$, V

$Z_{D\,base} \triangleq e_{D\,base} / i_{D\,base}$

$= S_{3\phi\,base} / i^2_{D\,base}$, Ω

$L_{D\,base} \triangleq Z_{D\,base} / \omega_{base}$, H

$\psi_{D\,base} \triangleq L_{D\,base} \times i_{D\,base}$, Wb·turns

$M_{D\,base} = L_{S\,base} \times i_{S\,base} / i_{D\,base}$

$i_{Q\,base} \triangleq \frac{L_{aq}}{M_Q} \times i_{S\,base}$, A

$e_{Q\,base} \triangleq S_{3\phi\,base} / i_{Q\,base}$, V

$Z_{Q\,base} = e_{Q\,base} / i_{Q\,base}$

$= S_{3\phi\,base} / i^2_{Q\,base}$, Ω

$L_{Q\,base} \triangleq Z_{Q\,base} / \omega_{base}$, H

$\psi_{Q\,base} \triangleq L_{Q\,base} \times i_{Q\,base}$, Wb·turns

$M_{Q\,base} = L_{S\,base} \times i_{S\,base} / i_{Q\,base}$

Formal Simplification of the Model

21 inductances
(many are θ-dependent)

P

14 constant inductances

L_{ad}-L_{aq} based P. U.

8 constant inductances

$L_d - L_l,\ M_F,\ M_D,\ \ldots \rightarrow L_{ad}$
$L_q - L_l,\ M_R,\ \ldots \rightarrow L_{aq}$

All mutual inductances in p.u. are equal to L_{ad} or L_{aq}, linked to the same element in an equivalent circuit.

$$\begin{bmatrix} e_a \\ e_b \\ e_c \\ e_F \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_a & 0 & 0 & 0 \\ 0 & 0 & 0 & R_F & 0 & 0 \\ 0 & 0 & 0 & 0 & R_D & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix} + \begin{bmatrix} l_{aa} & l_{ab} & l_{ac} & l_{aF} & l_{aD} & l_{aQ} \\ l_{ba} & l_{bb} & l_{bc} & l_{bF} & l_{bD} & l_{bQ} \\ l_{ca} & l_{cb} & l_{cc} & l_{cF} & l_{cD} & l_{cQ} \\ l_{Fa} & l_{Fb} & l_{Fc} & l_{FF} & l_{FD} & l_{FQ} \\ l_{Da} & l_{Db} & l_{Dc} & l_{DF} & l_{DD} & l_{DQ} \\ l_{Qa} & l_{Qb} & l_{Qc} & l_{QF} & l_{QD} & l_{QQ} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} -i_a \\ -i_b \\ -i_c \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ 0 & 0 & 0 & 0 & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_r \psi_q \\ 0 \\ 0 \\ \omega_r \psi_d \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ 0 & 0 & 0 & 0 & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_l + L_{ad} & L_{ad} & L_{ad} & 0 & 0 \\ 0 & L_{ad} & L_F & M_R & 0 & 0 \\ 0 & L_{ad} & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_l + L_{aq} & L_{aq} \\ 0 & 0 & 0 & 0 & L_{aq} & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_r \psi_q \\ 0 \\ 0 \\ \omega_r \psi_d \\ 0 \end{bmatrix}$$

Equivalent Circuits

Define differential operator $p=d/dt$

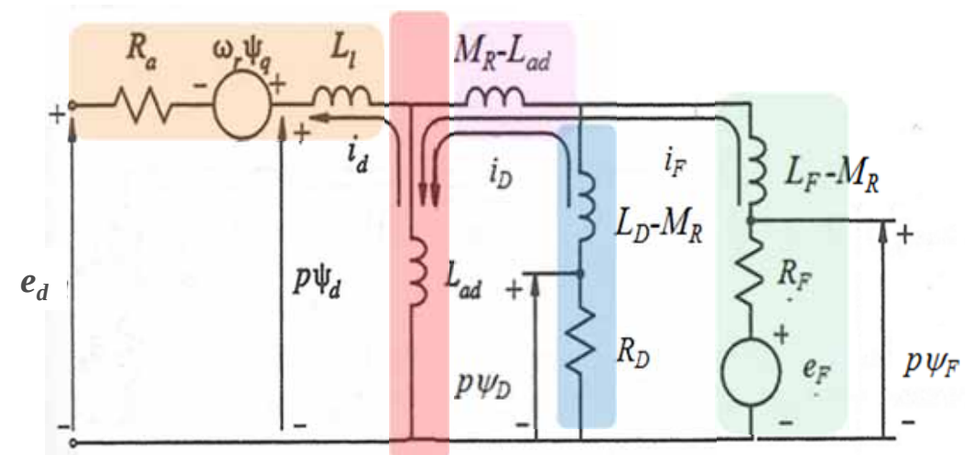
$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ e_D=0 \\ e_q \\ e_Q=0 \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & 0 & 0 \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ 0 & 0 & 0 & 0 & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_l+L_{ad} & L_{ad} & L_{ad} & 0 & 0 \\ 0 & L_{ad} & L_F & M_R & 0 & 0 \\ 0 & L_{ad} & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_l+L_{aq} & L_{aq} \\ 0 & 0 & 0 & 0 & L_{aq} & L_Q \end{bmatrix} p \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega_r\psi_q \\ 0 \\ 0 \\ \omega_r\psi_d \\ 0 \end{bmatrix}$$

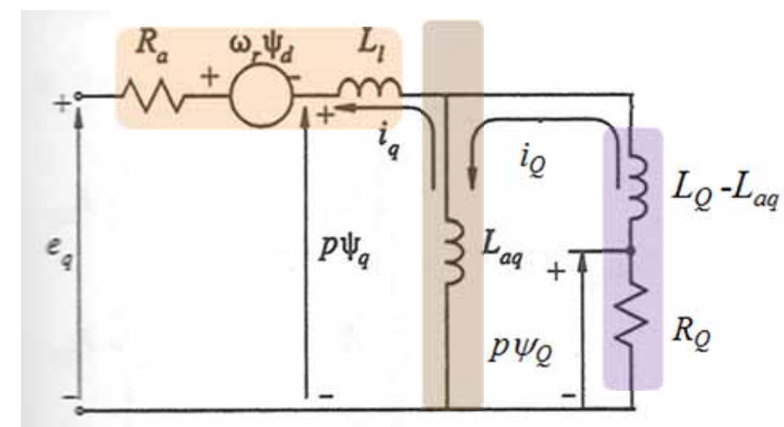
$$p \begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_F \\ \psi_D \\ \psi_q \\ \psi_Q \end{bmatrix} = \begin{bmatrix} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_l+L_{ad} & L_{ad} & L_{ad} & 0 & 0 \\ 0 & L_{ad} & L_F & M_R & 0 & 0 \\ 0 & L_{ad} & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_l+L_{aq} & L_{aq} \\ 0 & 0 & 0 & 0 & L_{aq} & L_Q \end{bmatrix} p \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}$$

$$\begin{aligned} e_d &= (L_l+L_{ad})\times(-pi_d) + L_{ad}\times pi_F + L_{ad}\times pi_D - \omega_r\psi_q - R_a i_d \\ &= L_{ad}\times p(-i_d+i_F+i_D) - L_l\times pi_d - \omega_r\psi_q - R_a i_d \\ p\psi_d &= (L_l+L_{ad})\times(-pi_d) + L_{ad}\times pi_F + L_{ad}\times pi_D = e_d + \omega_r\psi_q + R_a i_d \end{aligned}$$

$$\begin{aligned} e_F &= L_{ad}\times(-pi_d) + L_F\times pi_F + M_R\times pi_D + R_F i_F \\ &= L_{ad}\times p(-i_d+i_F+i_D) - L_{ad}\times p(i_D+i_F) + M_R\times pi_D + L_F\times pi_F + R_F i_F \\ &= L_{ad}\times p(-i_d+i_F+i_D) + (M_R-L_{ad})\times p(i_D+i_F) + (L_F - M_R)\times pi_F + R_F i_F \\ e_D &= 0 = L_{ad}\times(-pi_d) + M_R\times pi_F + L_D\times pi_D + R_D i_D \\ &= L_{ad}\times p(-i_d+i_F+i_D) + (M_R-L_{ad})\times p(i_D+i_F) + (L_D - M_R)\times pi_D + R_D i_D \end{aligned}$$

$$\begin{aligned} e_q &= (L_l+L_{aq})\times(-pi_q) + L_{aq}\times pi_Q + \omega_r\psi_d - R_a i_q \\ &= L_{aq}\times p(-i_q+i_Q) - L_l\times pi_q + \omega_r\psi_d - R_a i_q \\ p\psi_q &= (L_l+L_{aq})\times(-pi_q) + L_{aq}\times pi_Q = e_q - \omega_r\psi_d + R_a i_q \\ e_Q &= 0 = L_{aq}\times(-pi_q) + L_Q\times pi_Q + R_Q i_Q \\ &= L_{aq}\times p(-i_q+i_Q) + (L_Q-L_{aq})\times pi_Q + R_Q i_Q \end{aligned}$$





Equivalent Circuits with Multiple Damper Windings (e.g. Round-rotor Machines)

- d-axis:

$L_{fd} \triangleq L_F - M_R$ $\quad R_{fd} \triangleq R_F$

$\psi_{fd} \triangleq \psi_F$ $\quad e_{fd} \triangleq e_F$

$L_{1d} \triangleq L_D - M_R$ $\quad R_{1d} \triangleq R_D$

$\psi_{1d} \triangleq \psi_D$

M_R-$L_{ad} \approx 0$ is named L_{fkd1} in some literature to model rotor mutual flux leakage, i.e. the flux linking the rotor's field and damper windings but not stator windings)

- q-axis:

$L_{1q} \triangleq L_Q - L_{aq}$ $\quad R_{1q} \triangleq R_Q$

$\psi_{1q} \triangleq \psi_Q$

$L_{2q} \triangleq L_G - L_{aq}$ $\quad R_{2q} \triangleq R_G$

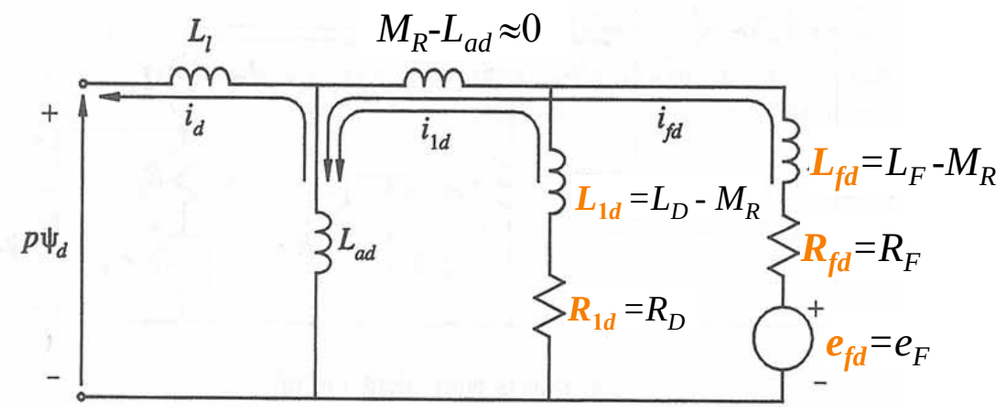
$\psi_{2q} \triangleq \psi_G$

Subscript Notations:

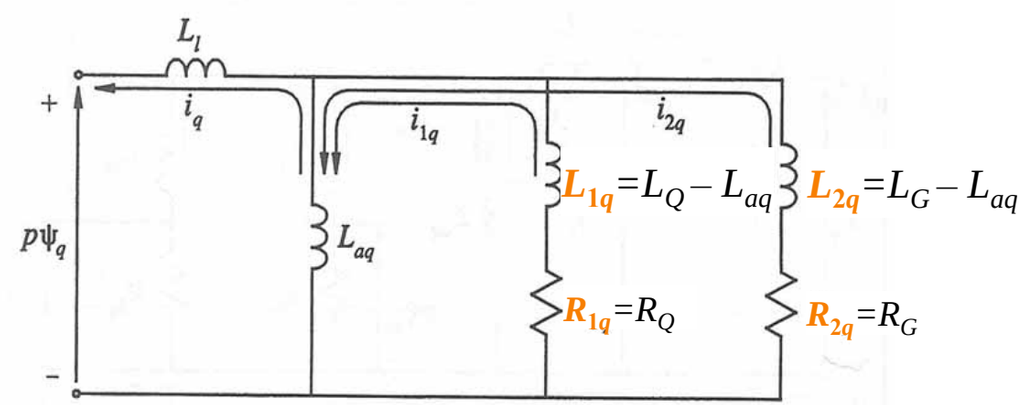
$(\ )_{fd}$ ~ field winding quantities

$(\ )_{kd}$ ~ k-th d-axis damper winding quantities

$(\ )_{kq}$ ~ k-th q-axis damper winding quantities



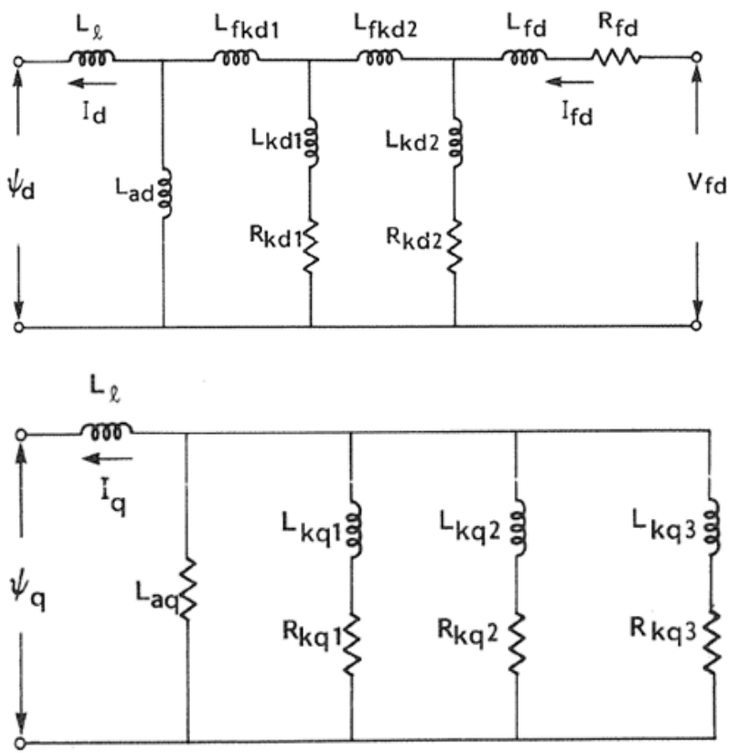
(a) d-axis equivalent circuit



(b) q-axis equivalent circuit

Example: a model with 3 rotor windings in each of d- and q-axis equivalent circuits

- EPRI Report EL-1424-V2, “Determination of Synchronous Machine Stability Study Constants, Volume 2”, 1980
  - The proposed equivalent circuits are expected to contain sufficient details to model all machines
  - Parameters are estimated by frequency response tests



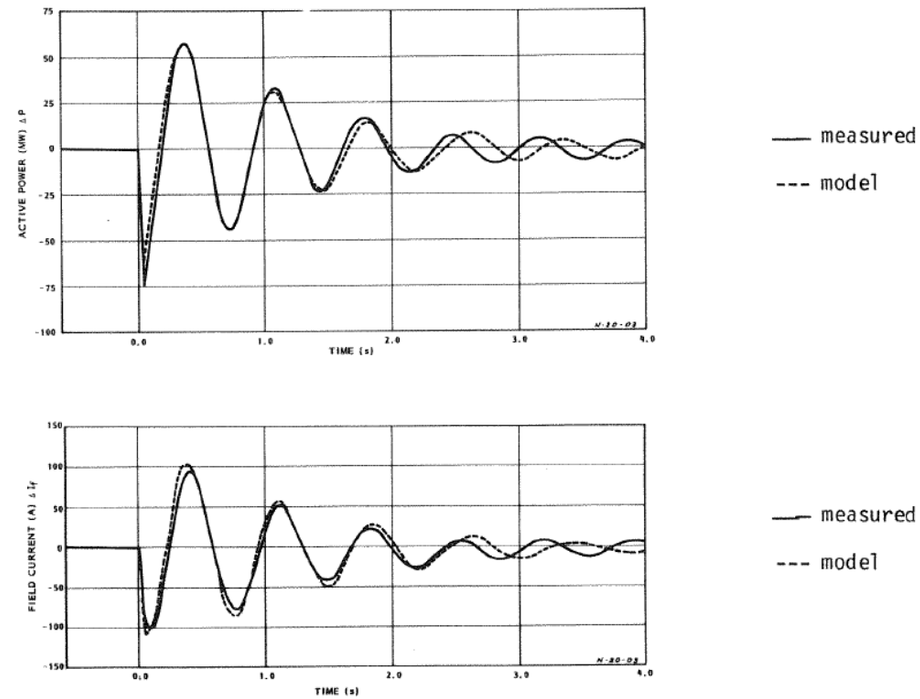


Figure S-2. Simulation of Nanticoke Generator Power and Field Current During a Transient Caused By Line Switching.

Example 3.1 (Kundur's book)

A 555 MVA, 24kV, 0.9p.f., 60Hz, 3 phase, 2 pole synchronous generator has the following inductances an resistances associated with the stator and field windings:

l_{aa}=3.2758+0.0458 cos(2θ)	mH	l_{ab}=-1.6379-0.0458 cos(2θ+π/3)	mH
l_{aF}=40.0 cosθ	mH	L_F=576.92	mH
R_a=0.0031	Ω	R_F=0.0715	Ω

a. Determine L_d and L_q in H

b. If the stator leakage inductance L_l is 0.4129mH, determine L_{ad} and L_{aq} in H

c. Using the machine rated values as the base values for the stator quantities, determine the per unit values of the following in the L_{ad} base reciprocal per unit system (assuming L_{ad}=M_F=M_R in per unit): L_l, L_{ad}, L_{aq}, L_d, L_q, M_F, L_F, L_{fd}, R_a and R_F

Solution:

a. $l_{aa} = L_s + L_m \cos 2\theta$ =3.2758+0.0458 cos2θ mH

$l_{ab} = -M_s - L_m \cos(2\theta+\pi/3)$ = -1.6379-0.0458 cos(2θ+π/3) mH

$l_{aF} = M_F \cos\theta$ =40.0 cosθ mH

$L_d = L_s + M_s + 3L_m/2$ $=3.2758+1.6379+\frac{3}{2}\times 0.0458$ =4.9825 mH

$L_q = L_s + M_s - 3L_m/2$ $=3.2758- 1.6379+\frac{3}{2}\times 0.0458$ =4.8451 mH

b. $L_{ad} = L_d - L_l$ =4.9825 – 0.4129 =4.5696 mH

$L_{aq} = L_q - L_l$ =4. 4851 – 0.4129 =4.432 mH

c. 3-phase VA_{base} = 555 MVA

$E_{RMS\ base}$	$=24/\sqrt{3}$		$=13.856$ kV	
$e_{s\ base}$ (peak)	$=\sqrt{2}\times E_{RMS\ base}=\sqrt{2}\times 13.856$		$=19.596$ kV	
$I_{RMS\ base}$	=3-phaseVAbase / $(3\times E_{RMS\ base})=555\times10^6/(3\times13.856\times10^3)=13.3512\times10^3$A			
$i_{s\ base}$ (peak)	$=\sqrt{2}\times I_{RMS\ base}=\sqrt{2}\times13.3512\times10^3$		$=18.8815\times10^3$A	
$Z_{s\ base}$	$=e_{s\ base}/i_{s\ base}$	$=19.596\times10^3/(18.8815\times10^3)$	$=1.03784\ \Omega$	
ω_{base}	$=2\pi\times60$	=377 elec. rad/s		
$L_{s\ base}$	$=Z_{s\ base}/\omega_{base}$	$=1.03784/377\times10^3$	=2.753	mH
$i_{F\ base}$	$=L_{ad}/M_F\times i_{s\ base}$	$=4.5696/40\times18.8815\times10^3$	=2158.0	A
$e_{F\ base}$	=3-phaseVAbase/i_{Fbase}	$= 555\times10^6/2158$	=257.183	kV
$Z_{F\ base}$	$= e_{F\ base} / i_{F\ base}$	$=257.183\times10^3/2158$	=119.18	Ω
$L_{F\ base}$	$= Z_{F\ base} / \omega_{base}$	$=119.18\times10^3/377$	=316.12	mH
$M_{F\ base}$	$= L_{S\ base} \times i_{S\ base} / i_{F\ base}$	=2.753×188815/2158	=241 mH	

Then per unit values are:

$L_l = L_l/L_{s\ base}$ =0.4129/2.753	=0.15 pu	$R_F=R_F/Z_{F\ base}$=0.0715/119.18	=0.0006 pu
L_{ad}=4.5696/2.753	=1.66 pu	$M_F=M_F/M_{F\ base}$=400/241	=1.66 pu
L_{aq}=4.432/2.753	=1.61pu	$L_F=L_F/L_{F\ base}$=576.92/316.12	=1.825 pu
$L_d= L_l + L_{ad}$=0.15+1.66	=1.81 pu	$L_{fd} = L_F-M_R =L_F - L_{ad}$=1.825 - 1.66	=0.165 pu
$L_q= L_l + L_{aq}$=0.15+1.61	=1.76 pu		
R_a=0.0031/1.03784	=0.003 pu		

Steady-state Analysis

- All derivatives (pX) are zero:

$p\omega_r=0 \rightarrow \omega_r=1$ and $L=X$ in p.u.

$p\psi_{fd}=0 \rightarrow e_{fd}=R_{fd}i_{fd}$

$p\psi_{1d}=0 \rightarrow i_{1d}=0 \qquad \psi_d=-L_d i_d+L_{ad}i_{fd}$

$p\psi_{1q}=0 \rightarrow i_{1q}=0 \qquad \psi_q=-L_q i_q$

$p\psi_d=0 \rightarrow e_d=-\psi_q-R_a i_d \Rightarrow e_d=L_q i_q-R_a i_d$

$p\psi_q=0 \rightarrow e_q=\psi_d-R_a i_q \qquad e_q=-L_d i_d+L_{ad}i_{fd}-R_a i_q$

Voltage and flux equations: 6+6 → 3+3

$e_{fd}=R_{fd}i_{fd}$	$\psi_{fd}=(L_{ad}+L_{fd})i_{fd}-L_{ad}i_d$
$e_d=X_q i_q-R_a i_d$	$\psi_{1d}=L_{ad}(i_{fd}-i_d)$
$e_q=-X_d i_d+X_{ad}i_{fd}-R_a i_q$	$\psi_{1q}=\psi_{2q}=-L_{aq}i_q$

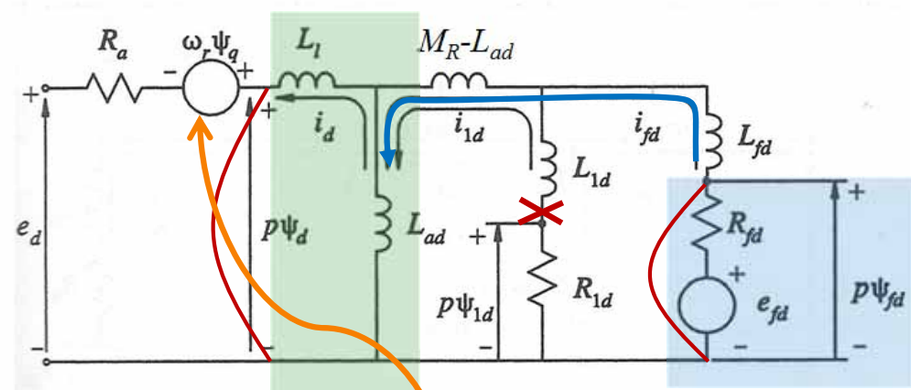
- Single equivalent circuit for both d and q axes:

$\tilde{I}_t=i_d+ji_q$

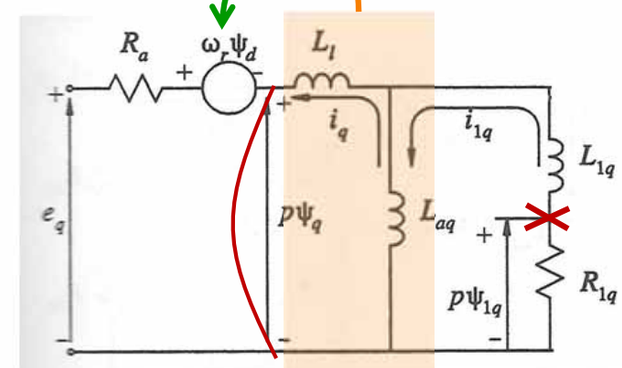
$\tilde{E}_t=e_d+je_q=X_q i_q-R_a i_d-jX_d i_d+jX_{ad}i_{fd}-jR_a i_q$

$=-R_a(i_d+ji_q)-jX_q\times ji_q-jX_d i_d+jX_{ad}i_{fd}$

$\tilde{E}_t=\tilde{E}_q-(R_a+jX_q)\tilde{I}_t$ where $\tilde{E}_q=j[X_{ad}i_{fd}-(X_d-X_q)i_d]$

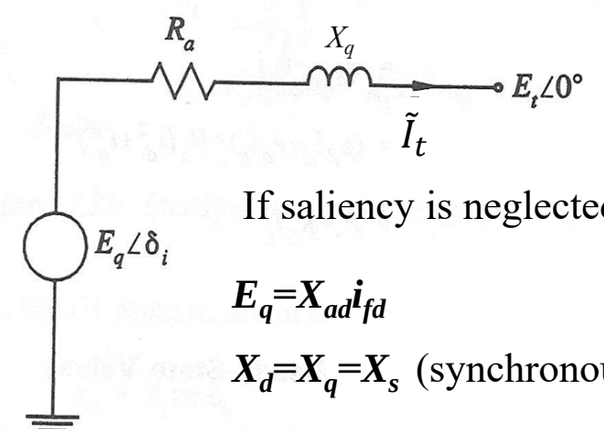


(a) d-axis equivalent circuit



(b) q-axis equivalent circuit

Under a steady-state condition, damper winding currents are all zero.



If saliency is neglected:

$E_q=X_{ad}i_{fd}$

$X_d=X_q=X_s$ (synchronous reactance)

Computing per-unit steady-state values

$$\omega_r = 1\text{p.u.}$$

- Active and Reactive Powers

$$S = \tilde{E}_t \tilde{I}_t^*$$
$$= (e_d + \mathrm{j}e_q)(i_d - \mathrm{j}i_q)$$
$$= (e_d i_d + e_q i_q) + \mathrm{j}(e_q i_d - e_d i_q)$$

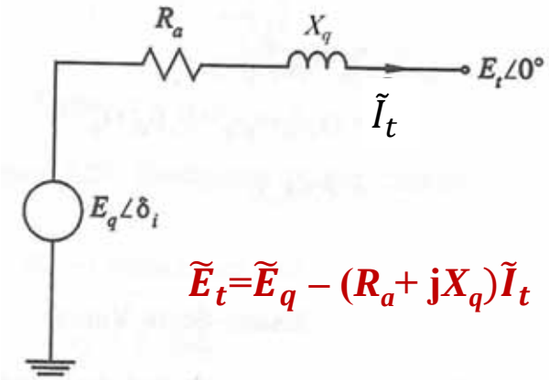
where $e_d = -\omega_r \psi_q - R_a i_d$

$$e_q = \omega_r \psi_d - R_a i_q$$

$$P_t = e_d i_d + e_q i_q$$
$$= \omega_r(\psi_d i_q - \psi_q i_d) - R_a(i_d^2 + i_q^2)$$
$$= P_e - R_a(i_d^2 + i_q^2)$$
$$Q_t = e_q i_d - e_d i_q$$

- Air-gap torque (or electric torque)

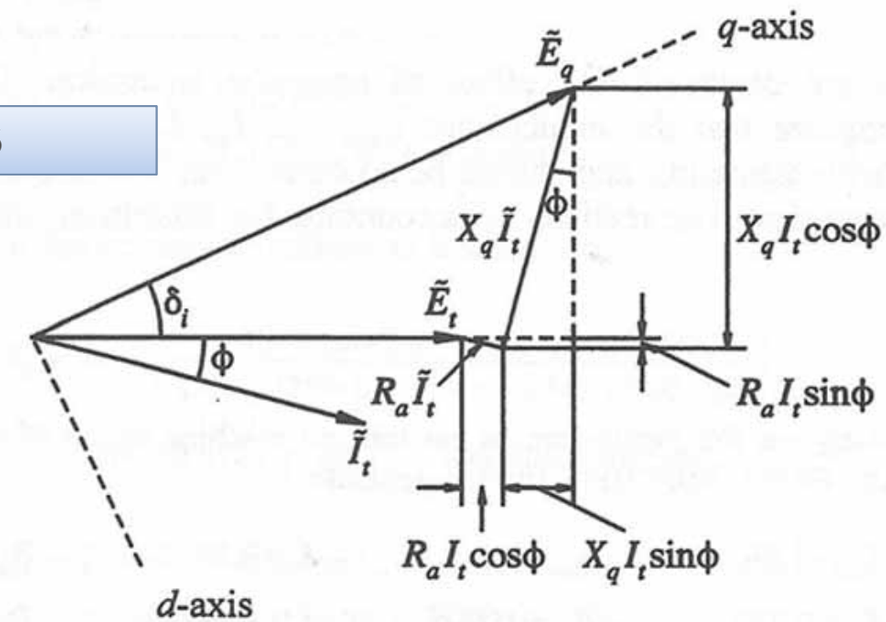
$$T_e = \frac{P_e}{\omega_r} = \psi_d i_q - \psi_q i_d$$
$$= P_t + R_a(i_d^2 + i_q^2)$$



$$\tilde{E}_t = \tilde{E}_q - (R_a + \mathrm{j}X_q)\tilde{I}_t$$

- Rotor angle δ

$$\delta = \delta_i + \angle\tilde{E}_t$$



$$I_t = \frac{\sqrt{P_t^2 + Q_t^2}}{E_t} \qquad \phi = \cos^{-1}\left(\frac{P_t}{E_t I_t}\right)$$

$$\boxed{\delta_i = \tan^{-1}\left(\frac{X_q I_t\cos\phi - R_a I_t\sin\phi}{E_t + R_a I_t\cos\phi + X_q I_t\sin\phi}\right)}$$

Representation of Magnetic Saturation

- Assumptions for stability studies
  - The leakage fluxes are not significantly affected by saturation of the iron portion, so L_l is constant and only L_{ad} and L_{aq} saturate in equivalent circuits
  - There is no magnetic coupling between d and q axes,so saturations on L_{ad} and L_{aq} can be modeled individually. Let L_{adu} and L_{aqu} denote their unsaturated values:

$$L_{ad}(i_{fd})=L_{adu},\quad L_{aq}(i_{fd})=L_{aqu} \text{ only if } i_{fd} \text{ or } MMF \text{ is not large}$$

  - The leakage fluxes do not contribute to the iron saturation. Thus, saturation is determined only by the air-gap flux linkage

$$\psi_d = -(L_l+L_{ad})i_d+L_{ad}i_{fd} \qquad \psi_q = -(L_l+L_{aq})i_q$$

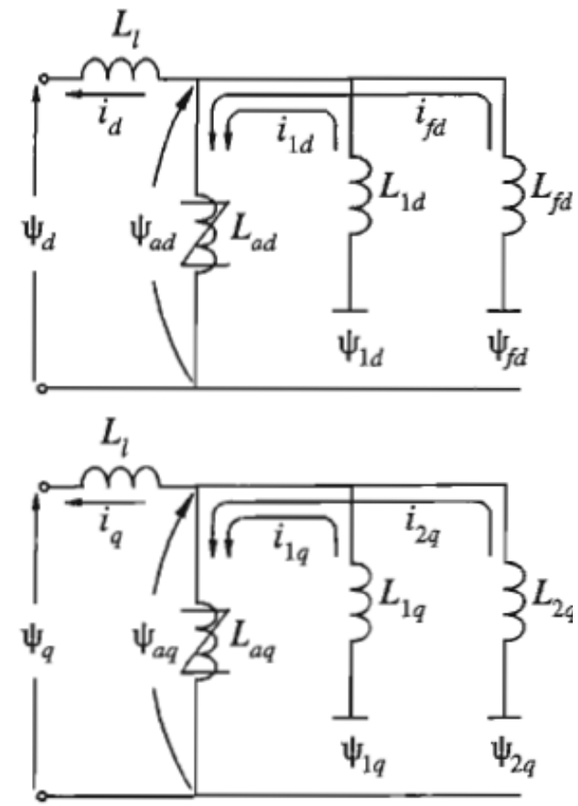
$$\tilde{\psi}_{at} = \psi_{ad} + j\psi_{aq}$$
$$= \psi_d + L_l i_d + j\psi_q + jL_l i_q$$
$$= L_{ad}(i_{fd} - i_d) - jL_{aq} i_q$$

$$\psi_{at} = \sqrt{\psi_{ad}^2 + \psi_{aq}^2}$$

First find ψ_{at} vs. i_{fd} and then estimate the function $L_{ad}(i_{fd})$ or $L_{aq}(i_{fd})$

  - Saturation relationship ψ_{at} vs. i_{fd} (or MMF) under loaded conditions is assumed to be the same as under no-load conditions (i_d=i_q=0), so only the open-circuit characteristic (OCC) is considered.

$$\tilde{\psi}_{at} = \psi_{at} = L_{ad} i_{fd}$$

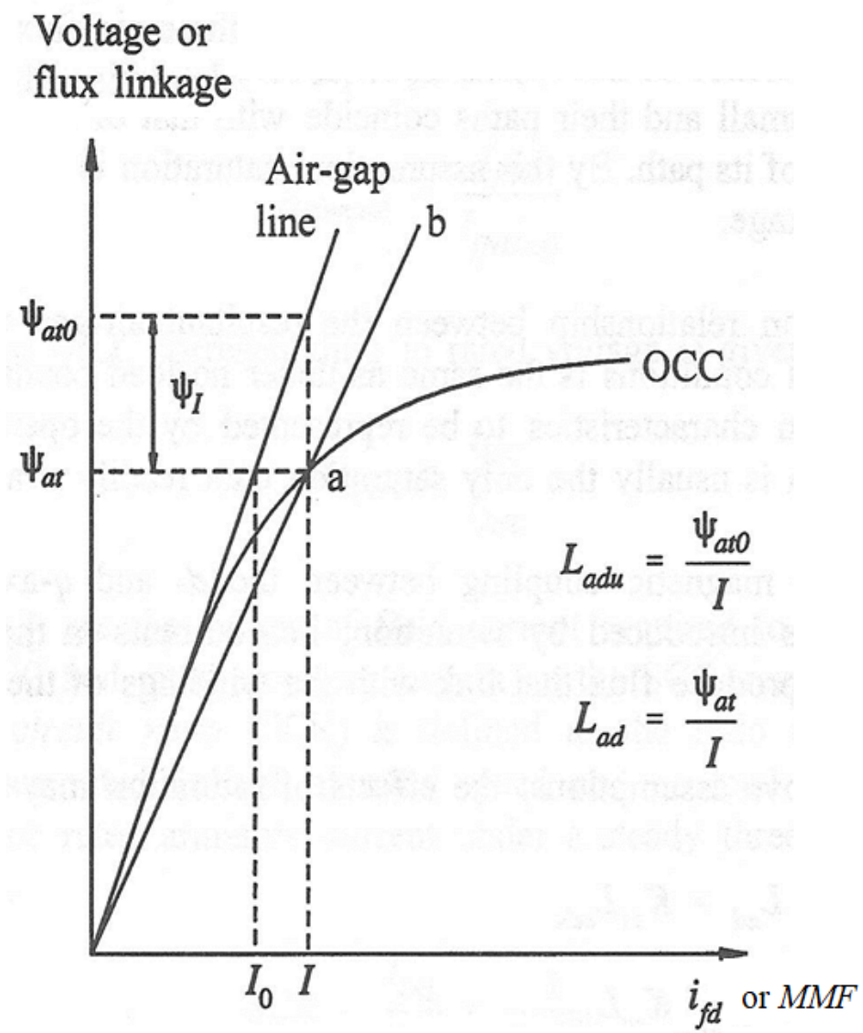


Estimating Saturation Factors K_{sd} and K_{sd}

$L_{ad}=K_{sd}L_{adu}$ $\quad$ $L_{aq}=K_{sq}L_{aqu}$

- Salient pole machines
  - The path for q-axis flux is largely in air, so L_{aq} does not vary significantly with saturation of the iron portion of the path
  - Assume $K_{sq}=1.0$ for all loading conditions.
- Round rotor machines
  - There is a magnetic saturation in both axes, but the saturation data in q axis is usually not available
  - Assume $K_{sq}=K_{sd}$
- Thus, we focus on estimating K_{sd}

$$K_{sd} = L_{ad}/L_{adu}$$
$$= \psi_{at}/\psi_{at0} = \psi_{at}/(\psi_{at}+\psi_I)$$
$$= I_0 / I$$



(See Kundur's Example 3.3 on Estimating K_{sd} for different loading conditions)

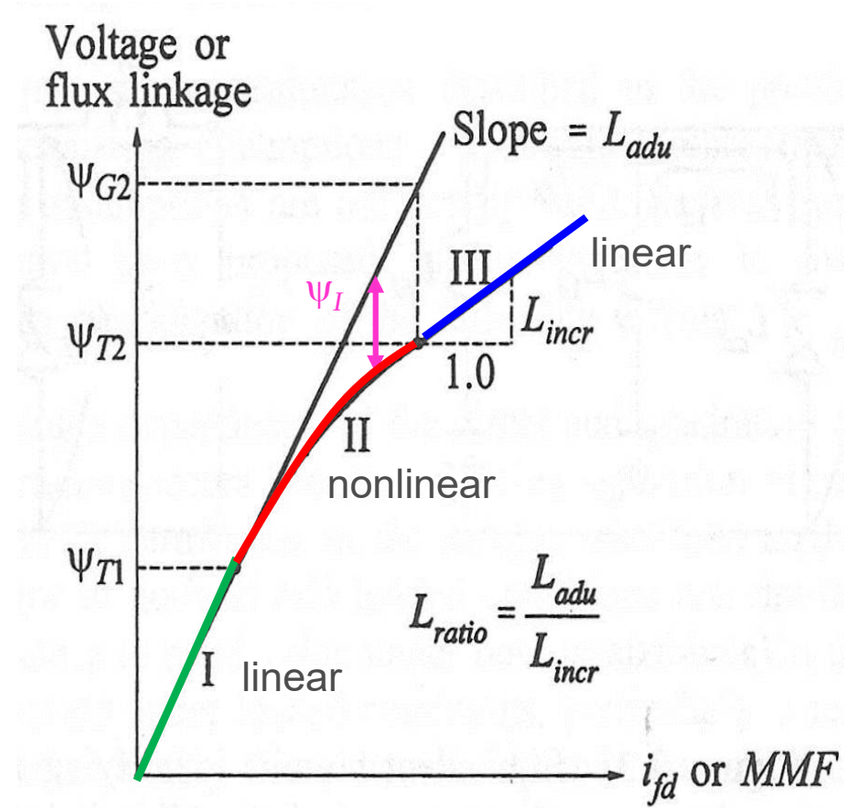
Modeling of the Saturation Characteristic

- ψ_I is modeled by 3 approximate functions
  - **Segment I** ($\psi_{at}<\psi_{T1}$): $\psi_I=\psi_{at0}-\psi_{at}=0$
  - **Segment II** ($\psi_{T1}<\psi_{at}<\psi_{T2}$): $\psi_I = A_{sat}e^{B_{sat}(\psi_{at}-\psi_{T1})}$
  - **Segment III** ($\psi_{at}>\psi_{T2}$): $\psi_I=\psi_{G2}+L_{ratio}(\psi_{at}-\psi_{T2})-\psi_{at}$

- Note: segments I and II are not connected since when $\psi_{at}=\psi_{T1}$, $\psi_I=A_{sat}\neq 0$ (usually small)
- Segments II and III are assumed to be connected at $\psi_{at}=\psi_{T2}$ to solve ψ_{G2}

$$A_{sat}e^{B_{sat}(\psi_{T2}-\psi_{T1})}=\psi_{G2}+L_{ratio}(\psi_{T2}-\psi_{T2})-\psi_{T2}$$

$$\psi_{G2}=\psi_{T2}+A_{sat}e^{B_{sat}(\psi_{T2}-\psi_{T1})}$$



Five independent parameters:
A_{sat}, B_{sat}, ψ_{T1}, ψ_{T2} and L_{ratio}

Synchronous Machine Model DG1S1

Model Descriptions

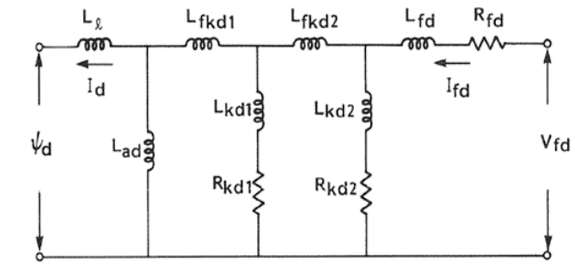
This model uses parameters in basic form and type 1 saturation model.

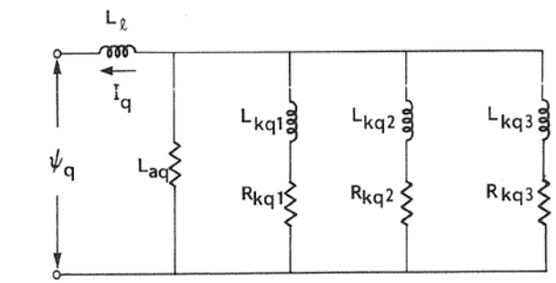
Data Format

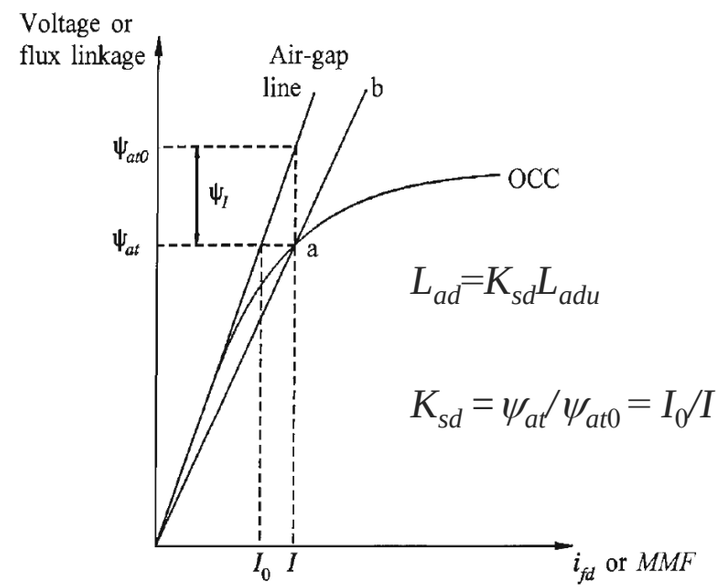
IBUS, 'DG1S1', I, MVA, X_{ad}, X_{aq}, X_l, R_a, X_{fd}, R_{fd}, X_{kq1}, R_{kq1}, X_{kd1}, R_{kd1}, X_{kq2}, R_{kq2}, X_{kd2}, R_{kd2}, X_{kq3}, R_{kq3}, H, K_D, α, A_{sat}, B_{sat}, ψ_L, ψ_M, RS /

Parameter Descriptions

IBUS	-	Bus number, name, or generator equipment name of the machine.
I	-	ID of the machine (may or may not be enclosed in single quotes).
MVA	-	MVA base of the machine. If not specified (i.e., no value or zero is entered), the MVA base of the matched generator in powerflow data will be used.
X_{ad}	-	Unsaturated direct axis mutual reactance in per unit on machine MVA base.
X_{aq}	-	Unsaturated quadrature axis mutual reactance in per unit on machine MVA base.
X_l	-	Leakage reactance in per unit on machine MVA base.
R_a	-	Armature resistance in per unit on machine MVA base.
X_{fd}	-	Field winding leakage reactance in per unit on machine MVA base.
R_{fd}	-	Field winding resistance in per unit on machines MVA base.
X_{kq1}	-	First quadrature axis damper winding leakage reactance in per unit on machine MVA base.
R_{kq1}	-	First quadrature axis damper winding resistance in per unit on machine MVA base.
X_{kd1}	-	First direct axis damper winding leakage reactance in per unit on machine MVA base.
R_{kd1}	-	First direct axis damper winding resistance in per unit on machine MVA base.
X_{kq2}	-	Second quadrature axis damper winding leakage reactance in per unit on machine MVA base.
R_{kq2}	-	Second quadrature axis damper winding resistance in per unit on machine MVA base.
X_{kd2}	-	Second direct axis damper winding leakage reactance in per unit on machine MVA base.
R_{kd2}	-	Second direct axis damper winding resistance in per unit on machine MVA base.
X_{kq3}	-	Third quadrature axis damper winding leakage reactance in per unit on machine MVA base.
R_{kq3}	-	Third quadrature axis damper winding resistance in per unit on machine MVA base.
H	-	Inertia time constant of the machine in MW-second/MVA.
K_D	-	Damping coefficient in (p.u. torque)/(p.u. speed deviation).
α	-	This parameter is used only for synchronous motor, as the exponential in the load characteristic of the motor: $T_m = K\omega^{\alpha}$ (K is determined by TSAT based on the initial condition). It is ignored for generator model.
A_{sat}	-	Coefficient in saturation characteristic.
B_{sat}	-	Coefficient in saturation characteristic.
ψ_L	-	Flux linkage on the saturation curve where the Region II characteristic starts.
ψ_M	-	Flux linkage on the saturation curve where the Region III characteristic starts.
RS	-	Ratio of the slopes of air-gap line and the Region III characteristic.







Here $\psi_L = \psi_{T1}$, $\psi_M = \psi_{T2}$ and RS=L_{ratio}

1.20 GENSAE

Salient Pole Generator Model (Exponential Saturation on Both A

This model is located at system bus #_____ IBUS,

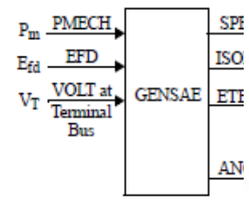
Machine identifier #_____ ID,

This model uses CONs starting with #_____ J,

and STATEs starting with #_____ K.

The machine MVA is _____ for each of _____ units = _____ MBASE.

ZSORCE for this machine is _____ + j _____ on the above MBASE.



CONs	#	Value	Description
J			T'_{do} (>0) (sec)
J+1			T''_{do} (>0) (sec)
J+2			T''_{qo} (>0) (sec)
J+3			H, Inertia
J+4			D, Speed damping
J+5			X_d
J+6			X_q
J+7			X'_d
J+8			$X''_d = X''_q$
J+9			X_l
J+10			S(1.0)
J+11			S(1.2)

Note: X_d, X_q, X'_d, X''_d, X''_q, X_l, H, and D are in pu, machine MVA base.

X''_q must be equal to X''_d.

STATEs	#	Description
K		E'_q
K+1		ψ''_q
K+2		ψkd
K+3		Δ speed (pu)
K+4		Angle (radians)

IBUS, 'GENSAE', ID, CON(J) to CON(J+11) /

From Anderson's book

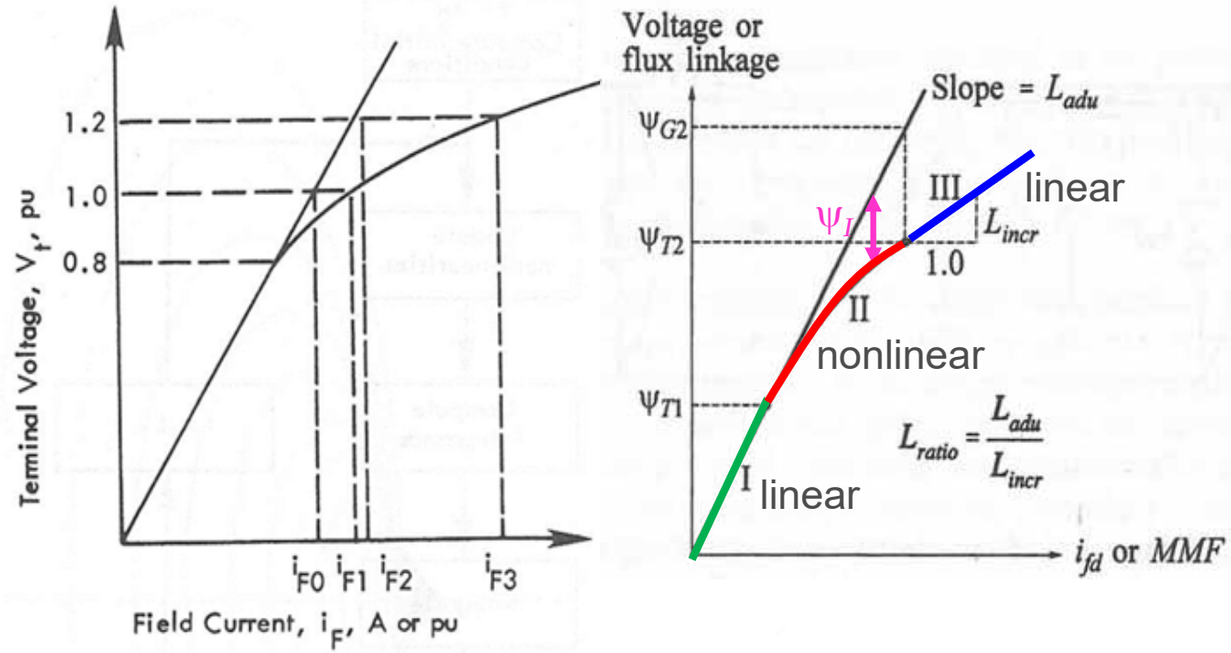


Fig. 5.25 Estimating saturation as an exponential function.

S(1.0)= $$S_{G1} = \frac{i_{F1} - i_{F0}}{i_{F0}}$$

S(1.2)= $$S_{G2} = \frac{i_{F3} - i_{F2}}{i_{F2}} = \frac{i_{F3} - 1.2 i_{F0}}{1.2 i_{F0}}$$

$$S_G V_t = A_G e^{B_G (V_t - 0.8)}$$

$$\psi_I = A_{sat} e^{B_{sat}(\psi_{at} - \psi_{T1})}$$

$$A_G = S_{G1}^2 / 1.2 S_{G2}$$

$A_{sat}=A_G$, $B_{sat}=B_G$, $\psi_{T1}=0.8$, $\psi_{T2}=1.2$

$$B_G = 5 \ln (1.2 S_{G2} / S_{G1})$$

$L_{ratio}=L_{ad}(i_{F3})/L_{adu}$

Example 3.2 in Kundur's Book

The following are the parameters in per unit on machine rating of a 555 MVA, 24 kV, 0.9 p.f., 60 Hz, 3600 RPM turbine-generator[1]:

$L_{ad}=1.66$	$L_{aq}=1.61$	$L_l=0.15$	$R_a=0.003$
$L_{fd}=0.165$	$R_{fd}=0.0006$	$L_{1d}=0.1713$	$R_{1d}=0.0284$
$L_{1q}=0.7252$	$R_{1q}=0.00619$	$L_{2q}=0.125$	$R_{2q}=0.02368$

M_R is assumed to be equal to L_{ad}.

(a) When the generator is delivering rated MVA at 0.9 p.f. (lag) and rated terminal voltage, compute the following:

(i) Internal angle δ_i in electrical degrees

(ii) Per unit values of e_d, e_q, i_d, i_q, i_{1d}, i_{1q}, i_{2q}, i_{fd}, e_{fd}, ψ_{fd}, ψ_{1d}, ψ_{1q}, ψ_{2q}

(iii) Air-gap torque T_e in per unit and in newton-meters

Assume that the effect of magnetic saturation at the given operating condition is to reduce L_{ad} and L_{aq} to 83.5% of the values given above.

(b) Compute the internal angle δ_i and field current i_{fd} for the above operating condition, using the approximate equivalent circuit of Figure 3.22. Neglect R_a.

Solution

(a) With the given operating condition, the per unit values of terminal quantities are

$P=0.9$, $Q=0.436$, $E_t=1.0$, $I_t=1.0$, $\phi=25.84°$

$R_{1d} >> R_{fd}$

$L_{fd}/R_{fd} >> L_{1d}/R_{1d}$

$L_{1q}/R_{1q} >> L_{2q}/R_{2q}$

$$\delta_i = \tan^{-1}\left(\frac{X_q I_t \cos\phi - R_a I_t \sin\phi}{E_t + R_a I_t \cos\phi + X_q I_t \sin\phi}\right)$$

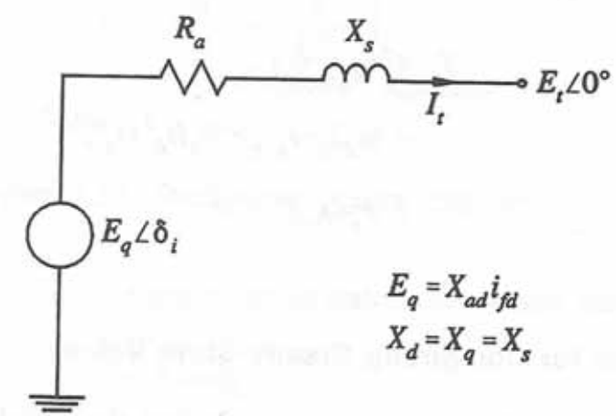


Figure 3.22

The saturated values of the inductances are

$$L_{ad} = 0.835\times1.66 = 1.386$$

$$L_{aq} = 0.835\times1.61 = 1.344$$

$$L_d = L_{ad}+L_l = 1.386+0.15 = 1.536$$

$$L_q = L_{aq}+L_l = 1.344+0.15 = 1.494$$

Following the procedure outlined in Section 3.6.5,

(i) $$\delta_i = \tan^{-1}\left(\frac{1.494\times1.0\times0.9-0.003\times1.0\times0.436}{1.0+0.003\times1.0\times0.9+1.494\times1.0\times0.436}\right)$$

$$= \tan^{-1}(0.812) = 39.1 \quad \text{electrical degrees}$$

(ii) $$e_d = E_t\sin\delta_i = 1.0\sin39.1 = 0.631 \quad \text{pu}$$

$$e_q = E_t\cos\delta_i = 1.0\cos39.1 = 0.776 \quad \text{pu}$$

$$i_d = I_t\sin(\delta_i+\phi) = 1.0\sin(39.1+25.84) = 0.906 \quad \text{pu}$$

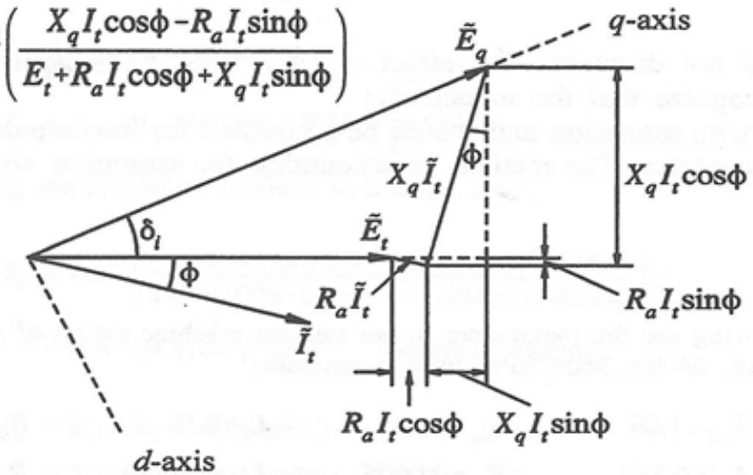
$$i_q = I_t\cos(\delta_i+\phi) = 1.0\cos(39.1+25.84) = 0.423 \quad \text{pu}$$

$$i_{fd} = \frac{e_q+R_a i_q+X_d i_d}{X_{ad}}$$

$$= \frac{0.776+0.003\times0.423+1.536\times0.906}{1.386}$$

$$= 1.565 \quad \text{pu}$$

$$\delta_i = \tan^{-1}\left(\frac{X_q I_t\cos\phi-R_a I_t\sin\phi}{E_t+R_a I_t\cos\phi+X_q I_t\sin\phi}\right)$$



$$e_{fd} = R_{fd}i_{fd} = 0.0006\times1.565$$

$$= 0.000939 \quad \text{pu}$$

$$\psi_{fd} = (L_{ad}+L_{fd})i_{fd}-L_{ad}i_d$$

$$= (1.386+0.165)\times1.565-1.386\times0.907$$

$$= 1.17 \quad \text{pu}$$

$$\psi_{1d} = L_{ad}(i_{fd}-i_d)$$

$$= 1.386\times(1.565-0.906)$$

$$= 0.913 \quad \text{pu}$$

$$\psi_{1q} = \psi_{2q} = -L_{aq}i_q = -1.344\times0.423$$

$$= -0.569 \quad \text{pu}$$

Under steady state,

$$i_{1d} = i_{1q} = i_{2q} = 0$$

(iii) Air-gap torque

$$T_e = P_t + I_t^2 R_a$$

$$= 0.9 + 1.0^2 \times 0.003$$

$$= 0.903 \quad \text{pu}$$

$$T_{base} = \frac{\text{MVA}_{\text{base}} \times 10^6}{\omega_{m\,base}}$$

$$= \frac{555 \times 10^6}{2\pi 60} = 1.472 \times 10^6 \quad \text{N·m}$$

Therefore,

$$T_e = 0.903 \times 1.472 \times 10^6$$

$$= 1.329 \times 10^6 \quad \text{N·m}$$

(b) Using the saturated value of X_{ad},

$$E_q = X_{ad} i_{fd} = 1.386 i_{fd}$$

$$X_s = X_{ad} + X_l = 1.386 + 0.15 = 1.536$$

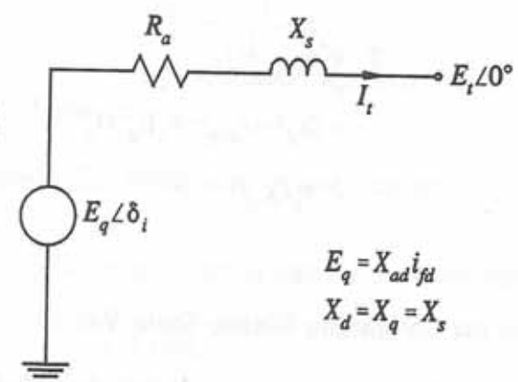
From the equivalent circuit of Figure 3.22, with $\tilde{E}_t$ as reference phasor,

$$\tilde{E}_q = \tilde{E}_t + jX_s \tilde{I}_t$$

$$= 1.0 + j1.536(0.9 - j0.43($$

$$= 1.670 + j1.382$$

$$= 2.17\angle 39.6° \quad \text{pu}$$



$$\delta_i = 39.6° \quad \approx 39.1°$$

Therefore,

$$i_{fd} = \frac{E_q}{X_{ad}} = \frac{2.17}{1.386} = 1.566 \quad \text{pu} \quad \approx 1.565\text{pu}$$

Sub-transient and Transient Analysis

- Following a disturbance, currents are induced in rotor circuits. Some of these induced rotor currents decay more rapidly than others.
  - Sub-transient parameters: influencing rapidly decaying (cycles) components
  - Transient parameters: influencing the slowly decaying (seconds) components
  - Synchronous parameters: influencing sustained (steady state) components

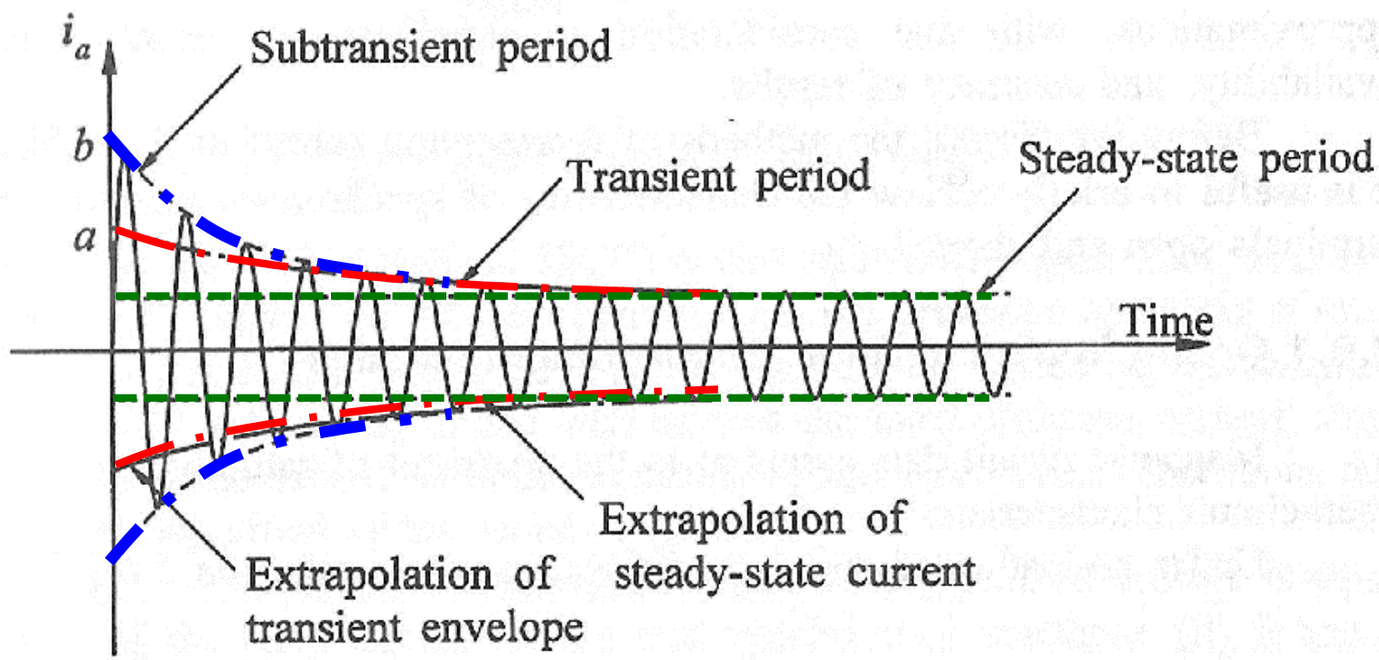


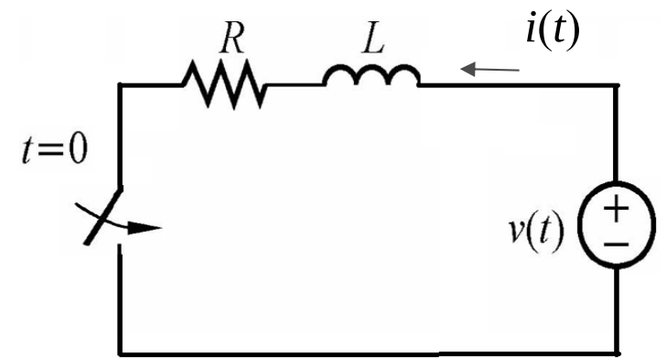
Figure 3.27 Fundamental frequency component of armature current

Transient Phenomena

- Study transient behavior of a simple RL circuit

$$v(t) = V_m \sin(\omega t + \alpha) \cdot u(t)$$ ← Unit step function

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$



- Apply Laplace Transform

$$RI(s) + L[sI(s) - i(0)] = V(s)$$

$$I(s) = \frac{V(s)}{R} \cdot \frac{1}{1+s\tau} = \frac{\Psi(s)}{R/s} \cdot \frac{1}{1+s\tau} \quad \text{where } \tau = \frac{L}{R}$$

(time to decay to $1/e \approx 37\%$)

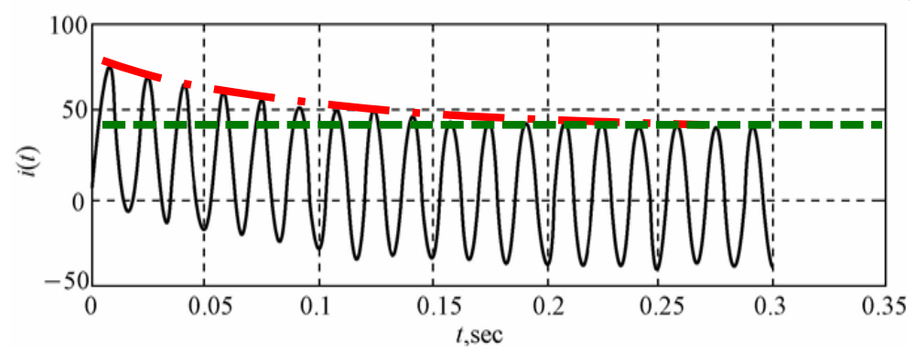
- Apply Inverse Laplace Transform to $I(s)$

$$i(t) = I_m \sin(\omega t + \alpha - \gamma) - I_m e^{-t/\tau} \sin(\alpha - \gamma)$$

where $I_m = V_m/\sqrt{R^2 + X^2}, \gamma = \tan^{-1}(\omega L/R)$

Steady-state (sinusoidal component) — dc offset (transient component)

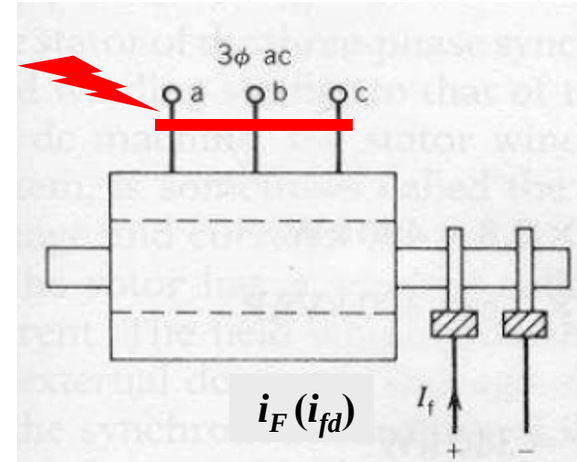
- Saadat's Example 8.1
  R=0.125Ω, L=10mH, V_m=151V
  $\tau = L/R$=0.08s



Saadat's Example 8.2: Balanced 3-Phase Short Circuit

A 500-MVA, 30-kV, 60-Hz synchronous generator operates at no-load with a constant excitation voltage of 400V. A 3-phase short circuit occurs at its armature terminals. Use its voltage equations to find transient waveforms for the current in each phase and the field current. Assume the short circuit to be applied at the instant when θ=0, i.e. the d-axis is along the magnetic axis of phase a. Also assume the rotor speed $\omega_r = \omega_s$.

Generator Parameters for Example 8.2		
$L_d = 0.0072$ H	$L_q = 0.0070$ H	$L_F = 2.500$ H
$L_D = 0.0068$ H	$L_Q = 0.0016$ H	$M_F = 0.100$ H
$M_D = 0.0054$ H	$M_Q = 0.0026$ H	$M_R = 0.125$ H
$r = 0.0020\ \Omega$	$r_F = 0.4000\ \Omega$	$r_D = 0.015\ \Omega$
$r_Q = 0.0150\ \Omega$	$L_0 = 0.0010$ H	



Solution:

The voltage equations are: $\mathbf{e} = \mathbf{R} \times \mathbf{i} + \mathbf{L}\dfrac{d}{dt}\mathbf{i}$

$$\begin{bmatrix} e_0 \\ e_d \\ e_F \\ 0 \\ e_q \\ 0 \end{bmatrix} = \begin{bmatrix} R_a+3R_n & 0 & 0 & 0 & 0 & 0 \\ 0 & R_a & 0 & 0 & -\omega_r L_q & -\omega_r kM_Q \\ 0 & 0 & R_F & 0 & 0 & 0 \\ 0 & 0 & 0 & R_D & 0 & 0 \\ 0 & \omega_r L_d & \omega_r kM_F & \omega_r kM_D & R_a & 0 \\ 0 & 0 & 0 & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix} + \begin{bmatrix} L_0+3L_n & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & kM_F & kM_D & 0 & 0 \\ 0 & kM_F & L_F & M_R & 0 & 0 \\ 0 & kM_D & M_R & L_D & 0 & 0 \\ 0 & 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & 0 & kM_Q & L_Q \end{bmatrix} \frac{d}{dt} \begin{bmatrix} -i_0 \\ -i_d \\ i_F \\ i_D \\ -i_q \\ i_Q \end{bmatrix}$$

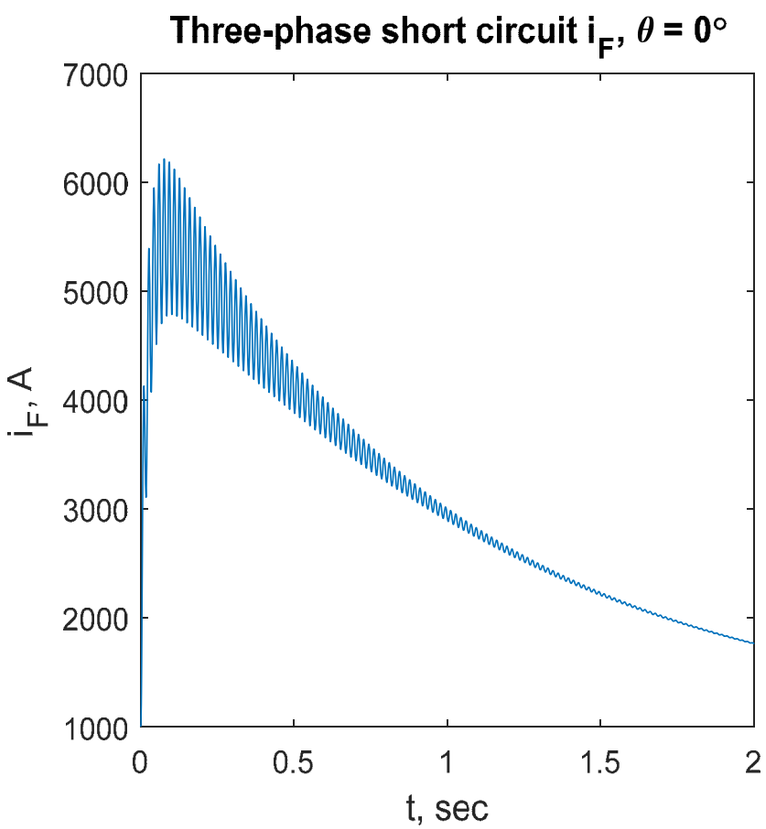
No-load condition: $i_a(0^+)=i_b(0^+)=i_c(0^+)=0 \rightarrow i_0(0^+)=i_d(0^+)=i_q(0^+)=0$

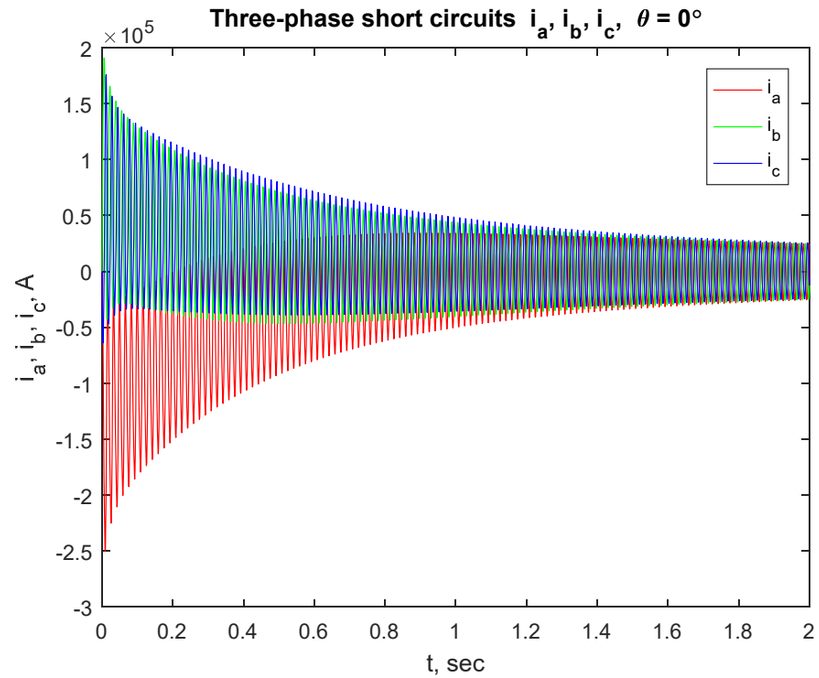
Balanced 3-phase fault: $e_a \equiv e_b \equiv e_c \equiv 0, \quad \rightarrow e_0 \equiv e_d \equiv e_q \equiv 0$

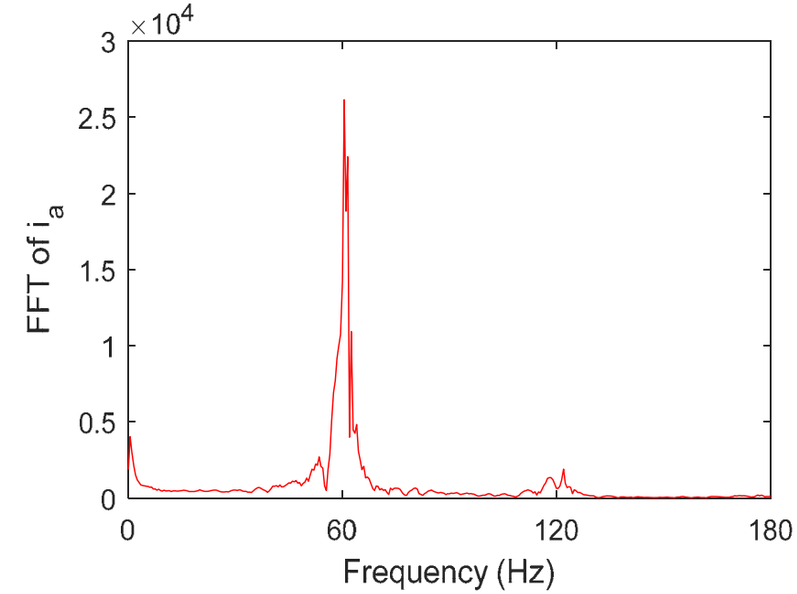
Initial rotor winding currents: $i_F(0^+)=e_F/R_F=400/0.4=1000$ A, $\quad i_D(0^+)=i_Q(0^+)=0$

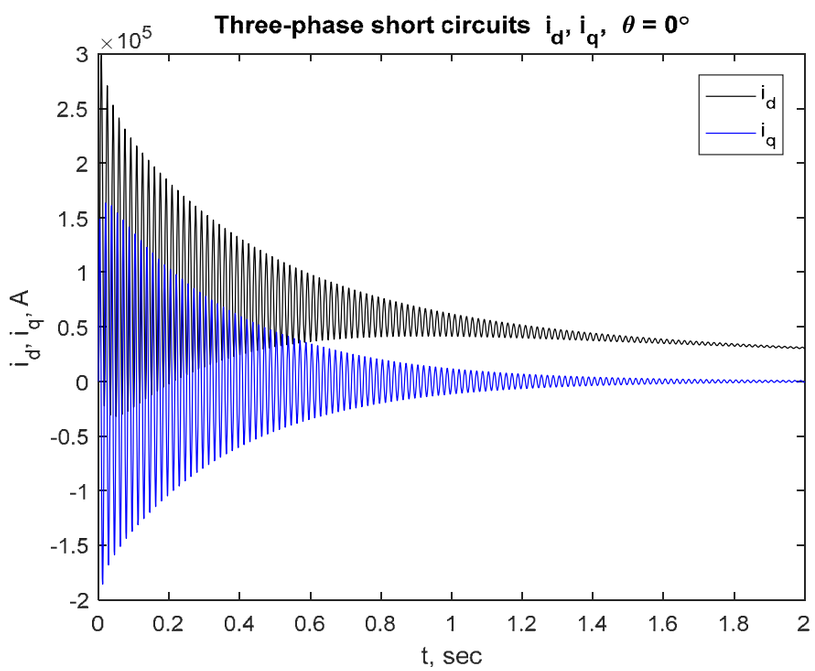
Then, solve the differential equations $\mathbf{i}(t)$ with these initial values $\mathbf{i}(0^+)$: $\dfrac{d}{dt}\mathbf{i} = \mathbf{L}^{-1}(\mathbf{e} - \mathbf{R} \times \mathbf{i})$

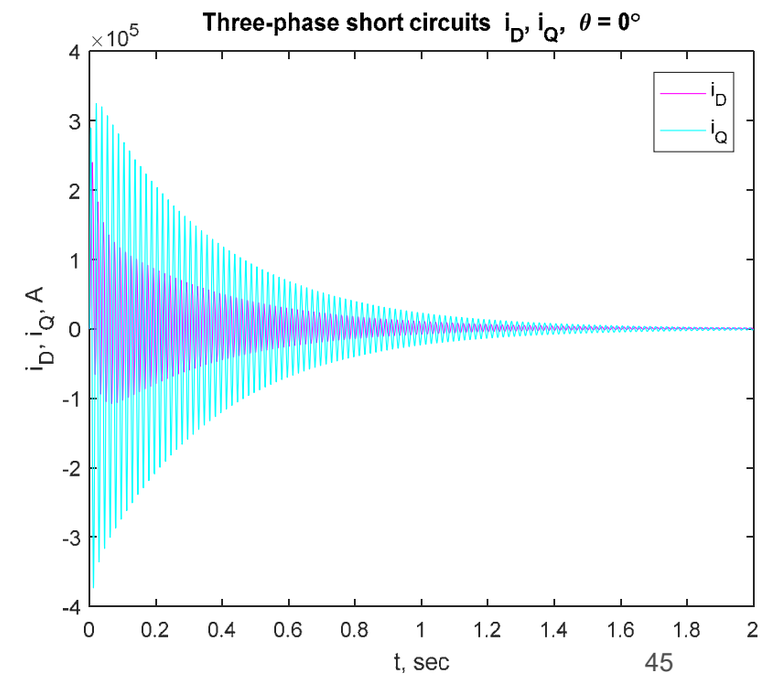
Results



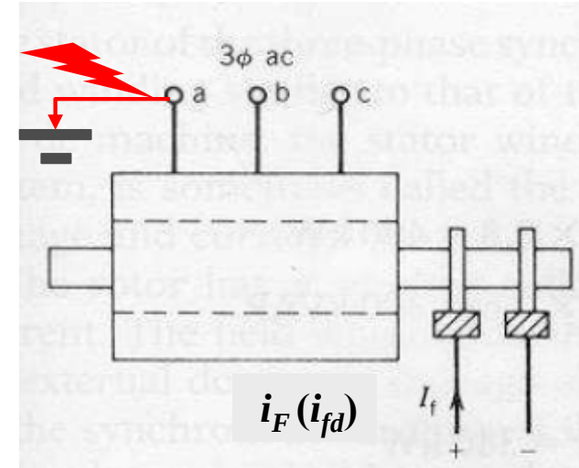








Saadat's Example 8.4: Line-to-Ground Short Circuit



Same as 8.2 except for a line-to-ground short circuit of phase *a*.

Solution: No-load condition: $i_b \equiv i_c \equiv 0$ Line-to-Ground fault: $e_a \equiv 0$

$$e_a = 0 = -R_a i_a + \frac{d}{dt}(-l_{aa} i_a - l_{ab} i_b - l_{ac} i_c + l_{aF} i_F + l_{aD} i_D + l_{aQ} i_Q)$$

$$= -R_a i_a - l_{aa}\frac{di_a}{dt} + l_{aF}\frac{di_F}{dt} + l_{aD}\frac{di_D}{dt} + l_{aQ}\frac{di_Q}{dt} - i_a\frac{dl_{aa}}{dt} + i_F\frac{dl_{aF}}{dt} + i_D\frac{dl_{aD}}{dt} + i_Q\frac{dl_{aQ}}{dt}$$

$$= -R_a i_a - (L_s + L_m\cos 2\theta)\frac{di_a}{dt} + M_F\cos\theta\frac{di_F}{dt} + M_D\cos\theta\frac{di_D}{dt} - M_Q\sin\theta\frac{di_Q}{dt} - (-2\omega_r L_m \sin 2\theta)\cdot i_a - \omega_r M_F\sin\theta\cdot i_F - \omega_r M_D\sin\theta\cdot i_D - \omega_r M_Q\cos\theta\cdot i_D$$

Solve these voltage equations:

$$\begin{bmatrix} e_a = 0 \\ e_F \\ e_D = 0 \\ e_Q = 0 \end{bmatrix} = \begin{bmatrix} R_a - 2\omega_r L_m \sin 2\theta & -\omega_r M_F \sin\theta & -\omega_r M_D \sin\theta & -\omega_r M_Q \cos\theta \\ -\omega_r M_F \sin\theta & R_F & 0 & 0 \\ -\omega_r M_D \sin\theta & 0 & R_D & 0 \\ -\omega_r M_Q \cos\theta & 0 & 0 & R_Q \end{bmatrix} \begin{bmatrix} -i_a \\ i_F \\ i_D \\ i_Q \end{bmatrix} + \begin{bmatrix} L_s + L_m\cos 2\theta & M_F\cos\theta & M_D\cos\theta & -M_Q\sin\theta \\ M_F\cos\theta & L_F & M_R & 0 \\ M_D\cos\theta & M_R & L_D & 0 \\ -M_Q\sin\theta & 0 & 0 & L_Q \end{bmatrix} \frac{d}{dt}\begin{bmatrix} -i_a \\ i_F \\ i_D \\ i_Q \end{bmatrix}$$

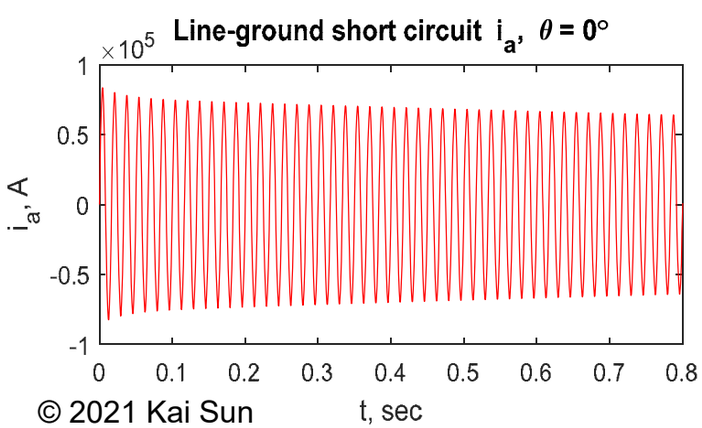
$$L_0 = L_s - 2M_s,$$

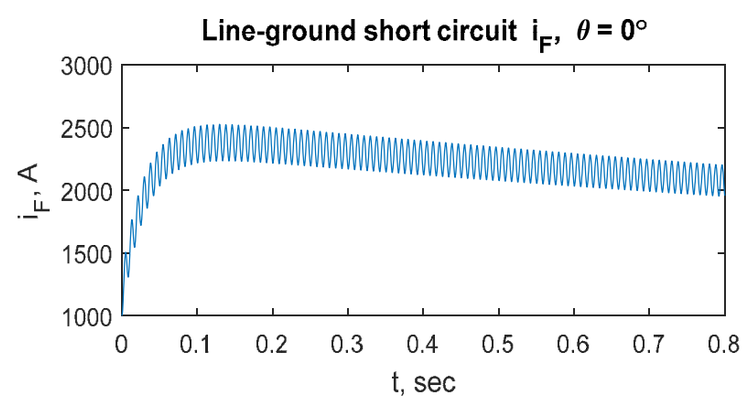
$$L_d = L_s + M_s + \frac{3}{2}L_m,$$

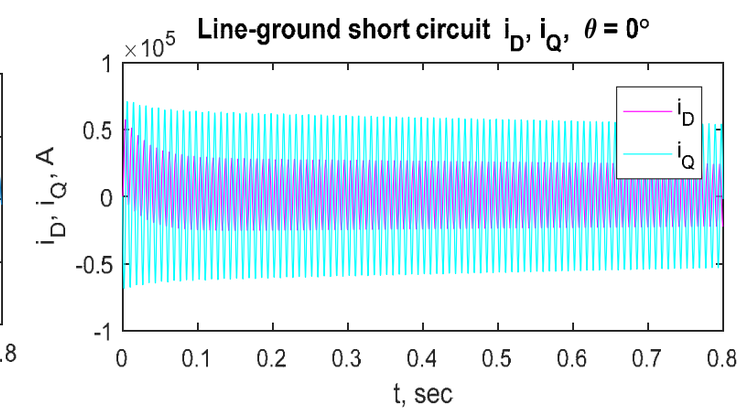
$$L_q = L_s + M_s - \frac{3}{2}L_m$$

$$L_s = (L_0 + L_d + L_q)/3,$$

$$L_m = (L_d - L_q)/3$$



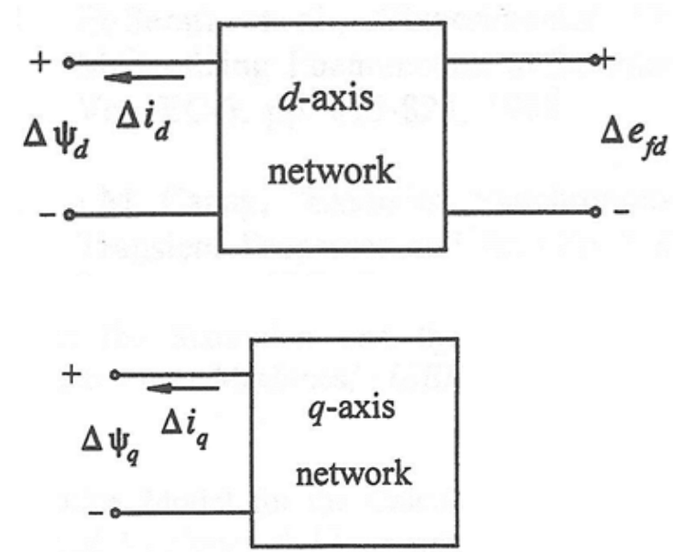




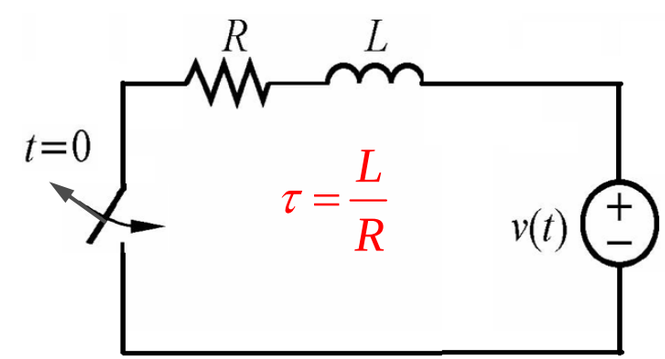
Finding short-circuit and open circuit time constants

Consider the d-axis network

- Short-circuit time constant τ
  - Instantaneous change on ψ_d
  - Delayed change on i_d (through $\frac{1}{1+s\tau}$)
- Open-circuit time constant τ_0
  - Instantaneous change on i_d
  - Delayed change on ψ_d (through $\frac{1}{1+s\tau 0}$)

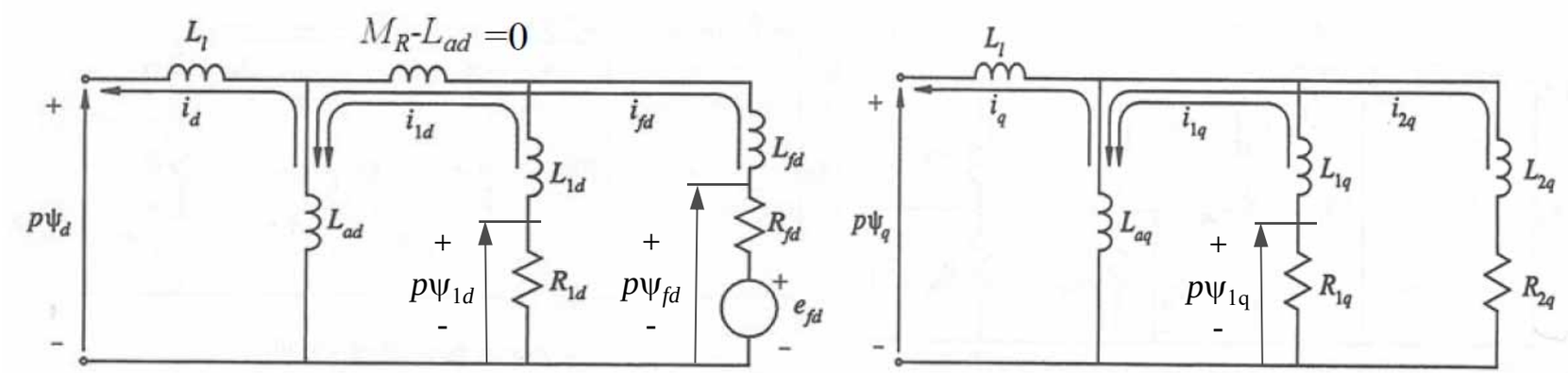


$$L_d(s) = -\left.\frac{\Delta\psi_d}{\Delta i_d}\right|_{\Delta e_{fd}=0} = L_d\frac{1+s\tau}{1+s\tau_0} \Rightarrow L_d(s) = L_d\frac{(1+sT_d')(1+sT_d'')}{(1+sT_{d0}')(1+sT_{d0}'')}$$



- Time constant τ or τ_0 is equal to the total inductance L divided by the total resistance R of the effective circuit

Transient and sub-transient parameters



(a) d-axis equivalent circuit (b) q-axis equivalent circuit

$R_{1d} >> R_{fd}$ $L_{fd}/R_{fd} >> L_{1d}/R_{1d}$

$L_{1q}/R_{1q} >> L_{2q}/R_{2q}$

	d axis circuit		q axis circuit	
Considered rotor windings	Only field Winding	Add the damper winding (ignoring R_{fd})	Only 1st damper winding	Add the 2nd damper winding (ignoring R_{1q})
Time constant (open circuit)	$T'_{d0} = \frac{L_{ad}+L_{fd}}{R_{fd}}$ 8.07(s)	$T''_{d0} = \frac{L_{ad}//L_{fd} + L_{1d}}{R_{1d}}$ 0.030(s)	$T'_{q0} = \frac{L_{aq}+L_{1q}}{R_{1q}}$ 1.0(s)	$T''_{q0} = \frac{L_{aq}//L_{1q} + L_{2q}}{R_{2q}}$ 0.07(s)
Time constant (short circuit)	$T'_d = \frac{L_{ad}//L_l + L_{fd}}{R_{fd}}$ 1.34(s)	$T''_d = \frac{L_{ad}//L_{fd}//L_l + L_{1d}}{R_{1d}}$ 0.023(s)	$T'_q = \frac{L_{aq}//L_l + L_{1q}}{R_{1q}}$ 0.37(s)	$T''_q = \frac{L_{aq}//L_{1q}//L_l + L_{2q}}{R_{2q}}$ 0.03(s)
Inductance (Reactance) $L_d(s)$ or $L_q(s)$ seen from the terminal	$L'_d = L_l + L_{ad}//L_{fd}$ 0.30(pu)	$L''_d = L_l + L_{ad}//L_{fd}//L_{1d}$ 0.23(pu)	$L'_q = L_l + L_{aq}//L_{1q}$ 0.65(pu)	$L''_q = L_l + L_{aq}//L_{1q}//L_{2q}$ 0.25(pu)

Based on Parameters in Kundur's Example 3.2

- Note: time constants are all in p.u. To be converted to seconds, they have to be multiplied by $t_{base}=1/\omega_{base}$ (i.e. 1/377 for 60Hz).

Table 4.1

Expressions for Standard Parameters of Synchronous Machine

Parameter	Classical Expression	Accurate Expression
T'_{d0}	T_1	T_1+T_2
T'_d	T_4	T_4+T_5
T''_{d0}	T_3	$T_3[T_1/(T_1+T_2)]$
T''_d	T_6	$T_6[T_4/(T_4+T_5)]$
L'_d	$L_d(T_4/T_1)$	$L_d(T_4+T_5)/(T_1+T_2)$
L''_d	$L_d(T_4T_6)/(T_1T_3)$	$L_d(T_4T_6)/(T_1T_3)$

with

$$T_1 = \frac{L_{ad}+L_{fd}}{R_{fd}} \qquad T_2 = \frac{L_{ad}+L_{1d}}{R_{1d}}$$

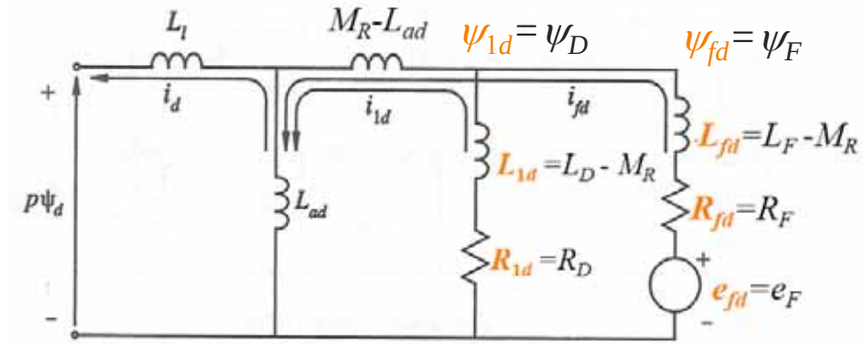
$$T_3 = \frac{1}{R_{1d}}\left(L_{1d}+\frac{L_{ad}L_{fd}}{L_{ad}+L_{fd}}\right) \qquad T_4 = \frac{1}{R_{fd}}\left(L_{fd}+\frac{L_{ad}L_l}{L_{ad}+L_l}\right)$$

$$T_5 = \frac{1}{R_{1d}}\left(L_{1d}+\frac{L_{ad}L_l}{L_{ad}+L_l}\right) \qquad T_6 = \frac{1}{R_{1d}}\left(L_{1d}+\frac{L_{ad}L_lL_{fd}}{L_{ad}L_l+L_{ad}L_{fd}+L_{fd}L_l}\right)$$

Notes:

1. Similar expressions apply to q-axis parameters.
2. All parameters are in per unit.
3. Time constants in seconds are obtained by dividing the per unit values given in the table by $\omega_0=2\pi f$.
4. All mutual inductances in d-axis assumed equal.

Deriving sub-transient and transient inductances (Classical)



(a) d-axis equivalent circuit

- Subtransient inductance L''_d

$$\psi_d = -L_d i_d + L_{ad} i_F + L_{ad} i_D \qquad = -L''_d\, i_d$$

$$\psi_F = 0 = -L_{ad} i_d + L_F i_F + M_R i_D$$

$$\psi_D = 0 = -L_{ad} i_d + M_R i_F + L_D i_D$$

$$L''_d = L_d - L_{ad}^2 \frac{L_F + L_D - 2M_R}{L_F L_D - M_R^2}$$

If $M_R = L_{ad}$

$$L''_d = L_d - L_{ad}^2 \frac{L_{fd} + L_{1d}}{L_{fd}L_{ad} + L_{fd}L_{1d} + L_{1d}L_{ad}} \qquad = L_l + \frac{L_{ad}L_{fd}L_{1d}}{L_{ad}L_{fd} + L_{ad}L_{1d} + L_{fd}L_{1d}}$$

If $M_R = L_D$ or $M_R M_D = M_F L_D$ as assumed in some textbooks

$$L''_d = L_d - L^2_{ad}/L_D$$

- Transient inductance L'_d: open circuit of the damping winding branch $i_D=0$

$$\psi_d = -L_d i_d + L_{ad} i_F \qquad = -L'_d\, i_d$$

$$\psi_F = 0 = -L_{ad} i_d + L_F i_F$$

$$L'_d = L_d - L^2_{ad}/L_F = L_l + \frac{L_{ad}L_{fd}}{L_{ad} + L_{fd}}$$

Synchronous, Transient and Sub-transient Inductances

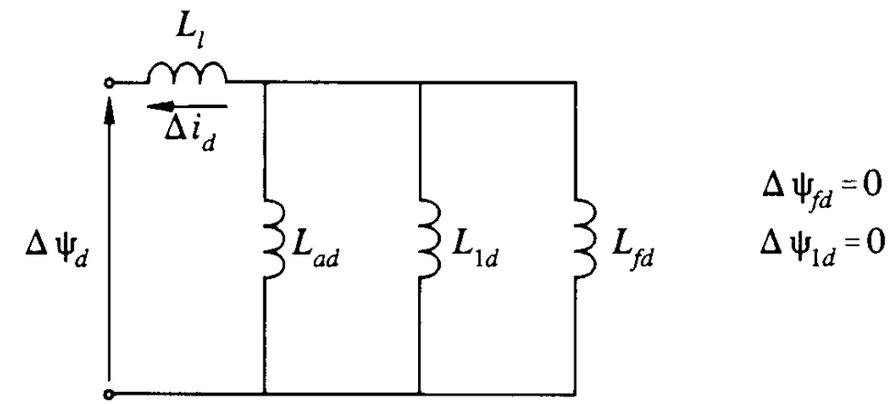


Figure 4.3 Equivalent circuit for incremental values, immediately following a disturbance

$$L_d(s) = L_d \frac{(1+sT'_d)(1+sT''_d)}{(1+sT'_{d0})(1+sT''_{d0})}$$

- Under steady-state condition: s=0 (t→∞)

  $L_d(0)=L_d$ (d-axis synchronous inductance)

- During a rapid transient: $s\to\infty$

$$L''_d = L_d(\infty) = L_d \frac{T'_d T''_d}{T'_{d0} T''_{d0}} = L_l + \frac{L_{ad}L_{fd}L_{1d}}{L_{ad}L_{fd} + L_{ad}L_{1d} + L_{fd}L_{1d}}$$ (d-axis sub-transient inductance)

- Without the damper winding: $s >> 1/T'_d$ and $1/T'_{d0}$ but $<< 1/T''_d$ and $1/T''_{d0}$

$$L'_d = L_d(\infty) = L_d \frac{T'_d}{T'_{d0}} = L_l + \frac{L_{ad}L_{fd}}{L_{ad} + L_{fd}}$$ (d-axis transient inductance)

Ordering of Time and Inductance Constants

The following are the parameters in per unit on machine rating of a 555 MVA, 24 kV, 0.9 p.f., 60 Hz, 3600 RPM turbine-generator[1]:

$L_{ad}=1.66$	$L_{aq}=1.61$	$L_l=0.15$	$R_a=0.003$
$L_{fd}=0.165$	$R_{fd}=0.0006$	$L_{1d}=0.1713$	$R_{1d}=0.0284$
$L_{1q}=0.7252$	$R_{1q}=0.00619$	$L_{2q}=0.125$	$R_{2q}=0.02368$

M_R is assumed to be equal to L_{ad}.

In general, there are:

$R_{1d} >> R_{fd}$, $L_{ad} >> L_{fd}$ and L_{1d}

$R_{2q} >> R_{1q}$, $L_{aq} >> L_{1q}$ and L_{2q}

$L_{1d}=L_D-L_{ad}$ $L_{fd}=L_F-L_{ad}$ $L_{1q}=L_Q-L_{aq}$ $L_{2q}=L_G-L_{aq}$

$$T'_{d0}=\frac{L_{ad}+L_{fd}}{R_{fd}} > T'_d=\frac{L_{ad}//L_l+L_{fd}}{R_{fd}} >> T''_{d0}=\frac{L_{ad}//L_{fd}+L_{1d}}{R_{1d}} > T''_d=\frac{L_{ad}//L_{fd}//L_l+L_{1d}}{R_{1d}}$$

$$T'_{q0}=\frac{L_{aq}+L_{1q}}{R_{1q}} > T'_q=\frac{L_{aq}//L_l+L_{1q}}{R_{1q}} >> T''_{q0}=\frac{L_{ad}//L_{1q}+L_{2q}}{R_{2q}} > T''_q=\frac{L_{ad}//L_{1q}//L_l+L_{2q}}{R_{2q}}$$

$L_d=L_l+L_{ad}$ > $L'_d=L_l+L_{ad}//L_{fd}=L_d-L^2_{ad}/L_F$ > $L''_d=L_l+L_{ad}//L_{fd}//L_{1d}$ (if assuming $M_R=L_{ad}$)

$=L_d-L^2_{ad}/L_D$ (in some books, if assuming $M_RM_D=M_FL_D$)

$L_q=L_l+L_{aq}$ > $L'_q=L_l+L_{aq}//L_{1q}=L_q-L^2_{aq}/L_Q$ > $L''_q=L_l+L_{aq}//L_{1q}//L_{2q}$

$=L_q-L^2_{aq}/L_G$ (in some books)

Table 4.2

Parameter		Hydraulic Units	Thermal Units
Synchronous Reactance	X_d	0.6 - 1.5	1.0 - 2.3
	X_q	0.4 - 1.0	1.0 - 2.3
Transient Reactance	X'_d	0.2 - 0.5	0.15 - 0.4
	X'_q	-	0.3 - 1.0
Subtransient Reactance	X''_d	0.15 - 0.35	0.12 - 0.25
	X''_q	0.2 - 0.45	0.12 - 0.25
Transient OC Time Constant	T'_{d0}	1.5 - 9.0 s	3.0 - 10.0 s
	T'_{q0}	-	0.5 - 2.0 s
Subtransient OC Time Constant	T''_{d0}	0.01 - 0.05 s	0.02 - 0.05 s
	T''_{q0}	0.01 - 0.09 s	0.02 - 0.05 s
Stator Leakage Inductance	X_l	0.1 - 0.2	0.1 - 0.2
Stator Resistance	R_a	0.002 - 0.02	0.0015 - 0.005

1. Reactance values are in per unit with stator base values equal to the corresponding machine rated values.
2. Time constants are in seconds.

$$X_d \geq X_q \geq X'_q \geq X'_d \geq X''_q \geq X''_d > X_l$$

$$T'_{d0} > T'_d >> T''_{d0} > T''_d$$

$$T'_{q0} > T'_q >> T''_{q0} > T''_q$$

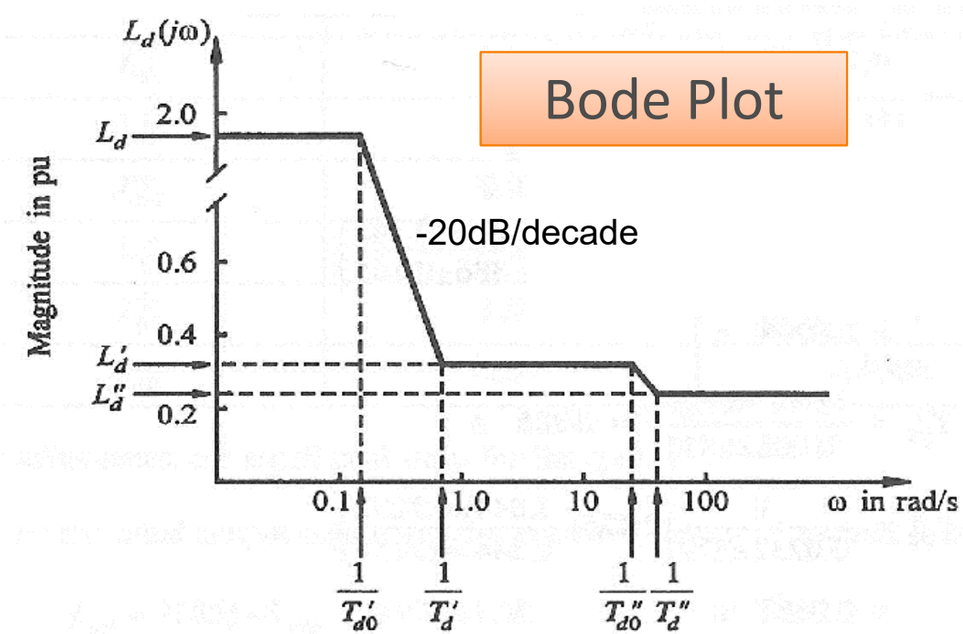
Parameter Estimation by Frequency Response Tests

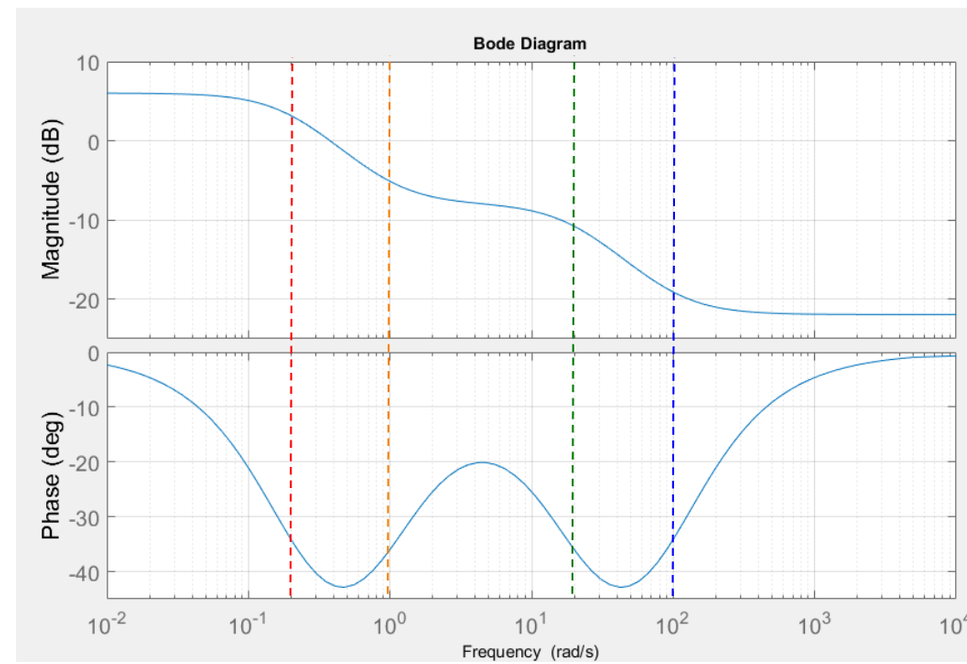
$$L_d(s) = L_d \frac{(1+sT_d')(1+sT_d'')}{(1+sT_{d0}')(1+sT_{d0}'')}$$

$T'_{d0} > T'_d >> T''_{d0} > T''_d$

$1/T'_{d0} < 1/T'_d << 1/T''_{d0} < 1/T''_d$

$T'_{d0}=5$, $T'_d=1$, $T''_{d0}=0.05$, $T''_d=0.01$ (s)





Swing Equations

$$J\frac{d\omega_m}{dt} = T_a = T_m - T_e$$

J	combined moment of inertia of generator and turbine, kg·m^2
ω_m	angular velocity of the rotor, mech. rad/s
t	time, s
T_a	accelerating torque in N.m
T_m	mechanical torque in N.m
T_e	electromagnetic torque in N.m

- Define per unit inertia constant

$$H = \frac{1}{2}\frac{J\omega_{0m}^2}{\mathrm{VA}_{base}} \quad \text{(s)}$$

$$J = \frac{2H}{\omega_{0m}^2}\mathrm{VA}_{base}$$

Type of generating unit	H (s)
Thermal unit (a) 3600 r/min (2-pole) (b) 1800 r/min (4-pole)	 2.5 to 6.0 4.0 to 10.0
Hydraulic unit	2.0 to 4.0

Some references define T_M or M=2H, called the mechanical starting time, i.e. the time required for rated torque to accelerate the rotor from standstill to rated speed

$$J\frac{d\omega_m}{dt} = T_a = T_m - T_e$$

$$\frac{2H}{\omega_{0m}^2}\mathrm{VA}_{base}\frac{d\omega_m}{dt} = T_m - T_e$$

$$2H\frac{d}{dt}\left(\frac{\omega_m}{\omega_{0m}}\right) = \frac{T_m - T_e}{\mathrm{VA}_{base}/\omega_{0m}}$$

$$2H\frac{d}{dt}\left(\frac{\omega_m}{\omega_{0m}}\right) = \frac{T_m - T_e}{T_{base}}$$

$$2H\frac{d\bar{\omega}_r}{dt} = \bar{T}_m - \bar{T}_e \quad \text{(in per unit)}$$

where

$$\omega_r \text{ (in per unit)} = \frac{\omega_m}{\omega_{0m}} = \frac{\omega_r / p_f}{\omega_0 / p_f} = \frac{\omega_r}{\omega_0}$$

Angular position of the rotor in electrical radian with respect to a synchronously rotating reference

$$\delta = \omega_r t - \omega_0 t + \delta_0 \quad \text{in rad}$$

$$\frac{d\delta}{dt} = \omega_r - \omega_0 = \Delta\omega_r \quad \text{in rad/s}$$

$$\frac{d^2\delta}{dt^2} = \frac{d\omega_r}{dt} = \frac{d(\Delta\omega_r)}{dt} \quad \text{in rad/s}^2$$

$$\frac{d^2\delta}{dt^2} = \omega_0\frac{d(\bar{\omega}_r)}{dt} = \omega_0\frac{d(\Delta\bar{\omega}_r)}{dt} \quad \text{in rad/s}^2$$

$$\Delta\bar{\omega}_r = \frac{\Delta\omega_r}{\omega_0} = \frac{1}{\omega_0}\frac{d\delta}{dt}$$

$$\boxed{\frac{2H}{\omega_0}\frac{d^2\delta}{dt^2} = \bar{T}_m - \bar{T}_e}$$

If adding a damping term proportional to speed deviation:

$$\boxed{\frac{2H}{\omega_0}\frac{d^2\delta}{dt^2} = \bar{T}_m - \bar{T}_e - K_D\Delta\bar{\omega}_r = \bar{T}_m - \bar{T}_e - \frac{K_D}{\omega_0}\frac{d\delta}{dt}}$$

State Space and Block diagram Representations

$$\frac{2H}{\omega_0}\frac{d^2\delta}{dt^2} = \bar{T}_m - \bar{T}_e - K_D\Delta\bar{\omega}_r = \bar{T}_m - \bar{T}_e - \frac{K_D}{\omega_0}\frac{d\delta}{dt}$$

$$2H\frac{d(\Delta\bar{\omega}_r)}{dt} = \bar{T}_m - \bar{T}_e - K_D\Delta\bar{\omega}_r$$

$$\frac{1}{\omega_0}\frac{d\delta}{dt} = \Delta\bar{\omega}_r$$

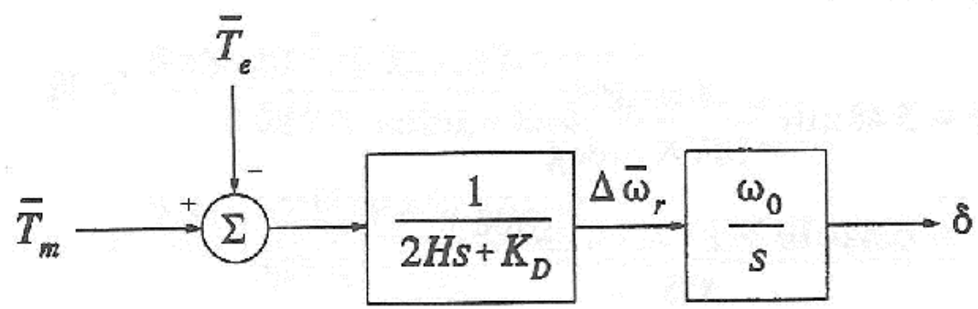


Figure 3.34 Block diagram representation of swing equations

State-Space Representation of a Synchronous Machine

So far, we modeled all critical dynamics about a synchronous machine:

- State variables (p**X**):
  - stator and rotor voltages, currents or flux linkages
  - swing equations (rotor angle and speed)
- Time constants:
  - Inertia: $2H$
  - Sub-transient and transient time constants, e.g. T'_{d0} and T''_{d0}
- Other parameters
  - Stator and rotor self- or mutual-inductances and resistances
  - Rotor mechanical torque T_m and stator electromagnetic torque T_e

State Space Model on a Salient-pole Machine

- Consider 5 windings: d, q, F (fd), D (1d) and Q (1q)
  - Voltage and flux equations: $\mathbf{e} = \mathbf{R}\times\mathbf{i} + \mathbf{L}\times\frac{d}{dt}\mathbf{i} + \mathbf{\Omega\Psi}$ $\quad \mathbf{\Psi} = \mathbf{L}\times\mathbf{i}$

$$\mathbf{e} = \begin{bmatrix} e_0 \\ e_d \\ e_{fd} \\ 0 \\ e_q \\ 0 \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} -i_0 \\ -i_d \\ i_{fd} \\ i_{1d} \\ -i_q \\ i_{1q} \end{bmatrix}, \quad \mathbf{\Psi} = \begin{bmatrix} \psi_0 \\ \psi_d \\ \psi_{fd} \\ \psi_{1d} \\ \psi_q \\ \psi_{1q} \end{bmatrix}, \quad \mathbf{\Omega\Psi} = \begin{bmatrix} 0 \\ -\omega_r\psi_q \\ 0 \\ 0 \\ \omega_r\psi_d \\ 0 \end{bmatrix}$$

$$\frac{d}{dt}\mathbf{\Psi} = -(\mathbf{R}\times\mathbf{L}^{-1} - \mathbf{\Omega})\times\mathbf{\Psi} + \mathbf{e}$$

  - Swing equations:

$$2H\frac{d\omega_r}{dt} = T_m - T_e = T_m - (\psi_d i_q - \psi_q i_d)$$

$$\frac{d\delta}{dt} = \omega_r - \omega_0 = \omega_r - 1$$

- Define state vector $\mathbf{x}=[\psi_d\ \psi_{fd}\ \psi_{1d}\ \psi_q\ \psi_{1q}\ \omega_r\ \delta]^T$
  Thus, the state-space model: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, e_{fd}, T_m, e_d, e_q)$

- e_{fd} and T_m are usually known but e_d and e_q are related to its loading conditions (the grid), so algebraic power-flow equations should be introduced.
- The grid model is a set of Differential-Algebraic Equations (DAEs)

Simplified Models

- Salient: $[\psi_d\ \psi_{fd}\ \psi_{1d}\ \psi_q\ \psi_{1q}\ \omega_r\ \delta]^T$
  Round: $[\psi_d\ \psi_{fd}\ \psi_{1d}\ \psi_q\ \psi_{1q}\ \psi_{2q}\ \omega_r\ \delta]^T$

Let $p\psi_d = p\psi_q = 0$ and $\omega_r\psi = \psi$

- Salient: $[\psi_{fd}\ \psi_{1d}\ \psi_{1q}\ \omega_r\ \delta]^T$ (**2.1 model**)
  Round: $[\psi_{fd}\ \psi_{1d}\ \psi_{1q}\ \psi_{2q}\ \omega_r\ \delta]^T$ (**2.2 model**)
  - Inertia $2H$ — $p\omega_r$
  - Transient T'_{d0} — $p\psi_{fd}$ T'_{q0} — $p\psi_{1q}$
  - Sub-transient T''_{d0} — $p\psi_{1d}$ T''_{q0} — $p\psi_{2q}$

Let $p\psi_{1d} = p\psi_{2q} = 0$ (neglect one damper winding in each axis)

- Salient: $[\psi_{fd}\ \omega_r\ \delta]^T$ (**1.0 model / 1-axis model**)
  Round: $[\psi_{fd}\ \psi_{1q}\ \omega_r\ \delta]^T$ (**1.1 model / 2-axis model**)
  - Inertia $2H$ — $p\omega_r$
  - Transient T'_{d0} — $p\psi_{fd}$ T'_{q0} — $p\psi_{1q}$

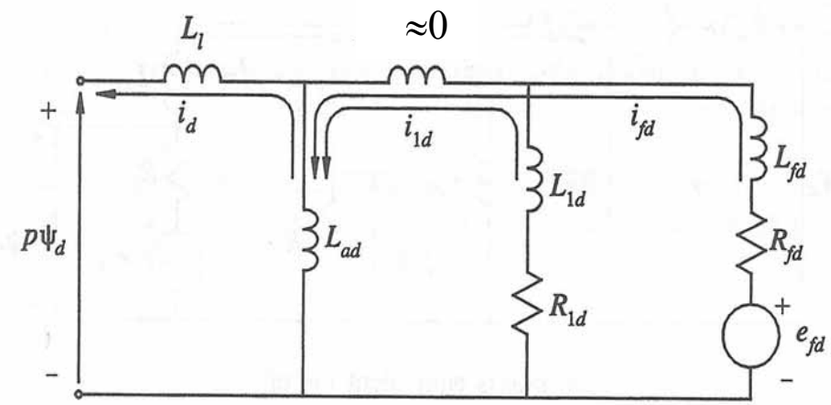
Constant flux linkages

- $[\omega_r\ \delta]^T$ (**0.0 model / classical model**)
  - Inertia $2H$ — $p\omega_r$

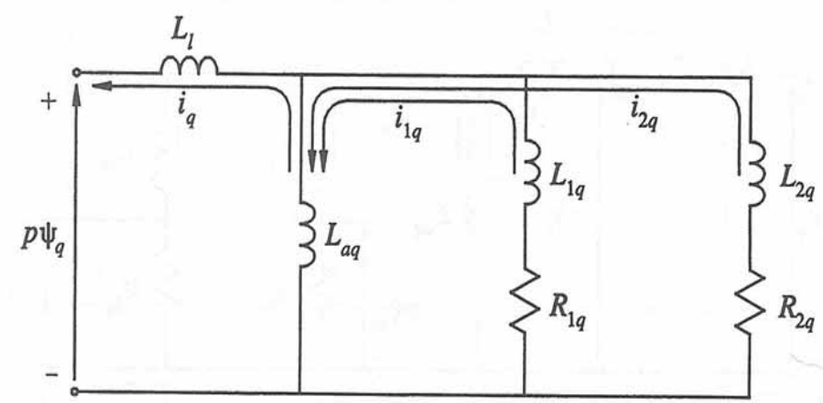
$$p\mathbf{\Psi} = -(\mathbf{R}\times\mathbf{L}^{-1} - \mathbf{\Omega})\times\mathbf{\Psi} + \mathbf{e}$$

$$2H \cdot p\omega_r = T_m - T_e$$

$$p\delta = \omega_r - 1$$



(a) d-axis equivalent circuit

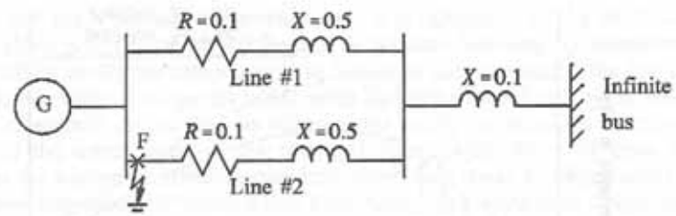


(b) q-axis equivalent circuit

Neglecting $p\psi_d$ and $p\psi_q$

$$e_d = -\omega_r \psi_q - R_a i_d + p\psi_d = -\omega_r \psi_q - R_a i_d$$

$$e_q = \omega_r \psi_d - R_a i_q + p\psi_q = \omega_r \psi_d - R_a i_q$$



Disturbance:
3-phase fault at F; cleared in 0.09 s by opening line #2

Generator parameters in per unit:

L_{ad} = 1.0	L_{aq} = 0.6	L_l = 0.18	L_{fd} = 0.13
L_{1d} = 0.11	L_{1q} = 0.13	R_a = 0.005	R_{fd} = 0.00075
R_{1d} = 0.02	R_{1q} = 0.04	H = 3.5	

Figure 5.1 System configuration and parameters

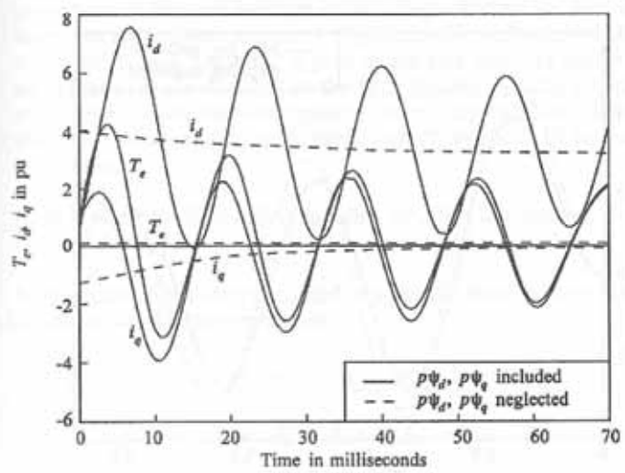


Figure 5.2 Effect of neglecting stator transients on air-gap torque and *d-q* components of stator currents

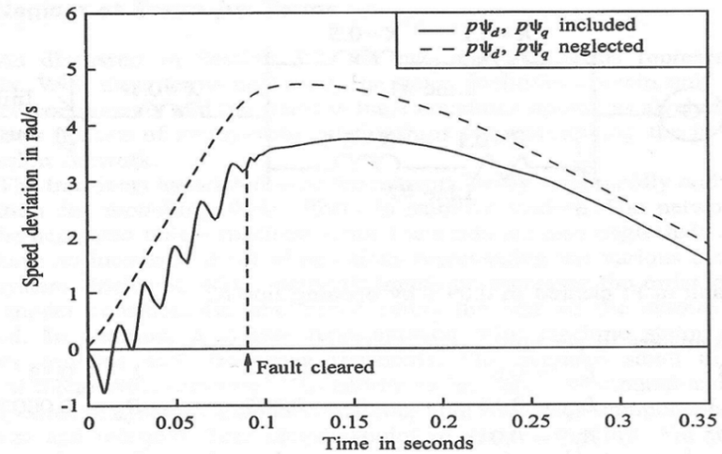


Figure 5.3 Effect of neglecting stator transients on speed deviation

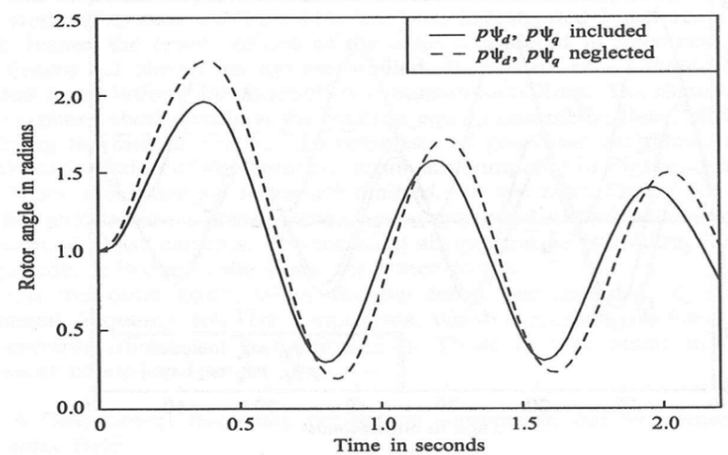


Figure 5.4 Effect of neglecting stator transients on rotor angle swings

Generator models neglecting $p\psi_d$ and $p\psi_q$

Flux equations (AEs):

$$\psi_d = -L_d i_d + L_{ad} i_{fd} + L_{ad} i_{1d}$$
$$\psi_q = -L_q i_q + L_{aq} i_{1q} + L_{aq} i_{2q}$$
$$\psi_{fd} = -L_{ad} i_d + L_F i_{fd} + M_R^d i_{1d}$$
$$\psi_{1d} = -L_{ad} i_d + M_R^d i_{fd} + L_D i_{1d}$$
$$\psi_{1q} = -L_{aq} i_q + L_Q i_{1q} + M_R^q i_{2q}$$
$$\psi_{2q} = -L_{aq} i_q + M_R^q i_{1q} + L_G i_{2q}$$

where $M_R^d \approx L_D \quad M_R^q \approx L_G$

Voltage equations (ODEs):

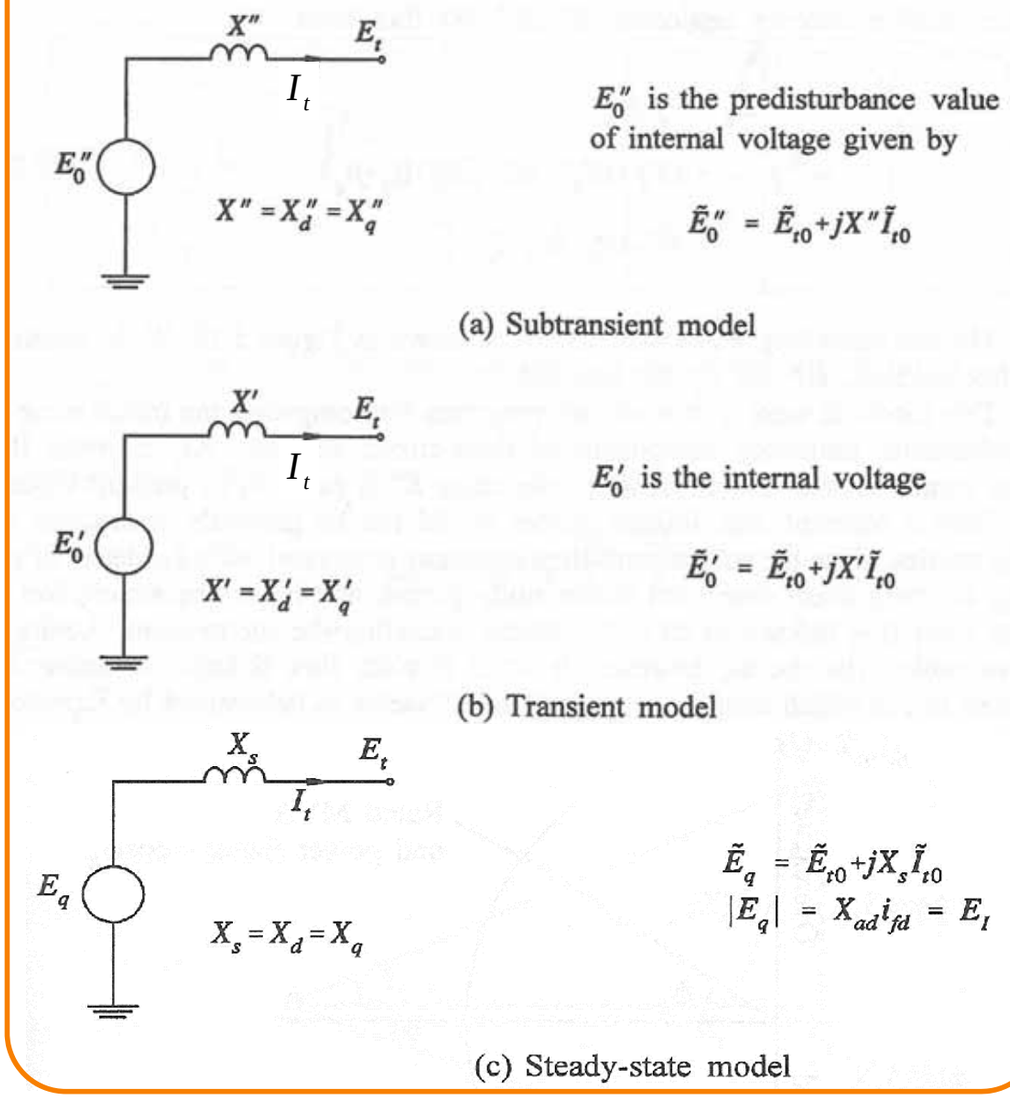
$$e_d = -R_a i_d - \omega_r \psi_q + p\psi_d$$
$$e_q = -R_a i_q + \omega_r \psi_d + p\psi_q$$
$$e_{fd} = R_{fd} i_{fd} + p\psi_{fd}$$
$$0 = R_{1d} i_{1d} + p\psi_{1d}$$
$$0 = R_{1q} i_{1q} + p\psi_{1q}$$
$$0 = R_{2q} i_{2q} + p\psi_{2q}$$

- There are four state variables $\psi_{fd}, \psi_{1d}, \psi_{1q}, \psi_{2q}$ in voltage equations.
- Our goal is to find four ODEs using only $X_d, X_q, X'_d, X'_q, X''_d, X''_q, T'_{d0}, T'_{q0}, T''_{d0}, T''_{q0}$ as parameters, which can be identified by frequency response tests.
- We prefer more meaningful state variables such as its internal voltages or emfs than flux linkages for a stability study on a power system having many generators.

$$e_d = -R_a i_d + X_q i_q + 0$$
$$e_q = -R_a i_q - X_d i_d + X_{ad} i_{fd}$$

Steady-state voltage equations indicate an internal voltage in the q-axis

Internal-voltage-behind-reactance model
appearing under different conditions



Voltage equations:

$$e_d = -R_a i_d - \omega_r \psi_q$$
$$e_q = -R_a i_q + \omega_r \psi_d$$
$$e_{fd} = R_{fd} i_{fd} + p\psi_{fd}$$
$$0 = R_{1d} i_{1d} + p\psi_{1d}$$
$$0 = R_{1q} i_{1q} + p\psi_{1q}$$
$$0 = R_{2q} i_{2q} + p\psi_{2q}$$

Flux equations:

$$\psi_d = -L_d i_d + L_{ad} i_{fd} + L_{ad} i_{1d}$$
$$\psi_q = -L_q i_q + L_{aq} i_{1q} + L_{aq} i_{2q}$$
$$\psi_{fd} = -L_{ad} i_d + L_F i_{fd} + M_R^d i_{1d}$$
$$\psi_{1d} = -L_{ad} i_d + M_R^d i_{fd} + L_D i_{1d}$$
$$\psi_{1q} = -L_{aq} i_q + L_Q i_{1q} + M_R^q i_{2q}$$
$$\psi_{2q} = -L_{aq} i_q + M_R^q i_{1q} + L_G i_{2q}$$

$$M_R^d \approx L_D \qquad M_R^q \approx L_G$$

D. W. Olive, Digital Simulation of Synchronous Machine Transients, IEEE Trans. Power Apparatus and Systems, Vol. PAS-87, No. 8, Aug. 1968

$$e_d = -R_a i_d - \omega_r \psi_q = -R_a i_d + X_q i_q - \omega_r L_{aq}(i_{1q} + i_{2q})$$
$$e_q = -R_a i_q + \omega_r \psi_d = -R_a i_q - X_d i_d + \omega_r L_{ad}(i_{fd} + i_{1d})$$

Steady-state:

$$e_d = -R_a i_d + X_q i_q + 0$$
$$e_q = -R_a i_q - X_d i_d + X_{ad} i_{fd}$$

- Find internal voltage $E'' = E''_d + jE''_q$ under sub-transient conditions:

$$e_d = -R_a i_d + X''_q i_q + E''_d$$
$$e_q = R_a i_q - X''_d i_d + E''_q$$

Assume: $E''_d = k_{2q}\psi_{2q}$

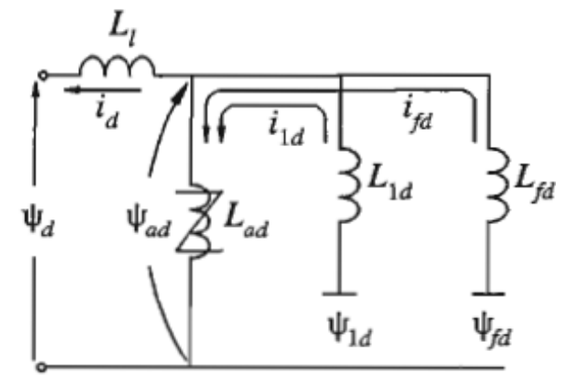
$E''_q = k_{1d}\psi_{1d}$

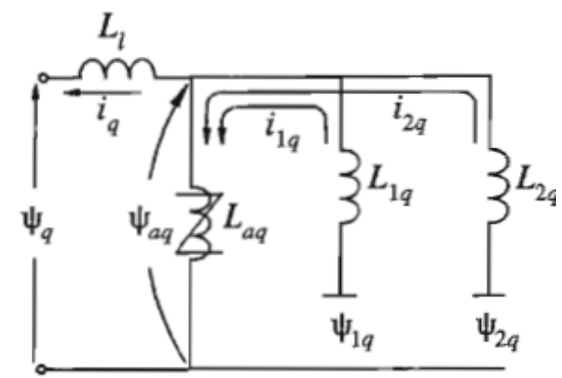
$$e_d + R_a i_d = X''_q i_q + E''_d = X''_q i_q + k_{2q}(-L_{aq} i_q + M_R^q i_{1q} + L_G i_{2q}) = X_q i_q - \omega_r L_{aq}(i_{1q} + i_{2q})$$
$$e_q + R_a i_q = -X''_d i_d + E''_q = -X''_d i_d + k_{1d}(-L_{ad} i_d + M_R^d i_{fd} + L_D i_{1d}) = -X_d i_d + \omega_r L_{ad}(i_{fd} + i_{1d})$$

$$E''_d = -\omega_r \frac{L_{ad}}{L_G}\psi_{2q}$$
$$E''_q = \omega_r \frac{L_{ad}}{L_D}\psi_{1d}$$

$$\psi''_q = -\frac{E''_d}{\omega_r} = \frac{L_{aq}}{L_G}\psi_{2q}$$
$$\psi''_d = \frac{E''_q}{\omega_r} = \frac{L_{ad}}{L_D}\psi_{1d}$$

$$\psi_q = \psi''_q - L''_q i_q$$
$$\psi_d = \psi''_d - L''_d i_d$$





Voltage equations:

$$e_d = -R_a i_d - \omega_r \psi_q$$
$$e_q = -R_a i_q + \omega_r \psi_d$$
$$e_{fd} = R_{fd} i_{fd} + p\psi_{fd}$$
$$0 = R_{1d} i_{1d} + p\psi_{1d}$$
$$0 = R_{1q} i_{1q} + p\psi_{1q}$$
$$0 = R_{2q} i_{2q} + p\psi_{2q}$$

Flux equations:

$$\psi_d = -L_d i_d + L_{ad} i_{fd} + L_{ad} i_{1d}$$
$$\psi_q = -L_q i_q + L_{aq} i_{1q} + L_{aq} i_{2q}$$
$$\psi_{fd} = -L_{ad} i_d + L_F i_{fd} + M_R^d i_{1d}$$
$$\psi_{1d} = -L_{ad} i_d + M_R^d i_{fd} + L_D i_{1d}$$
$$\psi_{1q} = -L_{aq} i_q + L_Q i_{1q} + M_R^q i_{2q}$$
$$\psi_{2q} = -L_{aq} i_q + M_R^q i_{1q} + L_G i_{2q}$$

$$M_R^d \approx L_D \qquad M_R^q \approx L_G$$

D. W. Olive, Digital Simulation of Synchronous Machine Transients, IEEE Trans. Power Apparatus and Systems, Vol. PAS-87, No. 8, Aug. 1968

$$e_d = -R_a i_d - \omega_r \psi_q = -R_a i_d + X_q i_q - \omega_r L_{aq}(i_{1q} + i_{2q})$$
$$e_q = -R_a i_q + \omega_r \psi_d = -R_a i_q - X_d i_d + \omega_r L_{ad}(i_{fd} + i_{1d})$$

- Find internal voltage $E' = E'_d + jE'_q$ under transient conditions ($i_{1d}=0$, $i_{2q}=0$):

$$e_d = -R_a i_d + X'_q i_q + E'_d$$
$$e_q = R_a i_q - X'_d i_d + E'_q$$

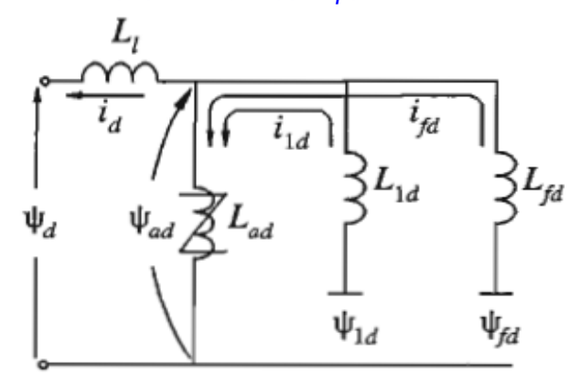
Assume: $E'_d = k_{1q}\psi_{1q}$

$E'_q = k_{fd}\psi_{fd}$

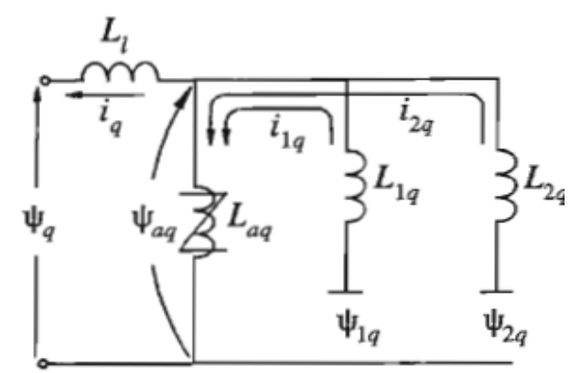
$$e_d + R_a i_d = X'_q i_q + k_{1q}(-L_{aq} i_q + L_Q i_{1q} + 0) = X_q i_q - \omega_r L_{aq}(i_{1q} + 0)$$
$$e_q + R_a i_q = -X'_d i_d + k_{fd}(-L_{ad} i_d + L_F i_{fd} + 0) = -X_d i_d + \omega_r L_{ad}(i_{fd} + 0)$$

$$E'_d = -\omega_r \frac{L_{aq}}{L_Q}\psi_{1q}$$
$$E'_q = \omega_r \frac{L_{ad}}{L_F}\psi_{fd}$$

$$\psi'_q = -\frac{E'_d}{\omega_r} = \frac{L_{aq}}{L_Q}\psi_{1q}$$
$$\psi'_d = \frac{E'_q}{\omega_r} = \frac{L_{ad}}{L_F}\psi_{fd}$$



$$\psi_q = \psi'_q - L'_q i_q$$
$$\psi_d = \psi'_d - L'_d i_d$$



Finding a detailed generator model

Voltage equations:

$$e_d = -R_a i_d - \omega_r \psi_q$$
$$e_q = -R_a i_q + \omega_r \psi_d$$
$$e_{fd} = R_{fd} i_{fd} + p\psi_{fd}$$
$$0 = R_{1d} i_{1d} + p\psi_{1d}$$
$$0 = R_{1q} i_{1q} + p\psi_{1q}$$
$$0 = R_{2q} i_{2q} + p\psi_{2q}$$

Flux equations:

$$\psi_d = -L_d i_d + L_{ad} i_{fd} + L_{ad} i_{1d}$$
$$\psi_q = -L_q i_q + L_{aq} i_{1q} + L_{aq} i_{2q}$$
$$\psi_{fd} = -L_{ad} i_d + L_F i_{fd} + M_R^d i_{1d}$$
$$\psi_{1d} = -L_{ad} i_d + M_R^d i_{fd} + L_D i_{1d}$$
$$\psi_{1q} = -L_{aq} i_q + L_Q i_{1q} + M_R^q i_{2q}$$
$$\psi_{2q} = -L_{aq} i_q + M_R^q i_{1q} + L_G i_{2q}$$

$$M_R^d \approx L_D \qquad M_R^q \approx L_G$$

$$E'_d = -\omega_r \frac{L_{aq}}{L_Q} \psi_{1q} \qquad T'_{d0} = \frac{L_F}{R_{fd}} \qquad X_d - X'_d = \omega_r \frac{L_{ad}^2}{L_F} \qquad i_d = \frac{E'_q - E''_q}{X'_d - X''_d}$$

$$E'_q = \omega_r \frac{L_{ad}}{L_F} \psi_{fd} \qquad T'_{q0} = \frac{L_Q}{R_{1q}} \qquad X_q - X'_q = \omega_r \frac{L_{aq}^2}{L_Q} \qquad i_q = \frac{E'_d - E''_d}{X'_q - X''_q}$$

$$E''_d = -\omega_r \frac{L_{ad}}{L_G} \psi_{2q} \qquad T''_{d0} = \frac{L_D - \frac{L_{ad}^2}{L_F}}{R_{1d}} \qquad X_d - X''_d = \omega_r \frac{L_{ad}^2}{L_D}$$

$$E''_q = \omega_r \frac{L_{ad}}{L_D} \psi_{1d} \qquad X_q - X''_q = \omega_r \frac{L_{aq}^2}{L_G}$$

$$T''_{q0} = \frac{L_G - \frac{L_{aq}^2}{L_Q}}{R_{2q}}$$

$$E_{fd} \stackrel{\text{def}}{=} \frac{\omega_r L_{ad}}{R_{fd}} e_{fd}$$

Eliminate all but E, X, T, i_d, i_q

(modeled as a **current source** having internal voltages behind a reactance)

$$T'_{d0} \frac{d}{dt} E'_q = -\frac{X_d - X''_d}{X'_d - X''_d} \cdot E'_q + \frac{X_d - X'_d}{X'_d - X''_d} \cdot E''_q + E_{fd}$$

$$T'_{q0} \frac{d}{dt} E'_d = -\frac{X_q - X''_q}{X'_q - X''_q} \cdot E'_d + \frac{X_q - X'_q}{X'_q - X''_q} \cdot E''_d$$

$$T''_{d0} \frac{d}{dt} E''_q = E'_q - E''_q - \left(X'_d - X''_d\right) \cdot i_d$$

$$T''_{q0} \frac{d}{dt} E''_d = E'_d - E''_d + \left(X'_q - X''_q\right) \cdot i_q$$

6th order (2.2) Detailed Model

$$
\begin{cases}
\dfrac{d\delta}{dt} = \Delta\omega_r \\
\dfrac{2H}{\omega_0}\dfrac{d\Delta\omega_r}{dt} = P_m - P_e - K_D\dfrac{\Delta\omega_r}{\omega_0} \\
T'_{d0}\dfrac{d}{dt}E'_q = -\dfrac{X_d - X''_d}{X'_d - X''_d}\cdot E'_q + \dfrac{X_d - X'_d}{X'_d - X''_d}\cdot E''_q + E_{fd} \\
T'_{q0}\dfrac{d}{dt}E'_d = -\dfrac{X_q - X''_q}{X'_q - X''_q}\cdot E'_d + \dfrac{X_q - X'_q}{X'_q - X''_q}\cdot E''_d \\
T''_{d0}\dfrac{d}{dt}E''_q = E'_q - E''_q - \left(X'_d - X''_d\right)\cdot i_d \\
T''_{q0}\dfrac{d}{dt}E''_d = E'_d - E''_d + \left(X'_q - X''_q\right)\cdot i_q
\end{cases}
$$

$$\tilde{E}_t = e_d + je_q = e^{-j(\delta-\pi/2)}\tilde{E}_g$$

$$\tilde{I}_t = i_d + ji_q = e^{-j(\delta-\pi/2)}\tilde{I}_g$$

$$e_d = -R_a i_d + X''_q i_q + E''_d$$

$$e_q = -R_a i_q - X''_d i_d + E''_q$$

$$P_e = e_d i_d + e_q i_q$$

δ – rad

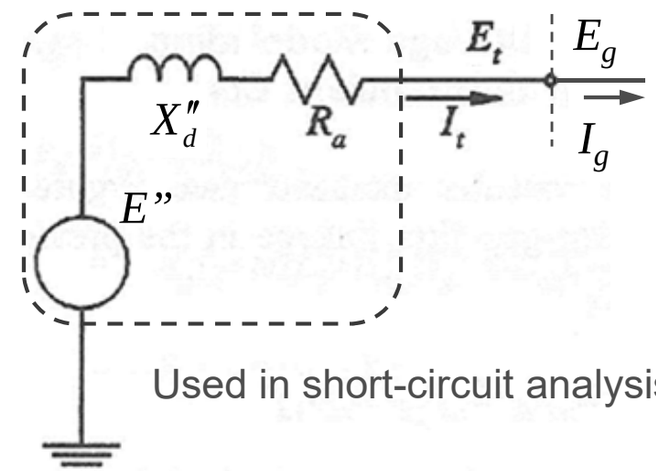
ω_r – rad/s

H, T', T'' – sec

P, E, V, I, i, X, L, R – pu

If $X''_d = X''_q$

$$\tilde{E}_t = \tilde{E}'' - (R_a + jX''_d)\tilde{I}_t$$



Used in short-circuit analysis

E_g and I_g are respectively E_t and I_t seen in the common reference of the system

4th order (1.1) Two-Axis Model

If T_{q0}'=0 , a 3rd order (1.0) 1-axis model is obtained.

- Ignore sub-transient dynamics (let i_{1d}=i_{2q}=0 or T''_{d0}=T''_{q0}=0)

$$\begin{cases} \dfrac{d\delta}{dt} = \Delta\omega_r \\ \dfrac{2H}{\omega_0}\dfrac{d\Delta\omega_r}{dt} = P_m - P_e - K_D\dfrac{\Delta\omega_r}{\omega_0} \\ T'_{d0}\dfrac{d}{dt}E'_q = -E'_q - (X_d - X'_d)\cdot i_d + E_{fd} \\ T'_{q0}\dfrac{d}{dt}E'_d = -E'_d + (X_q - X'_q)\cdot i_q \end{cases}$$

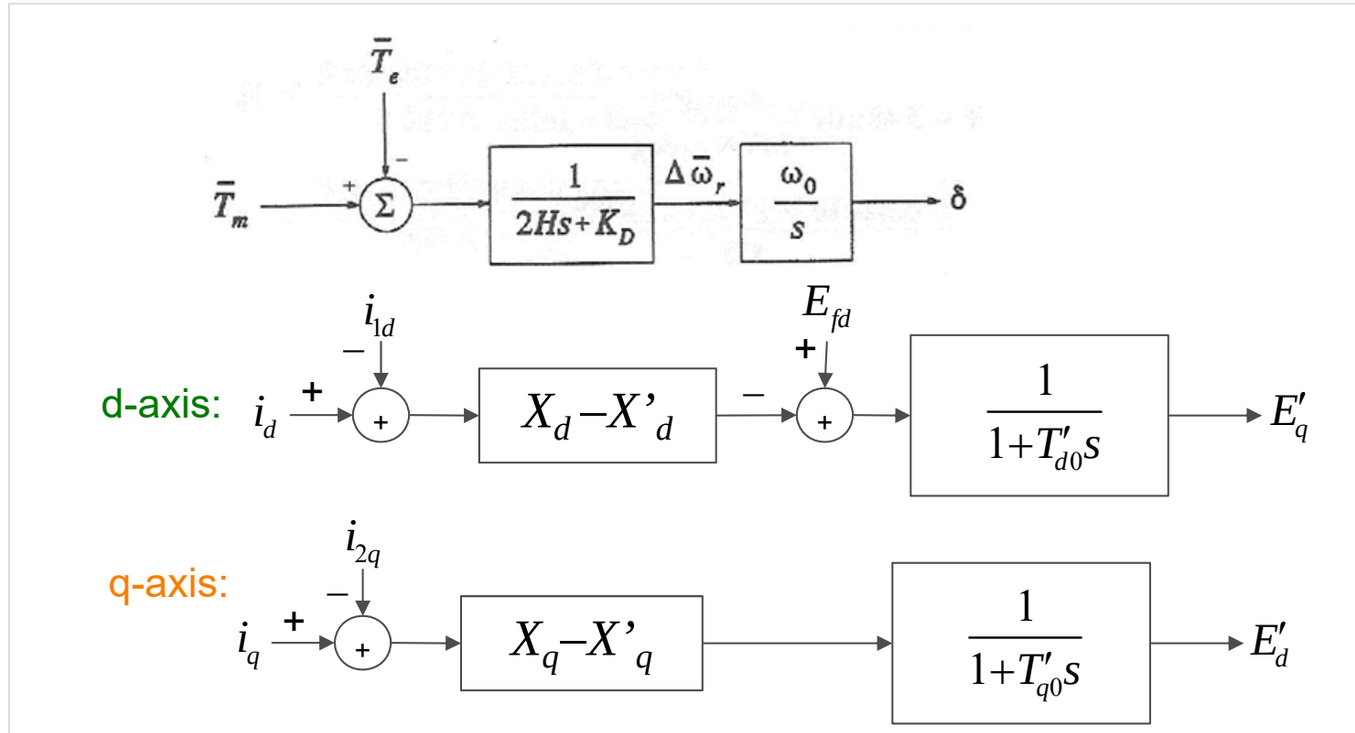
$$\tilde{E}_t = e_d + je_q = e^{-j(\delta-\pi/2)}\tilde{E}_g$$

$$\tilde{I}_t = i_d + ji_q = e^{-j(\delta-\pi/2)}\tilde{I}_g$$

$$e_d = -R_a i_d + X'_q i_q + E'_d$$

$$e_q = -R_a i_q - X'_d i_d + E'_q$$

$$P_e = e_d i_d + e_q i_q$$



δ – rad

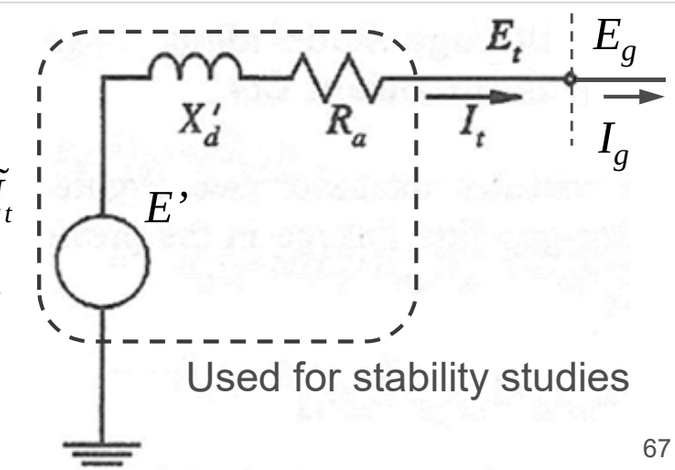
ω_r – rad/s

H, T', T'' – sec

P, E, V, I, i, X, R – pu

If $X'_d = X'_q$

$$\tilde{E}_t = \tilde{E}' - (R_a + jX'_d)\tilde{I}_t$$



2nd order (0.0) Classical Model

- Only keep swing equations as differential equations
- Assume $X'_d = X'_q$

$$\begin{cases} \dfrac{d\delta}{dt} = \Delta\omega_r \\ \dfrac{2H}{\omega_0}\dfrac{d\Delta\omega_r}{dt} = P_m - P_e - K_D\dfrac{\Delta\omega_r}{\omega_0} \end{cases}$$

$$\tilde{E}_g = e^{j(\delta-\pi/2)}\tilde{E}_t$$

$$\tilde{I}_g = e^{j(\delta-\pi/2)}\tilde{I}_t$$

$$\tilde{E}_t = \tilde{E}' - (R_a + jX'_d)\tilde{I}_t$$

$$P_e = real(\tilde{E}_t\tilde{I}_t^*)$$

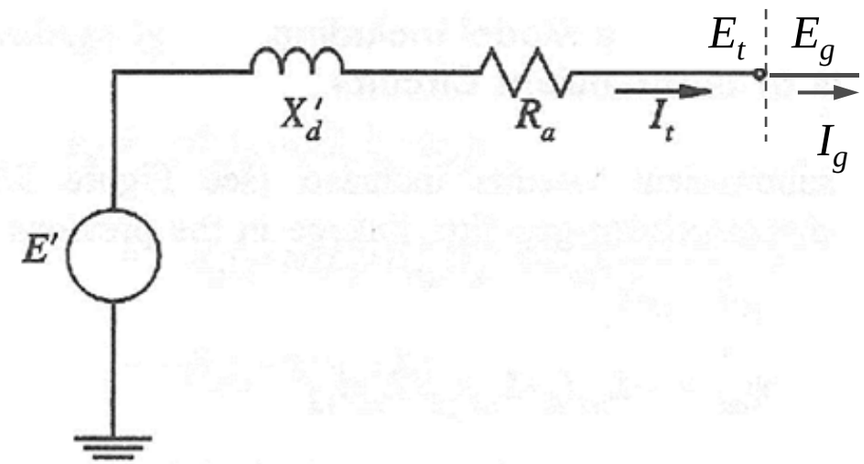
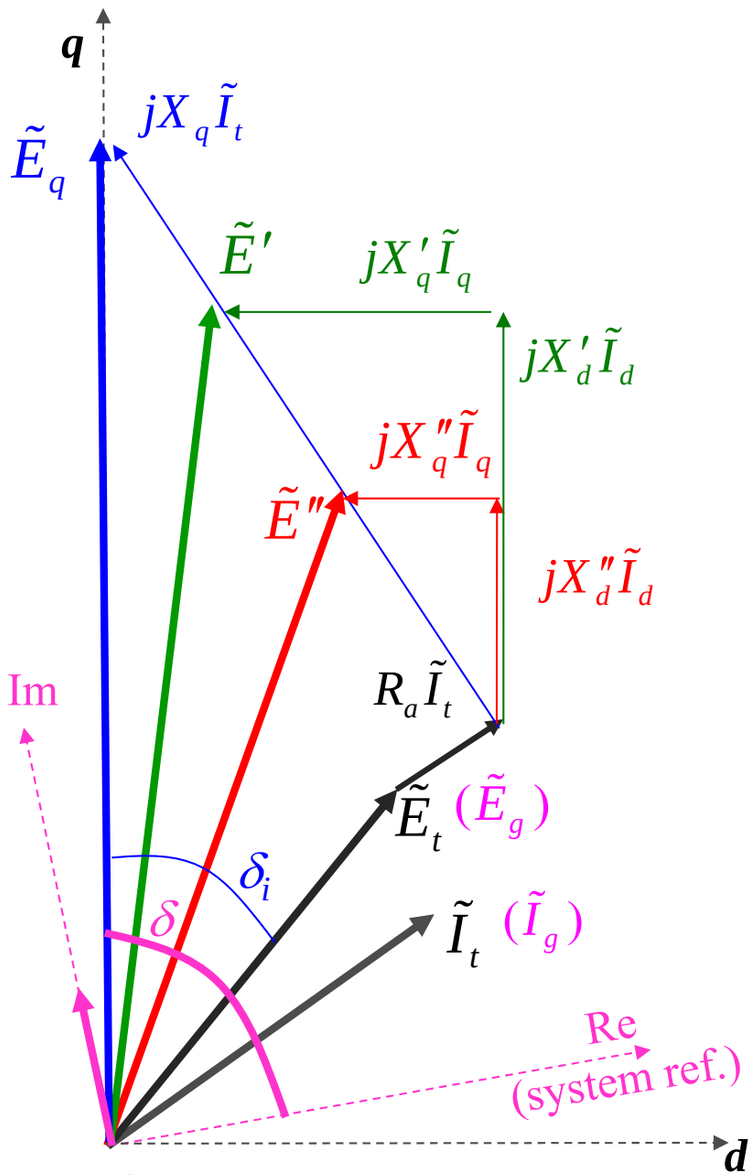


Figure 5.7 Simplified transient model

E' is constant and can be estimated by computing its pre-disturbance value

$$E' = |\tilde{E}_{t0} + (R_a + jX'_d)\tilde{I}_{t0}|$$

Phasor Diagram



$$\tilde{E}_g = \tilde{E}_t \times 1\angle(\delta - 90°)$$
$$\tilde{I}_g = \tilde{I}_t \times 1\angle(\delta - 90°)$$

Terminal voltage and current seen from the grid side

Steady state

$$e_d = -R_a i_d + X_q i_q$$
$$e_q = -R_a i_q - X_d i_d + X_{ad} i_{fd}$$

$$\tilde{E}_t = \tilde{E}_q - (R_a + jX_q)\tilde{I}_t$$

where $\tilde{E}_q = jX_{ad}i_{fd} - j(X_d - X_q)i_d = jX_{ad}i_{fd}$ (if $X_d = X_q = X_s$)

Transient dynamic

$$e_d = -R_a i_d + X'_q i_q + E'_d$$
$$e_q = -R_a i_q - X'_d i_d + E'_q$$

If $X'_d = X'_q$

$$\tilde{E}_t = \tilde{E}' - (R_a + jX'_d)\tilde{I}_t$$

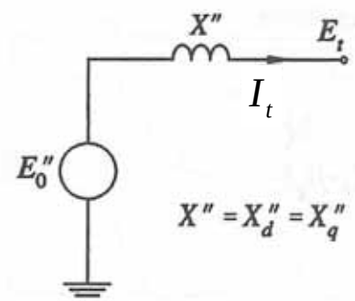
Sub-transient dynamic

$$e_d = -R_a i_d + X''_q i_q + E''_d$$
$$e_q = -R_a i_q - X''_d i_d + E''_q$$

If $X''_d = X''_q$

$$\tilde{E}_t = \tilde{E}'' - (R_a + jX''_d)\tilde{I}_t$$

Simplified models neglecting saliency effects

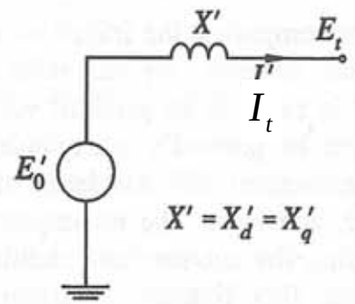


For short-circuit analysis

E_0'' is the predisturbance value of internal voltage given by

$$\tilde{E}_0'' = \tilde{E}_{t0} + jX''\tilde{I}_{t0}$$

(a) Subtransient model

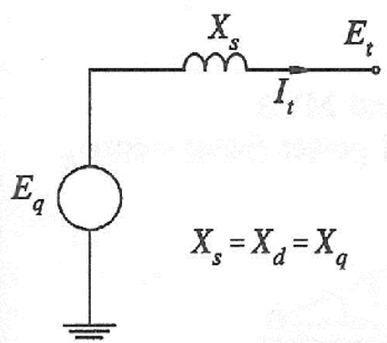


For stability studies

E_0' is the internal voltage

$$\tilde{E}_0' = \tilde{E}_{t0} + jX'\tilde{I}_{t0}$$

(b) Transient model



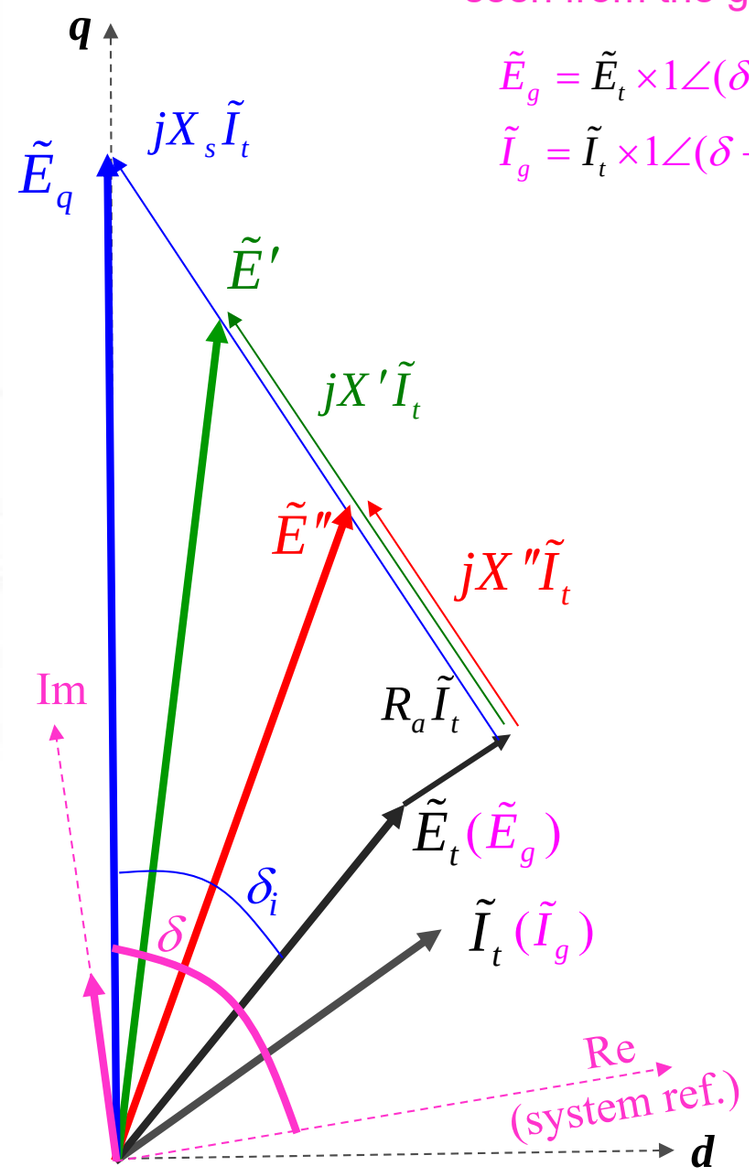
For steady-state analysis

$$\tilde{E}_q = \tilde{E}_{t0} + jX_s\tilde{I}_{t0}$$
$$|E_q| = X_{ad} i_{fd} = E_I$$

(c) Steady-state model

Terminal voltage and current seen from the grid side:

$$\tilde{E}_g = \tilde{E}_t \times 1\angle(\delta - 90°)$$
$$\tilde{I}_g = \tilde{I}_t \times 1\angle(\delta - 90°)$$



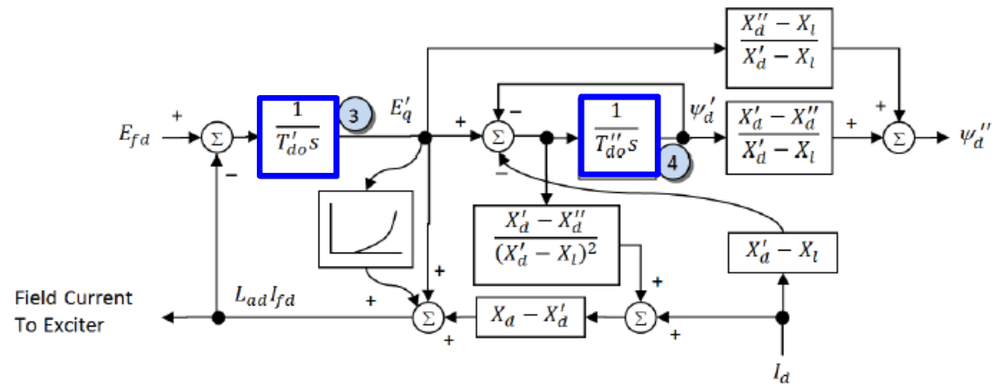
Comparison of PSS/E Generator Models

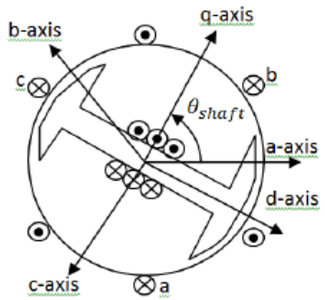
Table 1: Summary of Generator Models in Terms of Data Used

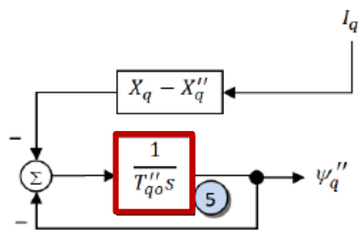
Reactance and Time Constants Used	Model				
	GENSAL and GENSAE	GENROU and GENROE	GENDCO	GENTRA	GENCLS
X_d	✔	✔	✔	✔	✔
X_q	✔	✔	✔	✔	
X'_d	✔	✔	✔	✔	✔
X'_q		✔	✔		
X''_d	✔	✔	✔		
X''_q	*	*	*		
X_l	✔	✔	✔	✔	
T'_{do}	✔	✔	✔	✔	
T'_{qo}		✔	✔		
T''_{do}	✔	✔	✔	✔	
T''_{qo}	✔	✔	✔	✔	
Saturation Factors	✔	✔	✔	✔	

*X''_q is assumed to be equal to X''_d.

2.1 Machine: GENSAL model with saturation.







Source: J. Weber, "Description of Machine Models GENROU, GENSAL, GENTPF and GENTPJ," PowerWorld, Oct 2015

Electrical Torque

$$\psi_q = \psi''_q - I_q X''_d$$

$$\psi_d = \psi''_d - I_d X''_d$$

$$T_{elec} = \psi_d I_q - \psi_q I_d = \psi''_d I_q - \psi''_q I_d$$

Mechanical Swing Equations

1. $\dot{\delta} = \omega * \omega_0$
2. $\dot{\omega} = \frac{1}{2H}\left(\frac{P_{mech} - D\omega}{1+\omega} - T_{elec}\right)$

ω = per unit speed deviation, so $\omega = 0$ means we are at synchronous speed and $\omega = 1$ would mean it's spinning at double synchronous speed

ω_0 = synchronous speed $2\pi f_0$ where f_0 is the nominal system frequency in Hz

Note: If option *Ignore Speed Effects in Generator Swing Equation* is true, then instead use

$$\dot{\omega} = \frac{1}{2H}(P_{mech} - D\omega - T_{elec})$$

Network Interface Equations

$$Z_{source} = R_a + jX''_d$$

$$Y_{source} = \frac{1}{R_a + jX''_d} = G + jB$$

$$V_d + jV_q = \frac{d\left(\psi''_d + j\psi''_q\right)}{dt}$$

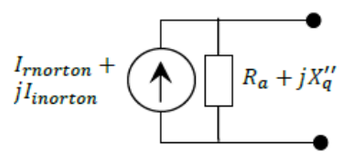
$$V_d + jV_q = j(1+\omega)\left(\psi''_d + j\psi''_q\right)$$

$$V_d + jV_q = \left(-\psi''_q + j\psi''_d\right)(1+\omega)$$

$$I_{dnorton} + jI_{qnorton} = (V_d + jV_q)(G + jB)$$

Convert to Network Reference

$$I_{rnorton} + jI_{inorton} = (I_d + jI_q)e^{j(\delta - \frac{\pi}{2})}$$



2.3 Machine: GENROU model with saturation

The GENROU model including saturation is presented below. The Mechanical Swing Equations, Electric Torque, and Network Interface Equations are all the same as the GENROU and GENSAL models previously.