ECE 522
Power Systems Analysis II

3 – Small-Signal Stability

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References:

– EPRI Dynamic Tutorial
– Chapters 12 and 17 of Kundur’s “Power System Stability and Control”
– Chapter 3 of Anderson’s “Power System Control and Stability”
– Other references
3.1 Small-Signal Stability: Overview and Model-based Methods
Power Oscillations

- The power system naturally enters periods of oscillation as it continually adjusts to new operating conditions or experiences other disturbances.
- Typically, oscillations have a small amplitude and do not last long.
- When the oscillation amplitude becomes large or the oscillations are sustained, a response is required:
  - A system operator may have the opportunity to respond and eliminate harmful oscillations or,
  - less desirably, protective relays may activate to trip system elements.

Figure 8-1. Rubber Band – Weight Analogy
Blackout on August 10, 1996

1. Initial event (15:42:03):
   Short circuit due to tree contact → Outages of 6 transformers and lines

2. Vulnerable conditions (minutes)
   Low-damped inter-area oscillations → Outages of generators and tie-lines

3. Blackouts (seconds)
   Unintentional separation → Loss of 24% load

Malin-Round Mountain #1 MW

- 0.276 Hz oscillation damping ratio >7%
- 0.264 Hz oscillation damping ratio =3.46%
- 0.252 Hz oscillation damping ratio ≈1%

Transient instability (blackouts)
Small Signal Stability

- **Small signal stability** (also referred to as small-disturbance stability) is the ability of a power system to maintain synchronism when subjected to small disturbances
  - In this context, a disturbance is considered to be small if the equations that describe the resulting response of the system may be linearized for the purpose of analysis
  - It is convenient to assume that the disturbances causing the changes disappear (details on the disturbance are not important)
  - The system is stable if it returns to its original state, i.e. a stable equilibrium point (only the behaviors in a neighborhood of the equilibrium are concerned)
  - Such a behavior can be determined in the linearized model of the power system
Characteristics of Small-Signal Stability Problems

- **Inter- or intra-area modes** (0.1-0.7Hz): machines in one part of the system swing against machines in other parts
  - **Inter-area model** (0.1-0.3Hz): involving all the generators in the system; the system is essentially split into two parts, with generators in one part swinging against machines in the other parts.
  - **Intra-area mode** (0.4-0.7Hz): involving subgroups of generators swinging against each other.

- **Local or machine-system modes** (0.7-2Hz): oscillations involve a small part of the system
  - **Local plant modes**: associated with rotor angle oscillations of a single generator or a single plant against the rest of the system; similar to the single-machine-infinite bus system
  - **Inter-machine or interplant modes**: associated with oscillations between the rotors of a few generators close to each other

- **Control or torsional modes** (2Hz – )
  - Due to inadequate tuning of the control systems, e.g. generator excitation systems, HVDC converters and SVCs, or torsional interaction (sub-synchronous resonance) with power system control
High & Low Frequency Oscillations

• Whenever power flows, $I^2R$ losses occur. These energy losses help to reduce the amplitude of the oscillation.

• The higher the frequency of the oscillation, the faster it is damped. High frequency ($>1.0$ HZ) oscillations are damped more rapidly than low frequency ($<1.0$ HZ) oscillations.

• Power system operators do not want any oscillations. However, when oscillations occur, it is better to have high frequency oscillations than low frequency oscillations.

• The power system can naturally dampen high frequency oscillations. Low frequency oscillations are more damaging to the power system, which may exist for a long time, become sustained (undamped) oscillations, and even trigger protective relays to trip elements.
SMIB System in the Classic Model

With resistance neglected:

\[ P_e + jQ_e = P_t + jQ_t \]
\[ E' \angle \delta \]
\[ (P_e = T_e) \]

Linearize swing equations at \( \delta = \delta_0 \):

\[ \Delta T_e = \frac{\partial T_e}{\partial \delta} \Delta \delta = \frac{E'E_B}{X_T} \cos \delta_0 \times \Delta \delta \]

Synchronizing torque coefficient:

\[ K_s = \frac{E'E_B}{X_T} \cos \delta_0 = P_{max} \cos \delta_0 \]

Note: all in p.u. except for \( \delta \) (rad), \( H \) (s) and time (s).
State-space representation

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} \Delta \omega_r \\ \Delta \delta \end{bmatrix} &= \begin{bmatrix} -\frac{K_D}{2H} & -\frac{K_S}{2H} \\ \omega_0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \omega_r \\ \Delta \delta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Delta T_m \\
\frac{d^2 \Delta \delta}{dt^2} + \frac{K_D}{2H} \frac{d \Delta \delta}{dt} + \frac{K_S \omega_0}{2H} \Delta \delta &= \frac{\omega_0 \Delta T_m}{2H}
\end{align*}
\]

- Apply Laplace Transform:

\[
(s^2 + \frac{K_D}{2H} s + \frac{K_S \omega_0}{2H}) \Delta \delta = \frac{\omega_0 \Delta T_m}{2H}
\]

- Characteristic equation:

\[
s^2 + \frac{K_D}{2H} s + \frac{K_S \omega_0}{2H} = 0
\]

Note: \( D \) (p.u.) = \( K_D \) for this SMIB system
Compared to a damped harmonic oscillator

\[ s^2 + \frac{K_D}{2H} s + \frac{K_S \omega_0}{2H} = 0 \quad (K_S = \frac{E' E_B}{X_T} \cos \delta_0 = P_{\text{max}} \cos \delta_0) \]

- General harmonic oscillator system:
  \[ s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \]
  \( \zeta \) - Damping ratio
  \( \omega_n \) - Nature frequency

- The zero-input response is a damped sinusoidal oscillation:
  \[ x(t) = Ae^{\sigma t} \sin(\omega t + \varphi) = Ae^{-\zeta \omega_n t} \sin(\omega_n t \sqrt{1-\zeta^2} + \varphi) \]
  \( \tau = 1 / \sigma = \frac{1}{\zeta \omega_n} \)

\( \sigma \) determines the time of decaying to \( 1/e = 36.8\% \)

Damping angle \( \theta = \cos^{-1} \zeta = \tan^{-1} \left( \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \)
Oscillation Frequency and Damping of an SMIB System

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]

\[ s_1, s_2 = \sigma \pm j\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \]

\[ s^2 + \frac{K_D}{2H} s + \frac{K_S\omega_0}{2H} = 0 \]

\[ (K_S = \frac{E'E_B}{X_T}\cos\delta_0 = P_{\text{max}}\cos\delta_0) \]

\[ \omega_n = \sqrt{\sigma^2 + \omega^2} = \sqrt{K_S \frac{\omega_0}{2H}} = \sqrt{\frac{\omega_0 E'E_B \cos\delta_0}{2HX_T}} \]

\[ \zeta = \frac{-\sigma}{\sqrt{\sigma^2 + \omega^2}} = \frac{1}{2} \frac{K_D}{\sqrt{K_S 2H\omega_0}} = K_D \sqrt{\frac{X_T}{8\omega_0 H E'E_B \cos\delta_0}} \]

\[ \omega = \omega_n\sqrt{1-\zeta^2} = \sqrt{K_S \frac{\omega_0}{2H} - \frac{K_D^2}{16H^2}} \]

\[ \sigma = -\zeta\omega_n = -\frac{K_D}{4H} \]

- How does \( \omega_n \) change?
  - If \( H \downarrow \) (lower inertia)
  - If \( X_T \downarrow \) (stronger transmission)
  - If \( \delta_0 \downarrow \) (lower loading)
System Response after a Small Disturbance

\[
\frac{d}{dt} \left[ \Delta \delta \right] = \begin{bmatrix} 0 & \omega_0 \\ -K_s / 2H & -K_d / 2H \end{bmatrix} \left[ \Delta \delta \right] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{\Delta T_m}{2H}
\]

\[
x_1 = \Delta \delta \quad x_2 = \Delta \omega_r = \frac{\Delta \delta}{\omega_0}
\]

\[
\Delta u = \frac{\Delta T_m}{2H}
\]

\[
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_n^2 / \omega_0 & -2\zeta \omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u
\]

\[
\dot{x}(t) = Ax(t) + B\Delta u(t)
\]

\[
y(t) = Cx(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

(Assuming the angle and speed to be directly measured)

Apply Laplace transform:

\[
sX(s) - x(0) = AX(s) + BU(s)
\]

\[
\Delta U(s) = \frac{\Delta u}{s}
\]

\[
Y(s) = X(s) = (sI - A)^{-1} \left[ x(0) + BU(s) \right]
\]

Zero-input Zero-state

\[
\begin{bmatrix} s + 2\zeta \omega_n & \omega_0 \\ -\omega_n^2 / \omega_0 & s \end{bmatrix}
\]

\[
\begin{bmatrix} x(0) + BU(s) \end{bmatrix}
\]

Zero-input Zero-state

\[
\begin{bmatrix} \Delta \delta(s) \\ \Delta \omega_r(s) \end{bmatrix} = \begin{bmatrix} s + 2\zeta \omega_n & \omega_0 \\ -\omega_n^2 / \omega_0 & s \end{bmatrix} \begin{bmatrix} \Delta \delta(0) \\ \Delta \omega_r(0) \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta u \end{bmatrix}
\]

(Assuming the angle and speed to be directly measured)
\[
\begin{bmatrix}
\Delta \delta(s) \\
\Delta \omega_r(s)
\end{bmatrix} = 
\begin{bmatrix}
s + 2\zeta \omega_n & \omega_0 \\
-\omega_n^2 / \omega_0 & s
\end{bmatrix}
\begin{bmatrix}
\Delta \delta(0) \\
\Delta \omega_r(0)
\end{bmatrix} + 
\begin{bmatrix}
0 \\
\Delta u/s
\end{bmatrix}
\]

\[
\Delta \omega_r(s) = \Delta \delta / \omega_0 = (\omega_r - \omega_0) / \omega_0 \quad \text{in pu}
\]
\[
\Delta u = \frac{\Delta T_m}{2H} \quad \text{pu}
\]

Zero-input response

- E.g. when the rotor is suddenly perturbed by a small angle \( \Delta \delta(0) \neq 0 \) and assume \( \Delta \omega_r(0) = 0 \)

\[
\Delta \delta(s) = \frac{(s + 2\zeta \omega_n)\Delta \delta(0)}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]
\[
\Delta \omega_r(s) = -\frac{\omega_n^2 \Delta \delta(0) / \omega_0}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

Zero-state response

- E.g. when there is a small increase in mechanical torque \( \Delta T_m = \Delta P_m \) in pu

\[
\Delta \delta(s) = \frac{\omega_0 \Delta u}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)}
\]
\[
\Delta \omega_r(s) = \frac{\Delta u}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

\[\Delta \delta \text{ in rad} = \frac{\Delta \delta(0)}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega t + \theta)\]
\[\Delta \omega_r \text{ in rad/s} = -\frac{\omega_n \Delta \delta(0)}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega t\]

\[\Delta \delta \text{ in rad} = \frac{\omega_0 \Delta T_m}{2H \omega_n^2} \left[ 1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin(\omega t + \theta) \right]\]
\[\Delta \omega_r \text{ in rad/s} = \frac{\omega_0 \Delta T_m}{2H \omega_n \sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega t\]

Damping angle: \( \theta = \cos^{-1} \zeta \)
Example

- Exp. 11.2 & 11.3 in Saadat’s book
- $H=9.94\,s$, $K_D=0.138\,pu$, $T_m=0.6\,pu$ with PF=0.8.
  Find the responses of the rotor angle and frequency under these disturbances

  (1) $\Delta \delta(0)=10^o=0.1745\,\text{rad}$
  (2) $\Delta P_e=0.2\,pu$

**Zero-input response:** $\Delta \delta(0)=10^o$

$\delta(0)=16.79+10=26.79^o$

**Zero-state response:** $\Delta P_e=0.2\,pu$

$\delta(\infty)=16.79+5.76=22.55^o$
Small-signal stability of a multi-machine system

A power system can be described by

\[ \dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r) \quad i = 1, 2, \ldots, n \]

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\quad
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_r
\end{bmatrix}
\quad
\begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_n
\end{bmatrix}
\]

\[ \dot{x} = f(x, u) \]
Linearization at the Equilibrium Point

\[
\dot{x}_0 = f(x_0, u_0) = 0
\]

Let us perturb the system from the above state by letting
\[
x = x_0 + \Delta x \quad u = u_0 + \Delta u
\]
\[
\dot{x} = \dot{x}_0 + \Delta \dot{x} = f\left((x_0 + \Delta x), (u_0 + \Delta u)\right)
\]

For small perturbations, the nonlinear function \( f \) can be expressed in terms of Taylor’s series expansion. Then, neglect 2\(^{nd}\) - and higher-order terms and only keep 1\(^{st}\) order partial derivatives:

\[
\dot{x}_i = \dot{x}_{i0} + \Delta \dot{x}_i = f_i\left((x_0 + \Delta x), (u_0 + \Delta u)\right)
\]
\[
= f_i(x_0, u_0) + \frac{\partial f_i}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f_i}{\partial x_n} \Delta x_n + \frac{\partial f_i}{\partial u_1} \Delta u_1 + \cdots + \frac{\partial f_i}{\partial u_r} \Delta u_r
\]

\[
\Delta \dot{x}_i = \frac{\partial f_i}{\partial x_1} \Delta x_1 + \cdots + \frac{\partial f_i}{\partial x_n} \Delta x_n + \frac{\partial f_i}{\partial u_1} \Delta u_1 + \cdots + \frac{\partial f_i}{\partial u_r} \Delta u_r \quad i = 1, 2, \cdots, n
\]
\[ \Delta \dot{x} = A \Delta x + B \Delta u \]

A is the Jacobin matrix of \( f \)

\[
A = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_r} \\
\frac{\partial f_2}{\partial u_1} & \cdots & \frac{\partial f_2}{\partial u_r} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_r}
\end{bmatrix}
\]

Take the Laplace transform to obtain the state equations in the frequency domain

\[ s \Delta X(s) - \Delta X(0) = A \Delta X(s) + B \Delta U(s) \]

\[ (sI - A) \Delta X(s) = \Delta X(0) + B \Delta U(s) \]

\[ \Delta X(s) = (sI - A)^{-1} \begin{bmatrix}
\Delta X(0) \\
\hline
Zero-Input \\
\end{bmatrix} + \begin{bmatrix}
\Delta U(s) \\
\hline
Zero-State \\
\end{bmatrix} \]
Eigen-values

\[ \Delta X(s) = (sI - A)^{-1} \left[ \Delta X(0) + B \Delta U(s) \right] \]

\[ = \frac{\text{adj}(sI - A)}{\det(sI - A)} \left[ \Delta X(0) + B \Delta U(s) \right] \]

\[ \det(A - \lambda I) = 0 \quad \text{i.e. Characteristic equation of } A \]

Poles of \( \Delta X(s) \) \( \rightarrow \) Eigenvalues of \( A \), i.e. \( \lambda = \lambda_1 \sim \lambda_n \)

For real \( A \), complex eigenvalues, if any, always occur in conjugate pairs.

- For the SMIB system:

\[
\begin{bmatrix}
\Delta \delta(s) \\
\Delta \omega_r(s)
\end{bmatrix} = \begin{bmatrix}
s + 2\zeta \omega_n & \omega_0 \\
-\omega_n^2/\omega_0 & s
\end{bmatrix} \begin{bmatrix}
\Delta \delta(0) \\
\Delta \omega_r(0)
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{\Delta u}{s}
\end{bmatrix} \quad \lambda_1, \lambda_2 = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2}
\]
Eigen-vectors

For any $\lambda_i$, the column vector $\phi_i$ satisfying $A\phi_i = \lambda_i \phi_i$ is called a right eigenvector of $A$ associated with $\lambda_i$.

Modal matrix $\Phi = [\phi_1, \phi_2, \cdots, \phi_n]$

$$A\Phi = \Phi \Lambda \quad \Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

$\Phi^{-1}A\Phi = \Lambda$ if $A$ has distinct eigenvalues (usually true for a real-world system)

Similarly the row vector $\psi_j$

$$\psi_j A = \lambda_j \psi_j \quad \rightarrow \quad \text{Left eigenvector associated with } \lambda_j$$

$$\Psi = \begin{bmatrix} \psi_1^T, \psi_2^T, \cdots, \psi_n^T \end{bmatrix}^T$$

The left and right eigenvectors corresponding to different eigenvalues are orthogonal (row vectors of $\Phi^{-1}$ are left eigenvectors of $A$, so we may let $\Psi = C \Phi^{-1}$ where $C$ is a diagonal matrix or simply equal to $I$ if normalized)

$$\Psi\Phi = I \quad \Leftrightarrow \quad \psi_j \phi_i = 0 \text{ if } i \neq j, \quad \text{or} \quad \psi_i \phi_i = 1$$
Free (zero-input) response

\[ \Delta \dot{x} = A \Delta x \rightarrow \text{Linearized system without external forcing} \]

In order to eliminate the cross-coupling between the state variables, consider a new state vector \( \mathbf{z} \)

\[ \dot{z}_i = \lambda_i z_i \rightarrow z_i(t) = z_i(0)e^{\lambda_i t} \]

\[ \dot{\mathbf{z}} = \Lambda \mathbf{z} = \Phi^{-1} \mathbf{A} \Phi \mathbf{z} \iff \Phi \dot{\mathbf{z}} = \mathbf{A} \Phi \mathbf{z} \]

\[ \iff \Delta \mathbf{x}(t) = \Phi \mathbf{z}(t) = [\phi_1 \phi_2 \cdots \phi_n] \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \sum_{i=1}^{n} \phi_i z_i(0)e^{\lambda_i t} \]

Free response is a linear combination of \( n \) modes.

\[ \Delta \mathbf{x}(t) = \sum_{i=1}^{n} \phi_i \psi_i \Delta \mathbf{x}(0)e^{\lambda_i t} = \sum_{i=1}^{n} \phi_i c_i e^{\lambda_i t} \]

\[ \Delta \mathbf{x}_k(t) = \sum_{i=1}^{n} \phi_{ki} c_i e^{\lambda_i t} = \phi_{k1} c_1 e^{\lambda_1 t} + \cdots + \phi_{kn} c_n e^{\lambda_n t} \]

\[ \mathbf{z}(t) = \Phi^{-1} \Delta \mathbf{x}(t) = \Psi \Delta \mathbf{x}(t) \]

\[ z_i(0) = \psi_i \Delta \mathbf{x}(0) = c_i \]

(Magnitude of excitation of the \( i \)th mode)
Eigen-values and stability

\[ \Delta x(t) = \sum_{i=1}^{n} \phi_i z_i(t) = \sum_{i=1}^{n} \phi_i z_i(0)e^{\lambda_i t} = \sum_{i=1}^{n} \phi_i c_i e^{\lambda_i t} \]

\[ \Delta x_k(t) = \sum_{i=1}^{n} \phi_{ki} c_i e^{\lambda_i t} = \phi_{ki} c_1 e^{\lambda_1 t} + \ldots + \phi_{kn} c_n e^{\lambda_n t} \]

- Each eigenvalue \( \lambda = \sigma \pm j\omega \)
  - A real eigenvalue \( (\omega = 0) \) corresponds to a non-oscillatory mode.
    - \( \sigma < 0 \): a decaying mode.
    - \( \sigma > 0 \): aperiodic instability.
  - Complex eigenvalues \( (\omega \neq 0) \) occur in conjugate pairs; each pair corresponds to an oscillatory mode
    - Frequency of oscillation in Hz: \( f = \omega / 2\pi \)
    - Damping ratio (rate of decay) of the oscillation amplitude
      \[ \xi = \frac{-\sigma}{\sqrt{\sigma^2 + \omega^2}} \]
<table>
<thead>
<tr>
<th>Eigenvalues ($\lambda = \sigma \pm j\omega$)</th>
<th>Trajectory</th>
<th>Type of singularity</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Eigenvalues Diagram" /></td>
<td><img src="image2" alt="Trajectory Diagram" /></td>
<td>Stable focus</td>
</tr>
<tr>
<td><img src="image3" alt="Eigenvalues Diagram" /></td>
<td><img src="image4" alt="Trajectory Diagram" /></td>
<td>Unstable focus</td>
</tr>
<tr>
<td><img src="image5" alt="Eigenvalues Diagram" /></td>
<td><img src="image6" alt="Trajectory Diagram" /></td>
<td>Vortex</td>
</tr>
</tbody>
</table>
Stable node

Unstable node

Saddle
Mode Shape and Mode Composition

\[ \Delta x(t) = \Phi z(t) = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix} \begin{bmatrix} z_1(t), z_2(t), \ldots, z_n(t) \end{bmatrix}^T \]

\[ \Delta x_k(t) = \sum_{i=1}^{n} \phi_{ki} z_i(t) \]

\[ z(t) = \Psi \Delta x(t) = \begin{bmatrix} \psi_1^T, \psi_2^T, \cdots, \psi_n^T \end{bmatrix}^T \begin{bmatrix} \Delta x_1(t), \Delta x_2(t), \ldots, \Delta x_n(t) \end{bmatrix}^T \]

\[ z_i(t) = \sum_{k=1}^{n} \psi_{ik} \Delta x_k(t) \]

• The variables \( \Delta x_1, \Delta x_2, \ldots, \Delta x_n \) are original state variables chosen to represent the dynamic performance of the system.

• Variables \( z_1, z_2, \ldots, z_n \) are transformed state variables; each is associated with only one mode. In other words, they are directly related to the modes.

• The right eigenvector \( \phi_i \) gives the **mode shape** of the \( i^{th} \) mode, i.e. the relative activity of the original state variables when the \( i^{th} \) mode is excited: The \( k^{th} \) element of \( \phi_i \), i.e. \( \phi_{ki} \), measures the activity of state variable \( x_k \) in the \( i^{th} \) mode.

• The left eigenvector \( \psi_i \) gives the **mode composition** of the \( i^{th} \) mode, i.e. what weighted composition of original state variables is needed to construct the mode: The \( k^{th} \) element of \( \psi_i \), i.e. \( \psi_{ik} \), weights the contribution of \( x_k \)'s activity to the \( i^{th} \) mode.
Participation factor

\[ \Psi \Phi = I \rightarrow (\Psi \Phi)_{ii} = \sum_{k=1}^{n} \psi_{ik} \phi_{ki} \triangleq \sum_{k=1}^{n} p_{ik} = 1 \]

- \( \phi_{ki} \) measures the activity of \( x_k \) in the \( i^{th} \) mode
- \( \psi_{ik} \) weights the contribution of this activity to the mode
- **Participation factor** \( p_{ki} = \psi_{ik} \phi_{ki} \) measures the participation of the \( k^{th} \) state variable \( x_k \) in the \( i^{th} \) mode.
- \( p_{ki} \) is dimensionless and hence invariant under changes of scale on the variables
- **Question:** considering \( \Psi = \Phi^{-1} \), why do we have to define the mode shape, mode composition, and participation factors, separately?

Learn Kundur’s Example 12.2 on a SMIB system
Oscillation Modes of a Multi-machine System in the Classic Model

\[
\frac{2H_i}{\omega_0} \frac{d^2 \delta_i}{dt^2} = P_{mi} - P_{ei} \quad i = 1, 2, \ldots, n \quad \text{(Ignoring damping)}
\]

\[
P_{ei} = E_i^2 G_{ii} + \sum_{j=1 \atop j \neq i}^{n} E_i E_j Y_{ij} \cos(\theta_{ij} - \delta_{ij}) = E_i^2 G_{ii} + \sum_{j=1 \atop j \neq i}^{n} E_i E_j \left( B_{ij} \sin \delta_{ij} + G_{ij} \cos \delta_{ij} \right)
\]

\[
\delta_{ij} = \delta_i - \delta_j, \quad B_{ij} = Y_{ij} \sin \theta_{ij}, \quad G_{ij} = Y_{ij} \cos \theta_{ij}
\]

Linearization at \( \delta_{ij0} \):

\[
P_{ei\Delta} \equiv P_{ei} - P_{ei0} \approx \sum_{j=1 \atop j \neq i}^{n} E_i E_j \left( B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right) \delta_{ij\Delta} = \sum_{j=1 \atop j \neq i}^{n} K_{sij} \delta_{ij\Delta}
\]

\[
K_{sij} \equiv \left. \frac{\partial P_{ij}}{\partial \delta_{ij}} \right|_{\delta_{ij0}} = E_i E_j \left( B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right) \quad \text{Synchronizing power coefficient}
\]

\[
\frac{2H_i}{\omega_0} \frac{d^2 \delta_{ij\Delta}}{dt^2} + \sum_{j=1 \atop j \neq i}^{n} E_i E_j \left( B_{ij} \cos \delta_{ij0} - G_{ij} \sin \delta_{ij0} \right) \delta_{ij\Delta} = 0 \quad i = 1, 2, \ldots, n
\]

\[
\Leftrightarrow \quad \frac{2H_i}{\omega_0} \frac{d^2 \delta_{ij\Delta}}{dt^2} + \sum_{j=1 \atop j \neq i}^{n} K_{sij} \delta_{ij\Delta} = 0 \quad i = 1, 2, \ldots, n
\]
\[
\frac{2H_i}{\omega_0} \frac{d^2 \delta_{i\Delta}}{dt^2} + \sum_{j=1\atop j \neq i}^{n} K_{sij} \delta_{ij\Delta} = 0 \quad i = 1, 2, \cdots, n
\]

**Note:** There are only \((n-1)\) independent equations because \(\sum \delta_{ij} = 0\), so we need to formulate the \((n-1)\) independent relative rotor angle equations with one reference machine, e.g., the \(n\text{-th}\) machine.

\[
\frac{d^2 \delta_{i\Delta}}{dt^2} - \frac{d^2 \delta_{n\Delta}}{dt^2} + \frac{\omega_0}{2H_i} \sum_{j=1\atop j \neq i}^{n} K_{sij} \delta_{ij\Delta} - \frac{\omega_R}{2H_n} \sum_{j=1}^{n-1} K_{snj} \delta_{nj\Delta} = 0, \quad i = 1, \cdots, n-1
\]

Consider each \(\delta_{in\Delta} = \delta_{i\Delta} - \delta_{n\Delta}\)

\[
\frac{d^2 \delta_{in\Delta}}{dt^2} + \left( \frac{\omega_0}{2H_i} \sum_{j=1\atop j \neq i}^{n} K_{sij} + \frac{\omega_0}{2H_n} K_{sni} \right) \delta_{in\Delta} + \sum_{j=1\atop j \neq i}^{n-1} \left( \frac{\omega_0}{2H_n} K_{snj} - \frac{\omega_0}{2H_i} K_{sij} \right) \delta_{jn\Delta} = 0, \quad i = 1, \cdots, n-1
\]

\[
\frac{d^2 \delta_{in\Delta}}{dt^2} + \sum_{j=1}^{n-1} \alpha_{ij} \delta_{jn\Delta} = 0 \quad i = 1, 2, \cdots, n-1
\]

\[
\alpha_{ii} = \frac{\omega_0}{2H_i} \sum_{j=1\atop j \neq i}^{n} K_{sij} + \frac{\omega_0}{2H_n} K_{sni} \quad \alpha_{ij} = \frac{\omega_0}{2H_n} K_{snj} - \frac{\omega_0}{2H_i} K_{sij}
\]
State-space representation

Let \( x_1, x_2, \ldots, x_{n-1} = \delta_{1n\Delta}, \delta_{2n\Delta}, \ldots, \delta_{(n-1)n\Delta} \) and

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n \\
\dot{x}_{n+1} \\
\vdots \\
\dot{x}_{2n-2}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n \\
x_{n+1} \\
\vdots \\
x_{2n-2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{n-1} \\
\dot{x}_n \\
\dot{x}_{n+1} \\
\vdots \\
\dot{x}_{2n-2}
\end{bmatrix}
= \begin{bmatrix}
-\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1(n-1)} \\
-\alpha_{21} & -\alpha_{22} & \cdots & -\alpha_{2(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
-\alpha_{(n-1)1} & -\alpha_{(n-1)2} & \cdots & -\alpha_{(n-1)(n-1)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n \\
x_{n+1} \\
\vdots \\
x_{2n-2}
\end{bmatrix}
\]

- Its characteristic equation \(|\lambda^2 I - A| = 0\) has \(2(n-1)\) imaginary roots, which occur in \((n-1)\) complex conjugate pairs
- This \(n\)-machine system has \((n-1)\) oscillation modes

Read Anderson’s Examples 3.2 and 3.3 about the linearization and eigen-analysis on the IEEE 9-bus system
Eigenanalysis of the IEEE 9-Bus System

• System model:
  - Classic generator model
  - Two loading conditions:
    ▪ Light load (LL)
    ▪ Heavy load (HL)
  - Two scenarios:
    ▪ N-0
    ▪ N-1 (line 5-7 is tripped)

• Linearized model
  - 6 state variables:
    \( (\Delta \omega_1, \Delta \delta_1, \Delta \omega_2, \Delta \delta_2, \Delta \omega_3, \Delta \delta_3) \)

• Cases summary

<table>
<thead>
<tr>
<th>Case 1</th>
<th>LL N-0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 2</td>
<td>LL N-1</td>
</tr>
<tr>
<td>Case 3</td>
<td>HL N-0</td>
</tr>
<tr>
<td>Case 4</td>
<td>HL N-1</td>
</tr>
</tbody>
</table>

Table I. Parameters and initial conditions for the system the classical model

<table>
<thead>
<tr>
<th></th>
<th>Generator 1</th>
<th>Generator 2</th>
<th>Generator 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>23.64</td>
<td>6.40</td>
<td>3.01</td>
</tr>
<tr>
<td>D</td>
<td>23.64</td>
<td>6.40</td>
<td>3.01</td>
</tr>
<tr>
<td>Ra</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>X'd</td>
<td>0.0608</td>
<td>0.1198</td>
<td>0.1813</td>
</tr>
</tbody>
</table>
Case 1: Light-load, N-0

- **Mode 1:** \( \lambda_{1,2} = \sigma \pm j\omega = -0.2500 \pm j8.7969 \)
  Damping ratio (\( \zeta \)) = 2.84 \%
  Frequency (\( \omega/2\pi \)) = 1.4001 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \omega_{r1} )</td>
<td>(</td>
<td>\phi_{k1}</td>
<td>) 0.326 180.00</td>
</tr>
<tr>
<td>( \Delta \delta_1 )</td>
<td>( \phi_{k1} ) 0.037 88.37</td>
<td>( \psi_{1k} ) 11.094 -91.63</td>
<td>( \psi_{1k} ) 0.411 -3.26</td>
</tr>
<tr>
<td>( \Delta \omega_{r2} )</td>
<td>1.000 0.00</td>
<td>1.000 0.00</td>
<td>1.000 0.00</td>
</tr>
<tr>
<td>( \Delta \delta_2 )</td>
<td>0.114 -91.63</td>
<td>8.800 88.37</td>
<td>1.000 -3.26</td>
</tr>
<tr>
<td>( \Delta \omega_{r3} )</td>
<td>0.556 0.00</td>
<td>0.261 0.00</td>
<td>0.145 0.00</td>
</tr>
<tr>
<td>( \Delta \delta_3 )</td>
<td>0.063 -91.63</td>
<td>2.294 88.37</td>
<td>0.145 -3.26</td>
</tr>
</tbody>
</table>

- **Mode 2:** \( \lambda_{3,4} = \sigma \pm j\omega = -0.2500 \pm j13.3564 \)
  Damping ratio (\( \zeta \)) = 1.87 \%
  Frequency (\( \omega/2\pi \)) = 2.1257 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \omega_{r1} )</td>
<td>(</td>
<td>\phi_{k1}</td>
<td>) 0.042 180.00</td>
</tr>
<tr>
<td>( \Delta \delta_1 )</td>
<td>0.003 88.93</td>
<td>4.476 -91.07</td>
<td>0.014 -2.14</td>
</tr>
<tr>
<td>( \Delta \omega_{r2} )</td>
<td>0.313 -180.00</td>
<td>0.665 180.00</td>
<td>0.208 0.00</td>
</tr>
<tr>
<td>( \Delta \delta_2 )</td>
<td>0.023 88.93</td>
<td>8.883 -91.07</td>
<td>0.208 -2.14</td>
</tr>
<tr>
<td>( \Delta \omega_{r3} )</td>
<td>1.000 0.00</td>
<td>1.000 0.00</td>
<td>1.000 0.00</td>
</tr>
<tr>
<td>( \Delta \delta_3 )</td>
<td>0.075 -91.07</td>
<td>13.359 88.93</td>
<td>1.000 -2.14</td>
</tr>
</tbody>
</table>
Mode shape in Case 1 (rotor angles)

• **Mode 1:** \( \lambda_{1,2} = \sigma \pm j \omega = -0.2500 \pm j 8.7969 \)
  Damping ratio \((\zeta)\) = 2.84 %,
  Frequency \((\omega/2\pi)\) = 1.4001 Hz

• **Mode 2:** \( \lambda_{3,4} = \sigma \pm j \omega = -0.2500 \pm j 13.3564 \)
  Damping ratio \((\zeta)\) = 1.87 %,
  Frequency \((\omega/2\pi)\) = 2.1257 Hz
Case 2: Light-load, N-1

• **Mode 1:** $\lambda_{1,2} = \sigma \pm j\omega = -0.25 \pm j6.4294$
  Damping ratio ($\zeta$) = 3.89%,
  Frequency ($\omega/2\pi$) = 1.0233 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \omega_r$</td>
<td>$</td>
<td>\phi_{k1}</td>
<td>$</td>
</tr>
<tr>
<td>$\Delta \omega_{r1}$</td>
<td>0.277</td>
<td>-180.00</td>
<td>1.243</td>
</tr>
<tr>
<td>$\Delta \delta_1$</td>
<td>0.043</td>
<td>87.77</td>
<td>7.999</td>
</tr>
<tr>
<td>$\Delta \omega_{r2}$</td>
<td>1.000</td>
<td>0.00</td>
<td>1.000</td>
</tr>
<tr>
<td>$\Delta \delta_2$</td>
<td>0.155</td>
<td>-92.23</td>
<td>6.434</td>
</tr>
<tr>
<td>$\Delta \omega_{r3}$</td>
<td>0.533</td>
<td>0.00</td>
<td>0.243</td>
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<tr>
<td>$\Delta \delta_3$</td>
<td>0.083</td>
<td>-92.23</td>
<td>1.565</td>
</tr>
</tbody>
</table>

• **Mode 2:** $\lambda_{3,4} = \sigma \pm j\omega = -0.25 \pm j13.3026$
  Damping ratio ($\zeta$) = 1.88%,
  Frequency ($\omega/2\pi$) = 2.1172 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \omega_r$</td>
<td>$</td>
<td>\phi_{k1}</td>
<td>$</td>
</tr>
<tr>
<td>$\Delta \omega_{r1}$</td>
<td>0.042</td>
<td>180.00</td>
<td>0.335</td>
</tr>
<tr>
<td>$\Delta \delta_1$</td>
<td>0.003</td>
<td>88.93</td>
<td>4.476</td>
</tr>
<tr>
<td>$\Delta \omega_{r2}$</td>
<td>0.313</td>
<td>-180.00</td>
<td>0.665</td>
</tr>
<tr>
<td>$\Delta \delta_2$</td>
<td>0.023</td>
<td>88.93</td>
<td>8.883</td>
</tr>
<tr>
<td>$\Delta \omega_{r3}$</td>
<td>1.000</td>
<td>0.00</td>
<td>1.000</td>
</tr>
<tr>
<td>$\Delta \delta_3$</td>
<td>0.075</td>
<td>-91.07</td>
<td>13.359</td>
</tr>
</tbody>
</table>
Mode shape in Case 2 (rotor angles)

- **Mode 1:** $\lambda_{1,2} = \sigma \pm j\omega = -0.25 \pm j6.4294$
  Damping ratio ($\zeta$) = 3.89%,
  Frequency ($\omega/2\pi$) = 1.0233 Hz

- **Mode 2:** $\lambda_{3,4} = \sigma \pm j\omega = -0.25 \pm j13.3026$
  Damping ratio ($\zeta$) = 1.88%,
  Frequency ($\omega/2\pi$) = 2.1172 Hz
Case 3: Heavy-load, N-0

- **Mode 1:** \( \lambda_{1,2} = \sigma \pm j\omega = -0.25 \pm j8.5816 \)
  - Damping ratio (\( \zeta \)) = 2.91 %,
  - Frequency (\( \omega/2\pi \)) = 1.3658 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \omega_1 )</td>
<td>0.313</td>
<td>180.00</td>
<td>1.244</td>
</tr>
<tr>
<td>( \Delta \delta_1 )</td>
<td>0.037</td>
<td>88.33</td>
<td>10.676</td>
</tr>
<tr>
<td>( \Delta \omega_2 )</td>
<td>1.000</td>
<td>0.00</td>
<td>1.000</td>
</tr>
<tr>
<td>( \Delta \delta_2 )</td>
<td>0.117</td>
<td>-91.67</td>
<td>8.585</td>
</tr>
<tr>
<td>( \Delta \omega_3 )</td>
<td>0.522</td>
<td>0.00</td>
<td>0.244</td>
</tr>
<tr>
<td>( \Delta \delta_3 )</td>
<td>0.061</td>
<td>-91.67</td>
<td>2.091</td>
</tr>
</tbody>
</table>

- **Mode 2:** \( \lambda_{3,4} = \sigma \pm j\omega = -0.25 \pm j13.3185 \)
  - Damping ratio (\( \zeta \)) = 1.88 %,
  - Frequency (\( \omega/2\pi \)) = 2.1197 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \omega_1 )</td>
<td>0.045</td>
<td>180.00</td>
<td>0.364</td>
</tr>
<tr>
<td>( \Delta \delta_1 )</td>
<td>0.003</td>
<td>88.92</td>
<td>4.847</td>
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<tr>
<td>( \Delta \omega_2 )</td>
<td>0.300</td>
<td>-180.00</td>
<td>0.636</td>
</tr>
<tr>
<td>( \Delta \delta_2 )</td>
<td>0.023</td>
<td>88.92</td>
<td>8.474</td>
</tr>
<tr>
<td>( \Delta \omega_3 )</td>
<td>1.000</td>
<td>0.00</td>
<td>1.000</td>
</tr>
<tr>
<td>( \Delta \delta_3 )</td>
<td>0.075</td>
<td>-91.08</td>
<td>13.321</td>
</tr>
</tbody>
</table>

Fig. 1. IEEE 9-bus system
Mode shape in Case 3 (rotor angles)

• **Mode 1:** \( \lambda_{1,2} = \sigma \pm j \omega = -0.25 \pm j8.5816 \)
  Damping ratio (\( \zeta \)) = 2.91 %,
  Frequency (\( \omega/2\pi \)) = 1.3658 Hz

• **Mode 2:** \( \lambda_{3,4} = \sigma \pm j \omega = -0.25 \pm j13.3185 \)
  Damping ratio (\( \zeta \)) = 1.88 %,
  Frequency (\( \omega/2\pi \)) = 2.1197 Hz
Case 4: Heavy-load, N-1

• **Mode 1:** $\lambda_{1,2} = \sigma \pm j\omega = -0.25 \pm j5.5509$
  Damping ratio ($\zeta$) = 4.50%,
  Frequency ($\omega/2\pi$) = 0.8835 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta\omega_{r1}$</td>
<td>0.226</td>
<td>180.00</td>
<td>1.235</td>
</tr>
<tr>
<td>$\Delta\delta_1$</td>
<td>0.041</td>
<td>87.42</td>
<td>6.863</td>
</tr>
<tr>
<td>$\Delta\omega_{r2}$</td>
<td>1.000</td>
<td>0.00</td>
<td>1.000</td>
</tr>
<tr>
<td>$\Delta\delta_2$</td>
<td>0.180</td>
<td>-92.58</td>
<td>5.557</td>
</tr>
<tr>
<td>$\Delta\omega_{r3}$</td>
<td>0.531</td>
<td>0.00</td>
<td>0.235</td>
</tr>
<tr>
<td>$\Delta\delta_3$</td>
<td>0.096</td>
<td>-92.58</td>
<td>1.305</td>
</tr>
</tbody>
</table>

• **Mode 2:** $\lambda_{3,4} = \sigma \pm j\omega = -0.25 \pm j13.1477$
  Damping ratio ($\zeta$) = 1.90%,
  Frequency ($\omega/2\pi$) = 2.0925 Hz

<table>
<thead>
<tr>
<th>State</th>
<th>Mode shape</th>
<th>Mode Comp.</th>
<th>Part. Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta\omega_{r1}$</td>
<td>0.044</td>
<td>180.00</td>
<td>0.382</td>
</tr>
<tr>
<td>$\Delta\delta_1$</td>
<td>0.003</td>
<td>88.91</td>
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<tr>
<td>$\Delta\omega_{r2}$</td>
<td>0.289</td>
<td>180.00</td>
<td>0.618</td>
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<tr>
<td>$\Delta\delta_2$</td>
<td>0.022</td>
<td>88.91</td>
<td>8.120</td>
</tr>
<tr>
<td>$\Delta\omega_{r3}$</td>
<td>1.000</td>
<td>0.00</td>
<td>1.000</td>
</tr>
<tr>
<td>$\Delta\delta_3$</td>
<td>0.076</td>
<td>-91.09</td>
<td>13.150</td>
</tr>
</tbody>
</table>
Mode shape in Case 4

- **Mode 1:** $\lambda_{1,2} = \sigma \pm j \omega = -0.25 \pm j 5.5509$
  Damping ratio ($\zeta$) = 4.50%,
  Frequency ($\omega/2\pi$) = 0.8835 Hz

- **Mode 2:** $\lambda_{3,4} = \sigma \pm j \omega = -0.25 \pm j 13.1477$
  Damping ratio ($\zeta$) = 1.90%,
  Frequency ($\omega/2\pi$) = 2.0925 Hz
Small Signal Stability of a Regulated Multi-Machine System

* Algebraic equations
** Differential equations

**Figure 12.18** Structure of the complete power system model
Formulation of General Multi-machine State Equations

• The linearized model of each dynamic device:

\[
\dot{x}_i = A_i x_i + B_i \Delta v \\
\Delta i_i = C_i x_i - Y_i \Delta v
\]

- \( x_i \) – Perturbed values of state variables
- \( i_i \) – Current injection into network from the device
- \( \Delta v \) – Vector of network bus voltages

\( B_i \) and \( Y_i \) have non-zero element corresponding only to the terminal voltage of the device and any remote bus voltages used to control the device.

\( \Delta i_i \) and \( \Delta v \) both have real and imaginary components.
• Such state equations for all the dynamic devices in the system may be combined into the form:

\[ \dot{x} = A_D x + B_D \Delta v \]

\[ \Delta i = C_D x - Y_D \Delta v \]

\( x \) is the vector of state variables of the complete system.

\( A_D \) and \( C_D \) are block diagonal matrices composed of \( A_i \) and \( C_i \) associated with the individual devices.

• Node equation of the transmission network:

\[ \Delta i = Y_N \Delta v \]

• The overall system state equation:

\[ \dot{x} = A_D x + B_D (Y_N + Y_D)^{-1} C_D x = Ax \]

\[ A = A_D + B_D (Y_N + Y_D)^{-1} C_D \]

• Read Kundur’s sec. 12.7 for other related information, e.g. load model linearization and selection of a reference rotor angle.
Model-based methods vs. Measurement-based methods

**Input:** DAE model
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\Delta u(t) \\
y(t) &= Cx(t)
\end{align*}
\]

**Output:**
\[
x_i(t) = A_i e^{\sigma_i t} \sin(\omega_i t + \theta_i)
\]

**Model-based methods:**
Eigen-analysis to find \( \lambda_i = \sigma_i \pm j\omega_i \), \( \phi_i, \psi_i, p_i \)

**Measurement-based methods:**
Measurements \( y(t) \)
Signal processing and decomposition to find a damped sinusoid with \( \sigma_i, \omega_i \) and \( \varphi_i \) to best fit \( y_i \)

\[
y_i(t) = B_i e^{\sigma_i t} \sin(\omega_i t + \varphi_i)
\]