ECE 522 - Power Systems Analysis II
Spring 2021

Transient Stability

Spring 2021

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Outline

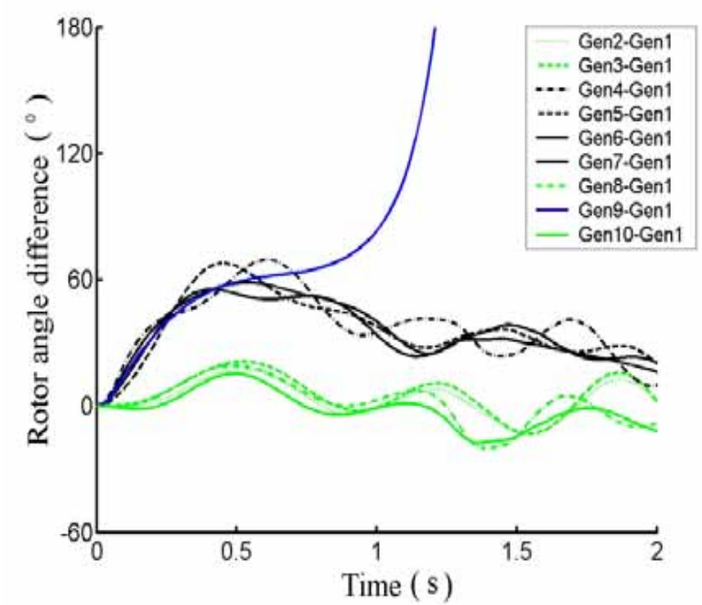
- Transient stability analysis using direct methods
  - Transient stability of an SMIB system
  - Direct methods for multi-machine systems

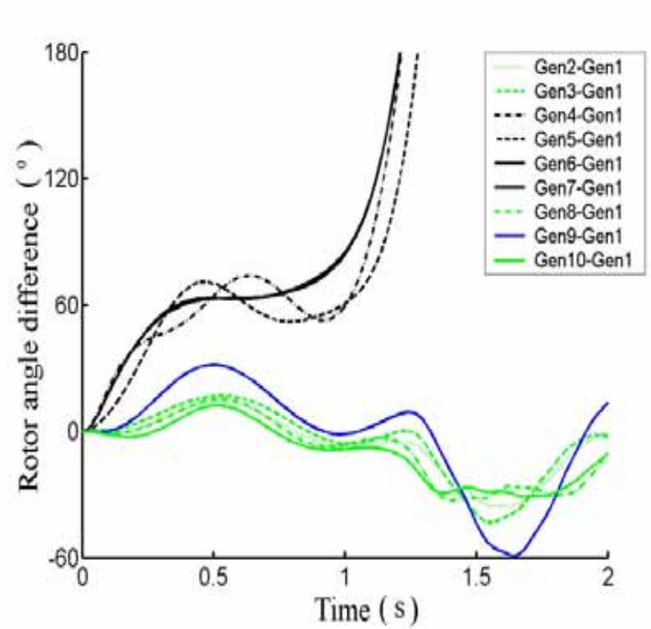
- Time-domain transient stability simulation
  - Explicit and implicit numerical integration techniques
  - Simulation of a multi-machine system

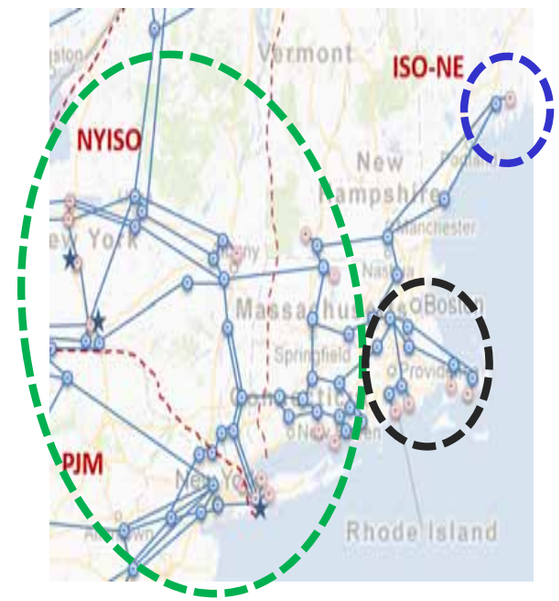
- References:
  - Kundur's Chapter 13
  - Saadat's Chapters 11.5-11.10

Transient Stability

- The ability of the power system to maintain synchronism when subjected to a severe disturbance such as a fault on transmission facilities, loss of generation or loss of a large load.
  - The system response to such disturbances involves large excursions of generator rotor angles, power flows, bus voltages, and other system variables.
  - Stability is influenced by the nonlinear characteristics of the system
  - If the resulting angular separation between the machines in the system remains within certain bounds, the system maintains synchronism.
  - If loss of synchronism due to transient instability occurs, it will usually be evident within 2-3 seconds after the initial disturbances







Two Approaches for Transient Stability Analysis

To analyze the stability of a nonlinear system modeled by DAEs (differential-algebraic equations) following a disturbance:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \qquad \text{DE}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y}, t) \qquad \text{AE}$$

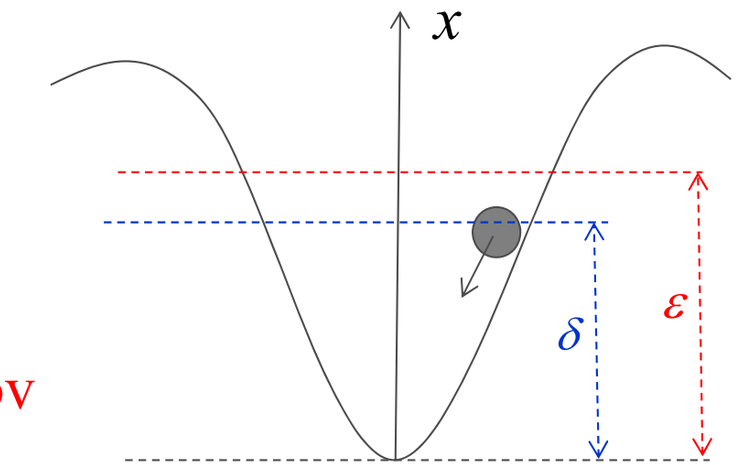
1. **Direct methods** (based on Lyapunov's second method for stability): Determine stability without explicitly solving the DAEs:
   1. Define a Transient Energy Function, which is an exact or approximate *Lyapunov function*.
   2. Compare the value of the function to a critical energy to judge whether the system state can stay inside a stability region of the equilibrium.
2. **Time-domain simulation** (indirect method): Solve an Initial Value Problem of the nonlinear **Differential-Algebraic Equations** (DAEs) with a given initial state $\mathbf{x}=\mathbf{x}_0$ at $t=t_0$ by using step-by-step numerical integration (explicit or implicit).

Transient Stability Analysis Using Direct Methods

Stability on a General Dynamical System

- In mathematics, stability theory addresses the stability of solutions of a set of differential equations, or in other words, stability of trajectories of a dynamical system under small perturbations of an initial condition.

- **Lyapunov Stability:** Consider a nonlinear dynamical system



$$\dot{x} = f(t, x) \qquad (1)$$

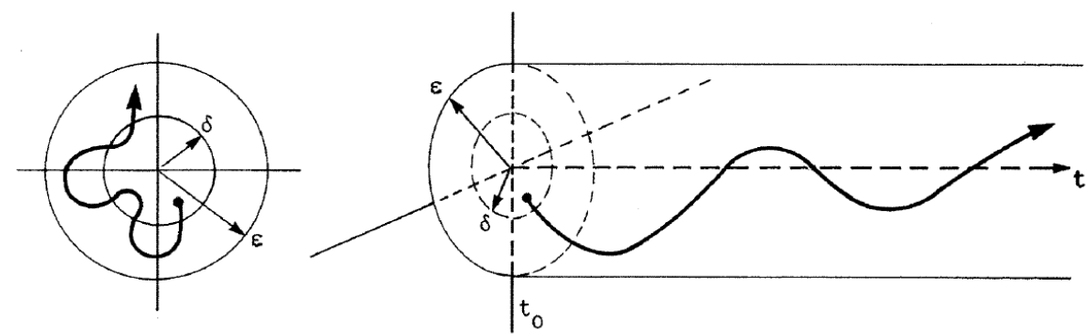
Assume origin x=0 is an equilibrium, i.e. $f(t, 0) = 0, \forall\, t \geq 0$

The equilibrium point x=0 is stable in the sense of Lyapunov

if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall\, t \geq t_0 \geq 0 \qquad (2)$$

In other words, the system variable will stay in any given small region (<ε) around the equilibrium point once becoming close enough (<δ) to that point.



Lyapunov Stability Criterion (Lyapunov's second method)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{y}, t) \qquad \text{DE}$$

$$\mathbf{0} = \mathbf{g}(\mathbf{x}, \mathbf{y}, t) \qquad \text{AE}$$

Its equilibrium $\mathbf{x}_s$, which satisfies $\mathbf{f}(\mathbf{x}_s)=\mathbf{0}$, is ***stable*** if there exists a positive definite function $V(\mathbf{x})$, called a ***Lyapunov function***, such that its time-derivative is not positive, i.e.

- $V(\mathbf{x}) \geq 0$ "=0" if and only if $\mathbf{x} = \mathbf{x}_s$ ($\mathbf{x}_S$ has the minimum V=0)
- $\dfrac{dV(\mathbf{x})}{dt} \leq 0$ (V does not increase with time)

Equilibrium $\mathbf{x}_s$ is ***asymptotically stable*** if the time-derivative of V is negative definite, i.e.

- $\dfrac{dV(\mathbf{x})}{dt} \leq 0$ "=0" if and only if $\mathbf{x} = \mathbf{x}_s$

Transient Energy Function (TEF) Method

- It is a special case of Lyapunov's second method that relaxes $V(\mathbf{x})$ to an approximate Lyapunov function, i.e. called a Transient Energy function (TEF)
- Consider "a rolling ball" analogy. two quantities are required to determine whether "ball" (state $\mathbf{x}$) can leave "bowl" (stability region of $\mathbf{x}_s$)
  - Initial kinetic energy of the ball
  - The height of the bowl rim at the crossing point (depending on the direction of the initial motion)

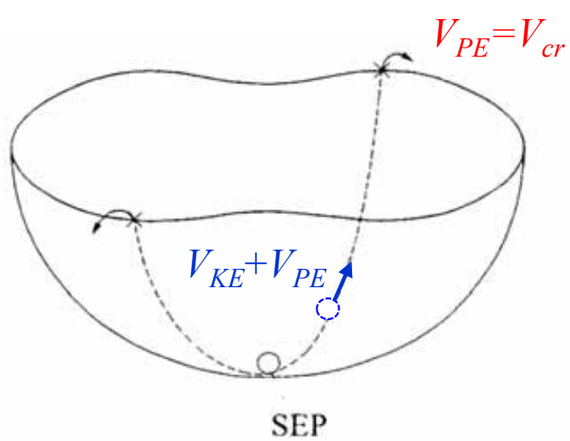


Figure 13.59 A ball rolling on the inner surface of a bowl

Application to a Power System

- Initially the system operates at stable equilibrium point (SEP) $\mathbf{x}_s$.
- If a fault occurs, the system gains kinetic energy V_{KE} and potential energy V_{PE} during the fault-on period and then moves away from $\mathbf{x}_s$.
- After fault clearing, V_{KE} is converted into V_{PE}.
- To avoid instability, the system must be capable of absorbing V_{KE} while moving towards the old SEP $\mathbf{x}_s$ or a new SEP.
- For a given post-disturbance network configuration, there is a critical, maximum amount of transient energy V_{cr} that the system can absorb without loss of synchronism
- Therefore, assessment of transient stability for the system state at fault clearing ($\mathbf{x}_{cl}$) requires:
  1. Define a TEF $V(\mathrm{x}_{cl})$ that adequately describes the transient energy responsible for separating one or more generators from the rest of the system
  2. Estimate the critical energy V_{cr} and calculate transient stability margin, i.e. $V_{cr} - V(\mathrm{x}_{cl})$.

Classic Model of a Single-Machine-Infinite-Bus System

$$\begin{cases} \dfrac{d\delta}{dt} = \omega_r - \omega_0 \\ \dfrac{2H}{\omega_0}\dfrac{d\omega_r}{dt} = P_m - P_e - \dfrac{K_D}{\omega_0}(\omega_r - \omega_0) \end{cases}$$

$$P_e = P_{\max}\sin\delta = \frac{E'E_B}{X_T}\sin\delta$$

$$\frac{2H}{\omega_0}\ddot{\delta} + \frac{K_D}{\omega_0}\dot{\delta} + P_{\max}\sin\delta - P_m = 0$$

$\omega_0 = 2\pi\times 60 \approx 377$ rad/s

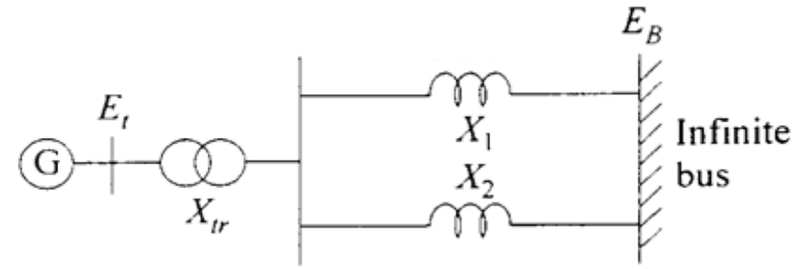
H in s

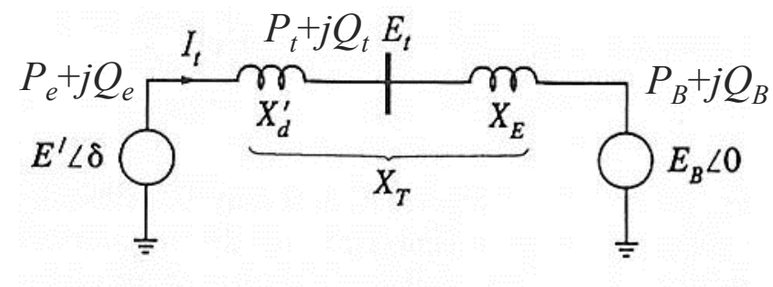
ω_r in rad/s

δ in rad

K_D in p.u

With all resistances neglected:

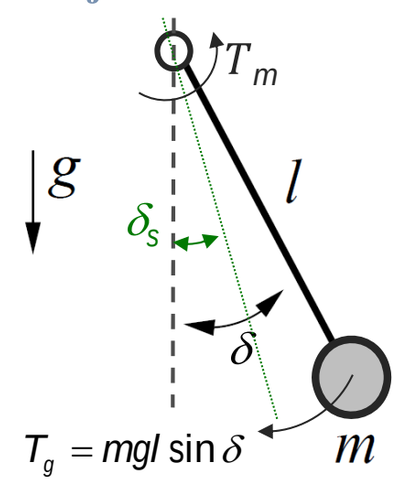




Equilibria satisfy $P_{\max}\sin\delta - P_m = 0$

- $\delta_s = \arcsin(P_m/P_{\max})$
  stable equilibrium point (SEP)
- $\delta_u = 180^\circ - \delta_s$
  unstable equilibrium point (UEP)

Pendulum subject to a constant torque



$$ml^2\cdot\ddot{\delta} + D\cdot\dot{\delta} + mgl\cdot\sin\delta - T_m = 0$$

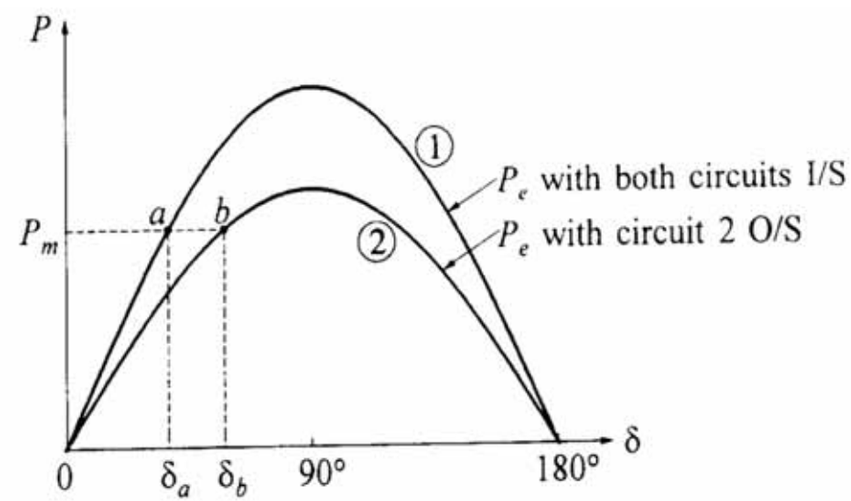
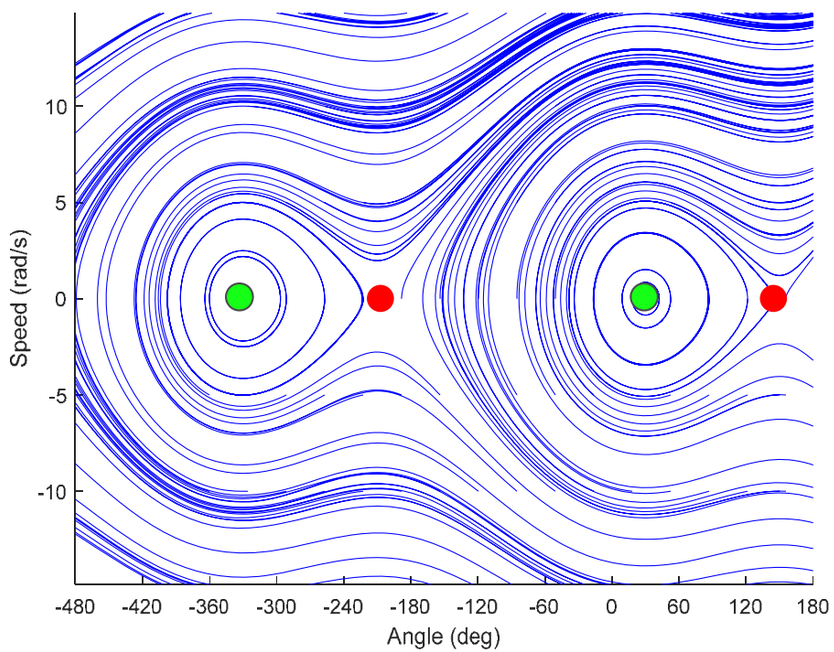


Figure 13.3 Power-angle relationship

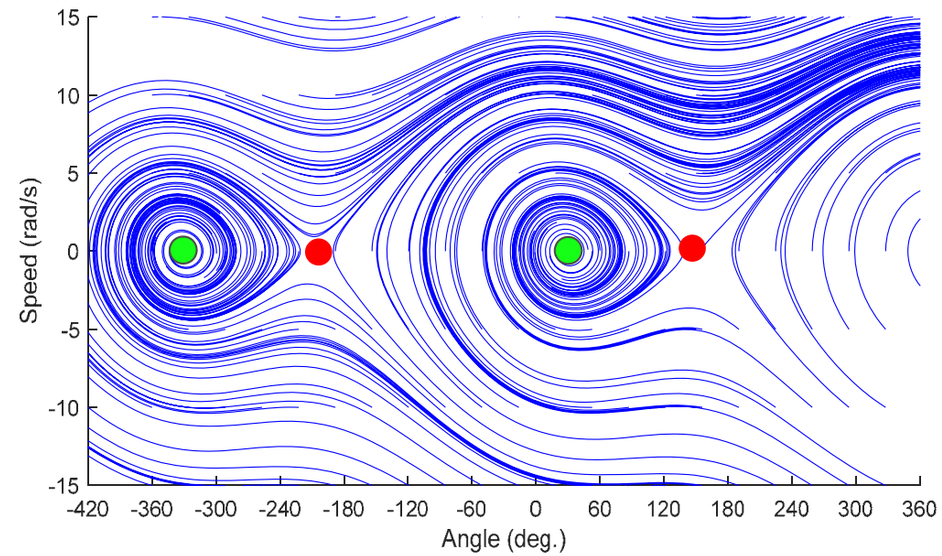
Phase-Space Trajectories

$$\frac{2H}{\omega_0}\ddot{\delta} + \frac{K_D}{\omega_0}\dot{\delta} + P_{\max}\sin\delta - P_m = 0$$

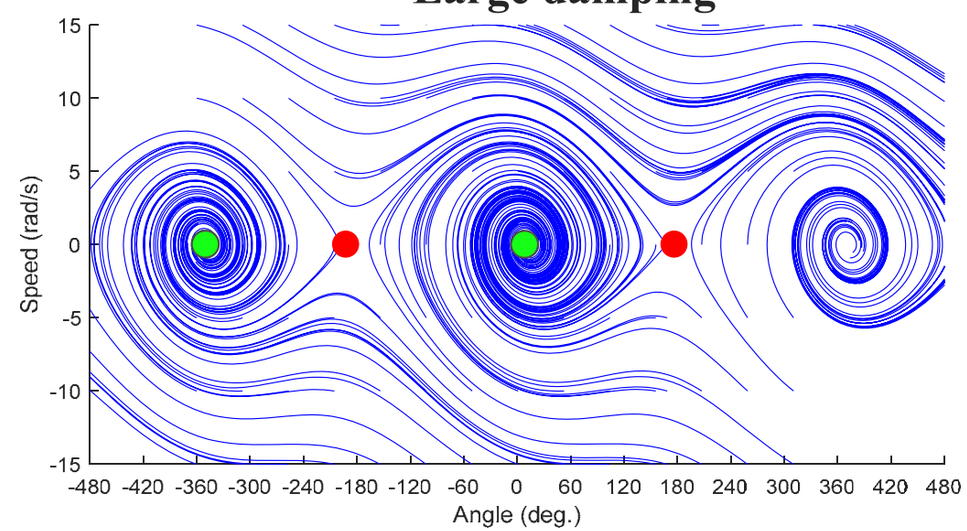
No damping (K_D=0)



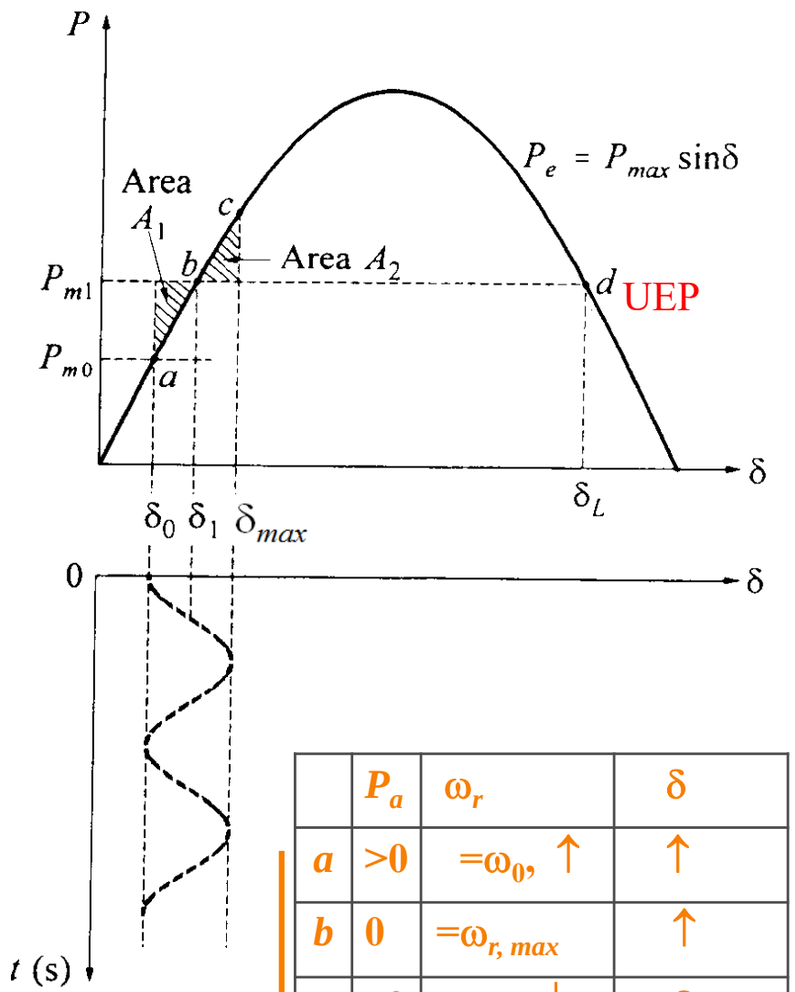
Small damping



Large damping



Response to a step change in P_m



	P_a	ω_r	δ
a	>0	$=\omega_0$, ↑	↑
b	0	$=\omega_{r,\,max}$	↑
c	<0	$=\omega_0$, ↓	$=\delta_{max}$

Ignore all losses (K_D=0):

$$\frac{2H}{\omega_0}\frac{d^2\delta}{dt^2} = P_m - P_e = P_m - P_{\max}\sin\delta = P_a \text{ (accelerating power)}$$

Consider a sudden increase in P_m: $P_{m0} \rightarrow P_{m1}$.

- New SEP *b* satisfying $P_e(\delta_1)=P_{m1}$
- ***a*→ *b***: Due to the rotor's inertia, δ cannot jump from δ_0 to δ_1, so $P_a=P_{m1}-P_e(\delta_0)>0$ and ω_r increases from ω_0. **When *b* is reached, P_a=0 but $\omega_r > \omega_0$**, and δ continues to increase.
- ***b*→ *c***: $\delta>\delta_1$ and $P_a<0$, so ω_r decreases while δ increases until *c*. **At *c*, $\omega_r=\omega_0$ and δ reaches its maxmum δ_{max}.**
- **← *c***: Starting from *c*, the rotor starts to decelerate (since $P_a<0$), so **$\omega_r<\omega_0$ and δ decreases.**
- With all resistances (damping) neglected, δ and ω_r oscillate around new SEP *b* with a constant amplitude.

Question 1: Does δ have a sinusoidal waveform in time?

Question 2: When does ω_r reach the minimum speed?

Nonlinear Oscillation of an SMIB System

$$\frac{2H}{\omega_0}\ddot{\delta}+\frac{K_D}{\omega_0}\dot{\delta}+P_{\max}\sin\delta-P_m=0$$

Let $\Delta\delta=\delta-\delta_s$, $\alpha=K_D/(2H)$
$\beta=\omega_0 P_{\max}/(2H)$

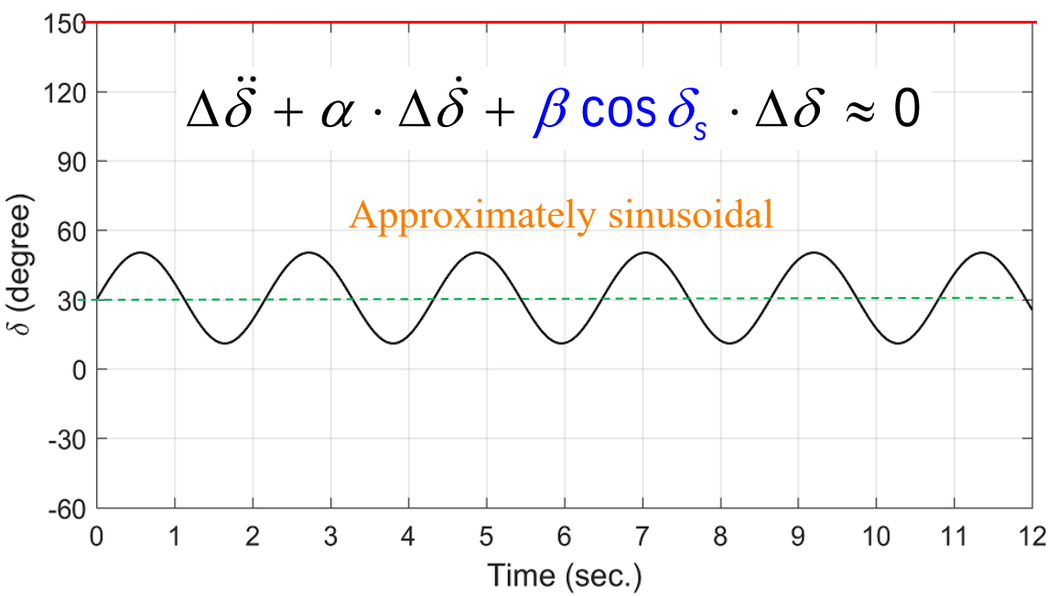
$$\ddot{\delta}+\alpha\cdot\dot{\delta}+\beta\cdot(\sin\delta-\sin\delta_s)=0$$

$$\Delta\ddot{\delta}+\alpha\cdot\Delta\dot{\delta}+\beta\cdot\mathrm{sinc}\left(\frac{\Delta\delta}{2}\right)\cos\left(\delta_s+\frac{\Delta\delta}{2}\right)\cdot\Delta\delta=0$$

$2H/\omega_0=0.1$, $D=0$, $P_{\max}=1$, $P_m=0.5 \Rightarrow \alpha=0$, $\beta=10$, $\delta_s=30^{\circ}$

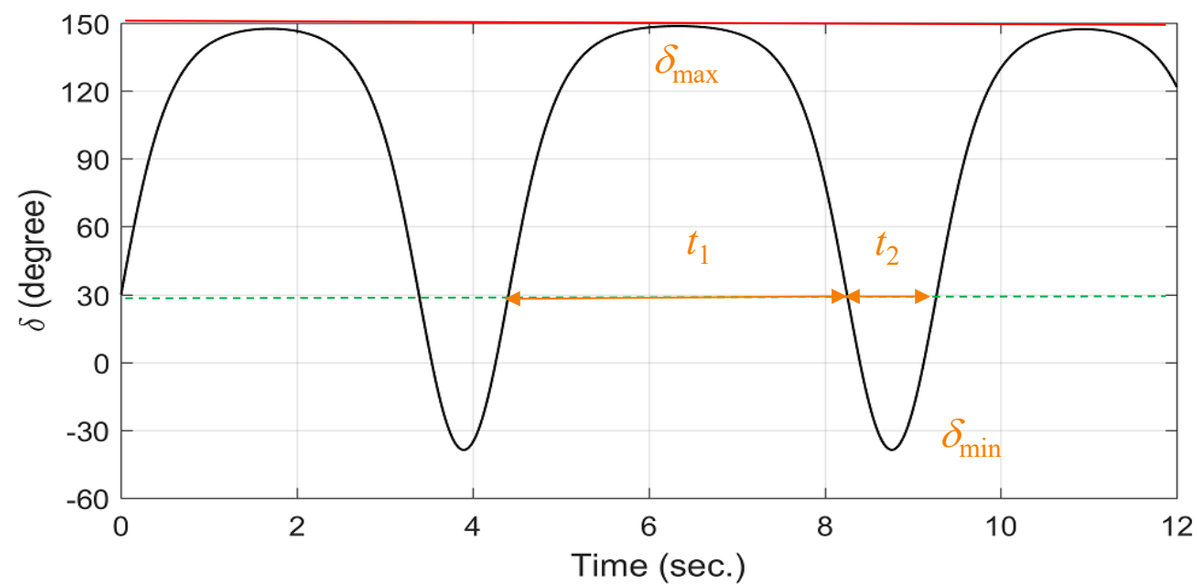
Small disturbance

$$f\approx\frac{1}{2\pi}\sqrt{\beta\cos\delta_s}$$



Marginally stable

$$f=\frac{1}{t_1(\delta_{\max})+t_2(\delta_{\min})}$$



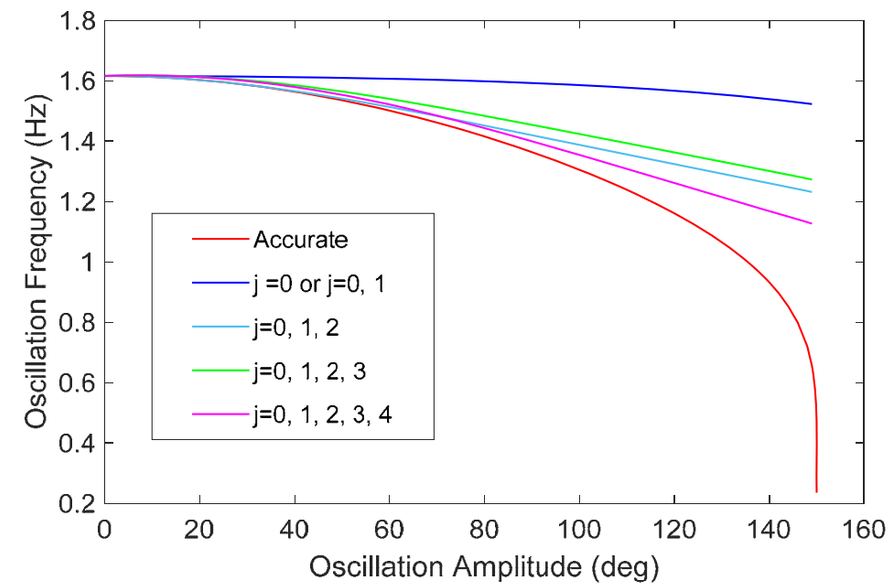
Frequency-Amplitude (F-A) Curve

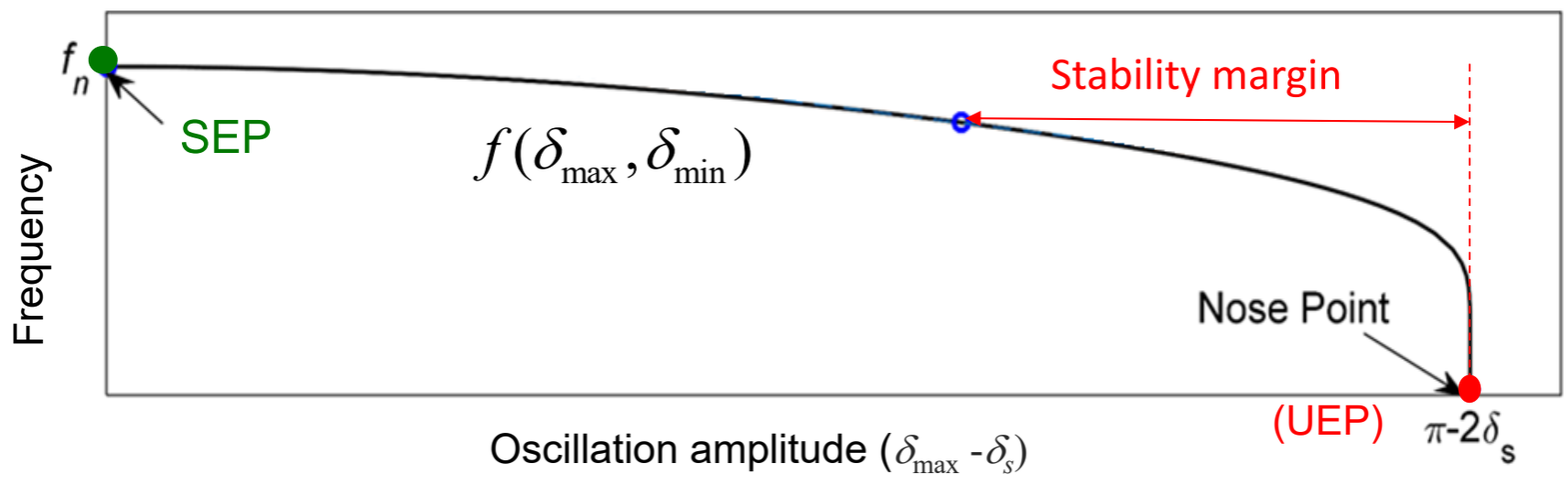
$$f(\delta_{\max},\delta_{\min}) = \frac{1}{t_1(\delta_{\max}) + t_2(\delta_{\min})}$$

$$t_i(x) = \sqrt{\frac{2H}{\omega_0 P_{\max}}} \int_0^{x-\delta_s} \frac{d\Delta\delta}{\sqrt{\cos(\Delta\delta+\delta_s) - \cos x + (\Delta\delta + \delta_s - x)\sin\delta_s}}$$

$$= \sum_{j=0}^{\infty} \left[t_{i,2j}(x) + t_{i,2j+1}(x) \right]$$

Define oscillation amplitude as $\Delta\delta_{\max} = \delta_{\max} - \delta_s$ or $\Delta\delta_{\min} = \delta_s - \delta_{\min}$.





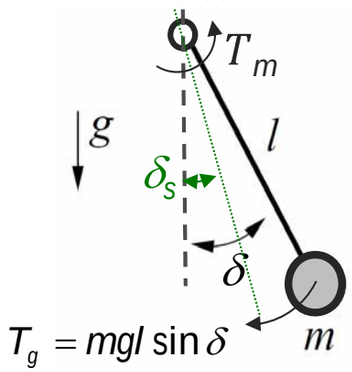
[1] B. Wang, K. Sun, "Formulation and Characterization of Power System Electromechanical Oscillations," IEEE Trans. Power Systems, Nov. 2016

[2] B. Wang, X. Su, K. Sun, "Properties of the Frequency-Amplitude Curve," IEEE Trans. Power Systems, Jan. 2017

Equal-Area Criterion (EAC)

$$\frac{2H}{\omega_0}\ddot{\delta}+\frac{K_D}{\omega_0}\dot{\delta}+P_{\max}\sin\delta-P_m=0$$

Pendulum with a constant torque



$T_g = mgl\sin\delta$

$$ml^2\cdot\ddot{\delta}+D\cdot\dot{\delta}+mgl\cdot\sin\delta-T_m=0$$

- Ignore all losses (K_D=0):

$$\frac{d\delta^2}{dt^2}=\frac{\omega_0}{2H}(P_m-P_e)$$

$$\frac{d}{dt}\left[\frac{d\delta}{dt}\right]^2=2\frac{d\delta}{dt}\cdot\frac{d\delta^2}{dt^2}=\frac{d\delta}{dt}\frac{\omega_0}{H}(P_m-P_e)$$

$$\left[\frac{d\delta}{dt}\right]^2=\int_{\delta_0}^{\delta}\frac{\omega_0(P_m-P_e)}{H}d\delta$$

$$\Delta\omega_r^2=\frac{\omega_0}{H}\int_{\delta_0}^{\delta}(P_m-P_e)d\delta$$

Kinetic energy:

$$V_{ke}=\frac{1}{2}ml^2\cdot\dot{\delta}^2$$

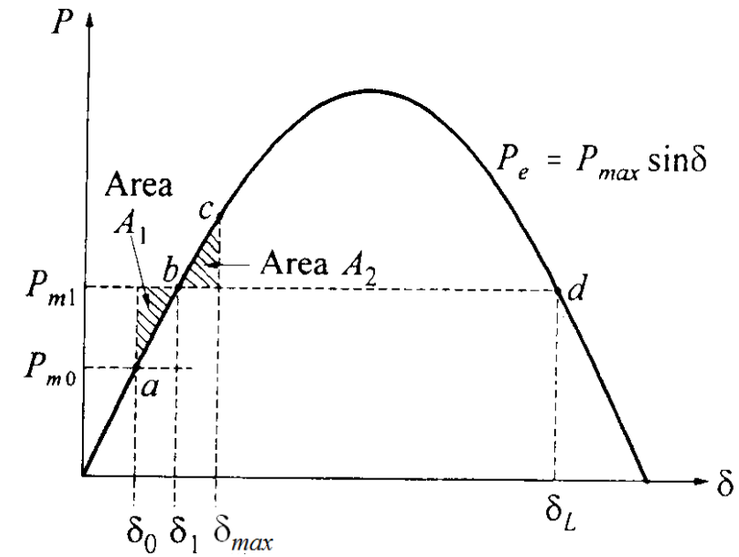
$$V_{ke}=\frac{1}{2}(2H\omega_0)\cdot\left(\frac{\Delta\omega_r}{\omega_0}\right)^2=\int_{\delta_0}^{\delta}(P_m-P_e)d\delta$$

Moment of inertia in p.u.

Potential energy (ref: δ_s)

$$V_{pe}=mgl(\cos\delta_s-\cos\delta)-T_m(\delta-\delta_s)$$

$$V_{pe}=P_{\max}(\cos\delta_s-\cos\delta)-P_m(\delta-\delta_s)$$



At $\delta_{\max}$ exists, $\Delta\omega_r=d\delta/dt=0$:

$$\boxed{0=\int_{\delta_0}^{\delta_{\max}}(P_m-P_e)d\delta}$$

$$=\int_{\delta_0}^{\delta_1}(P_m-P_e)d\delta-\int_{\delta_1}^{\delta_{\max}}-(P_m-P_e)d\delta$$

$$=|A_1|-|A_2|=0$$

- $|A_1|=V_{ke}$ gained from a to b
- $|A_2|=P_{\max}(\cos\delta_1-\cos\delta_{\max})-P_m(\delta_{\max}-\delta_1)$

  $=V_{pe}$ at c relative to b

Question: What if K_D>0?

Equal-Area Criterion (EAC):

The stability is maintained only if a decelerating area $|A_2| \geq$ accelerating area $|A_1|$ can be located above P_m.

- When δ reaches UEP, if $|A_2|<|A_1|$, δ will continues to increase (since $\omega_r>\omega_0$) and accelerate to lose stability.
- Transient stability depends on the size of the step change in P_m.

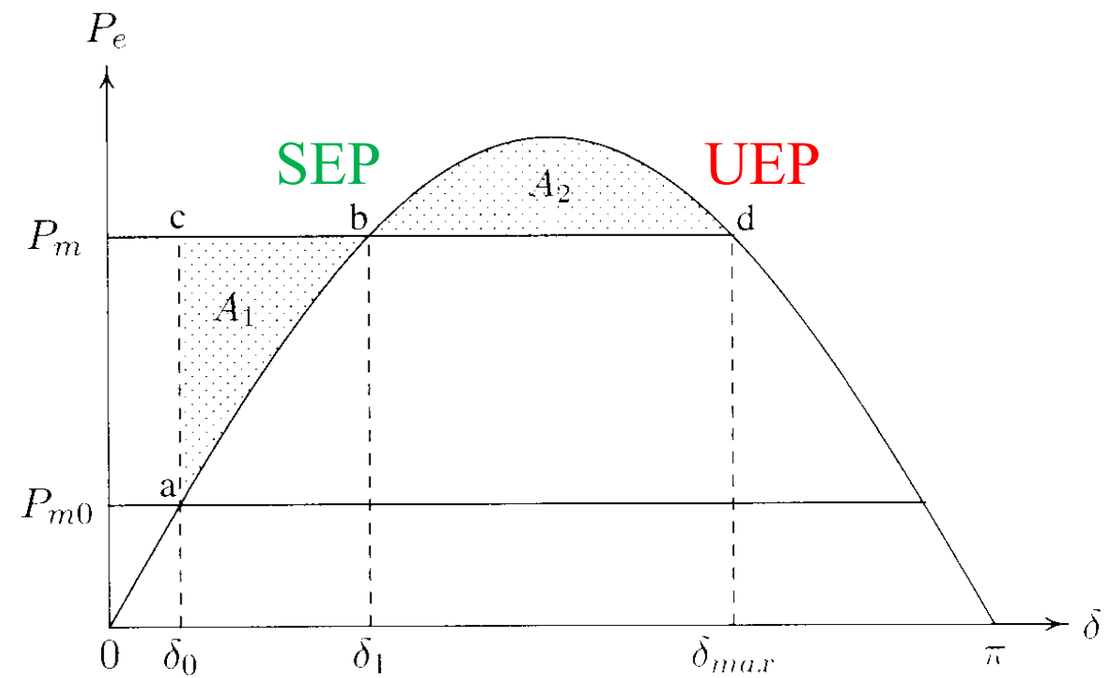


FIGURE 11.12
Equal-area criterion—maximum power limit.

Transient stability limit for a step change of P_m

Following a step change $P_{m0} \rightarrow P_m$, solve the transient stability limit of P_m:

- Assume $|A_1| = |A_2|$ in order to solve the limit of P_m

$$P_m(\delta_1 - \delta_0) - \int_{\delta_0}^{\delta_1} P_{\max} \sin\delta d\delta = \int_{\delta_1}^{\delta_{\max}} P_{\max} \sin\delta d\delta - P_m(\delta_{\max} - \delta_1)$$

$$-P_m\delta_0 + P_{\max}(\cos\delta_1 - \cos\delta_0) = -P_{\max}(\cos\delta_{\max} - \cos\delta_1) - P_m\delta_{\max}$$

$$(\delta_{\max} - \delta_0)P_m = P_{\max}(\cos\delta_0 - \cos\delta_{\max})$$

- Since at the UEP $P_m = P_{\max}\sin\delta_{\max}$, there is

$$(\delta_{\max} - \delta_0)\sin\delta_{\max} + \cos\delta_{\max} = \cos\delta_0$$

- Solve $\delta_{\max}$.
- Thus, transient stability limit for a step change of P_m

$$\Delta P_m \le P_{\max}\sin\delta_{\max} - P_{m0}$$

- The new SEP:

$$\delta_1 = \pi - \delta_{\max}$$

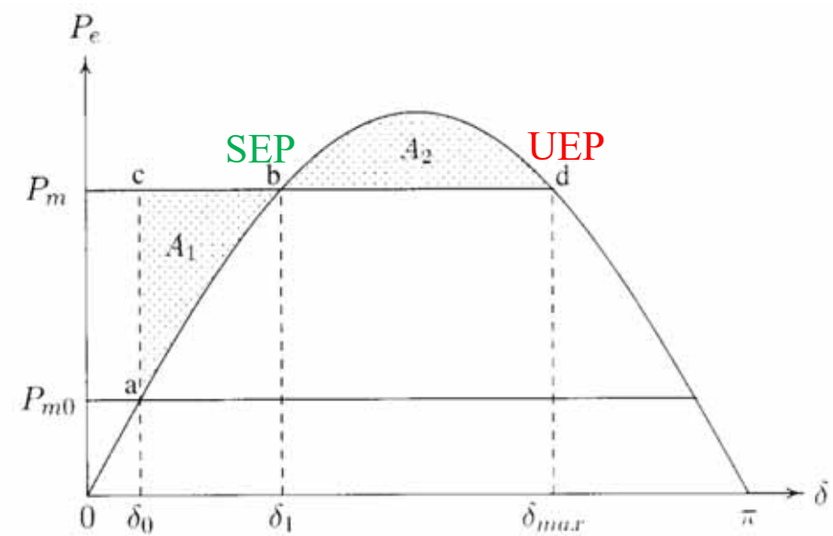


FIGURE 11.12
Equal-area criterion—maximum power limit.

Question: From P_{m0}, is there anyway to increase P_m beyond $P_{max}\sin\delta_{\max}$?

Solve δ_{max} by the Newton-Raphson method

$$(\delta_{\max} - \delta_0)\sin\delta_{\max} + \cos\delta_{\max} = \cos\delta_0$$

- Define nonlinear function: $f(\delta_{\max}) = (\delta_{\max} - \delta_0)\sin\delta_{\max} + \cos\delta_{\max} - \cos\delta_0$
- Taylor expansion at an estimate: $f(\delta_{\max}^{(k)}) + \left.\frac{df}{d\delta_{\max}}\right|_{\delta_{\max}^{(k)}} \Delta\delta_{\max}^{(k)} + \frac{1}{2!}\left.\frac{d^2 f}{d\delta_{\max}^2}\right|_{\delta_{\max}^{(k)}} (\Delta\delta_{\max}^{(k)})^2 + \cdots = 0$
- Select an initial estimate: $\pi/2 < \delta_{\max}^{(0)} < \pi$
- Calculate iterative solutions by the N-R algorithm:

$$\delta_{\max}^{(k+1)} = \delta_{\max}^{(k)} + \Delta\delta_{\max}^{(k)} \quad \text{where } \Delta\delta_{\max}^{(k)} = \frac{-f(\delta_{\max}^{(k)})}{\left.\frac{df}{d\delta_{\max}}\right|_{\delta_{\max}^{(k)}}} = \frac{-f(\delta_{\max}^{(k)})}{(\delta_{\max}^{(k)} - \delta_0)\cos\delta_{\max}^{(k)}}$$

- Give a solution when a specific accuracy ε is reached.

$$\left|\delta_{\max}^{(k+1)} - \delta_{\max}^{(k)}\right| \le \varepsilon$$

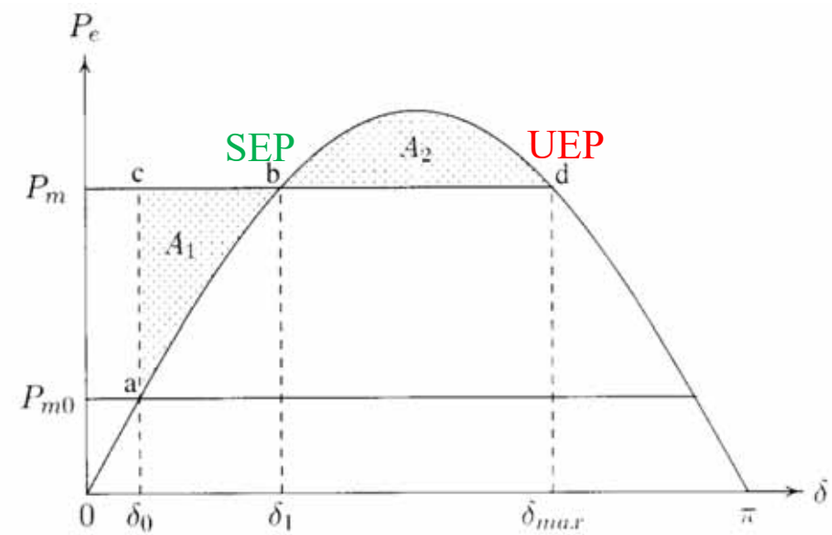
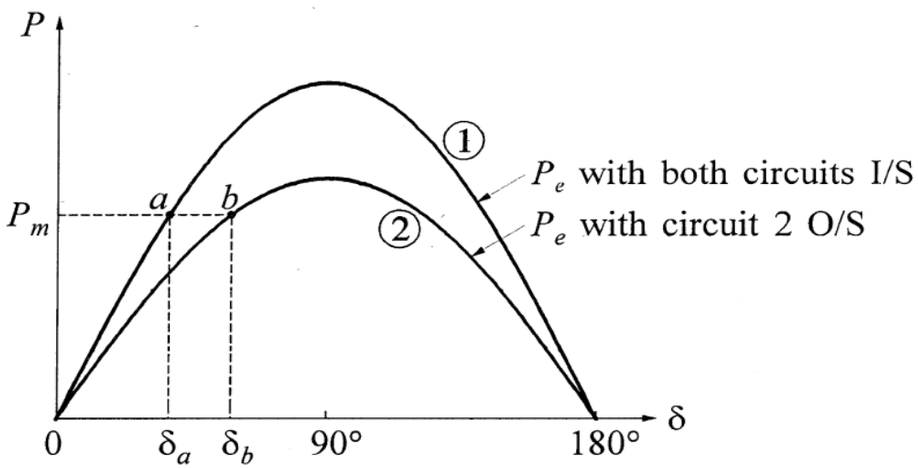
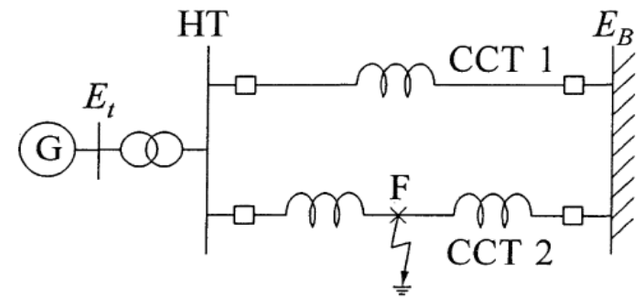


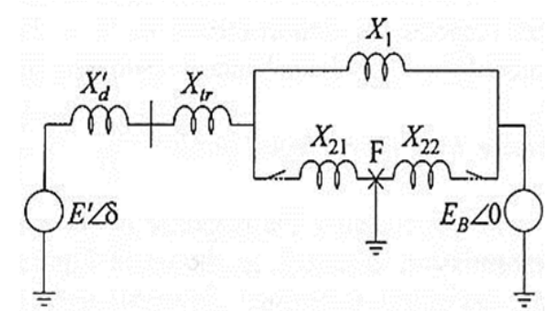
FIGURE 11.12
Equal-area criterion—maximum power limit.

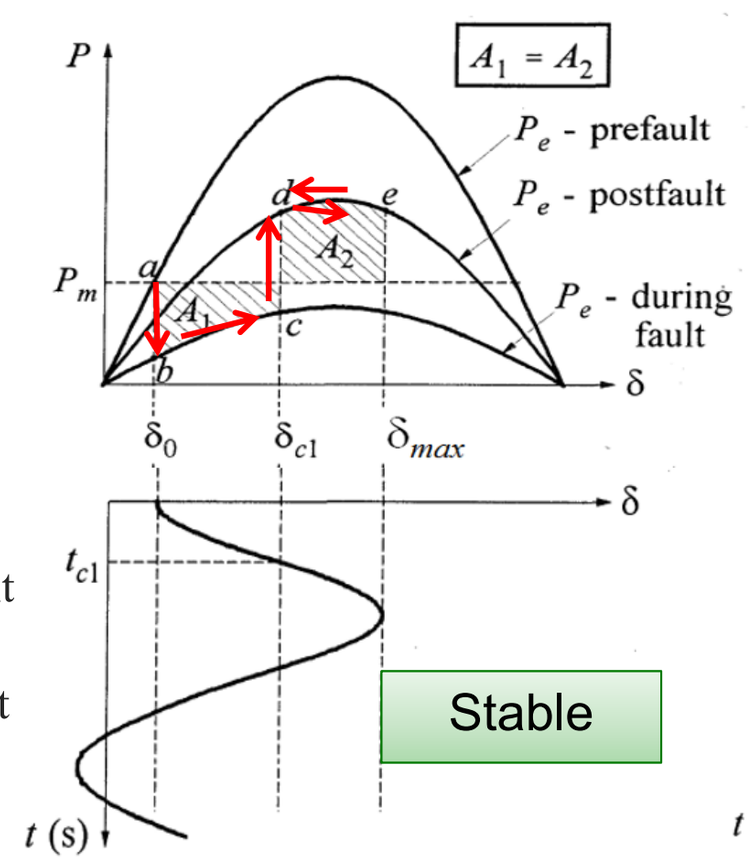
Response to a three-phase fault

$$P_e = P_{\max}\sin\delta = \frac{E'E_B}{X_T}\sin\delta$$

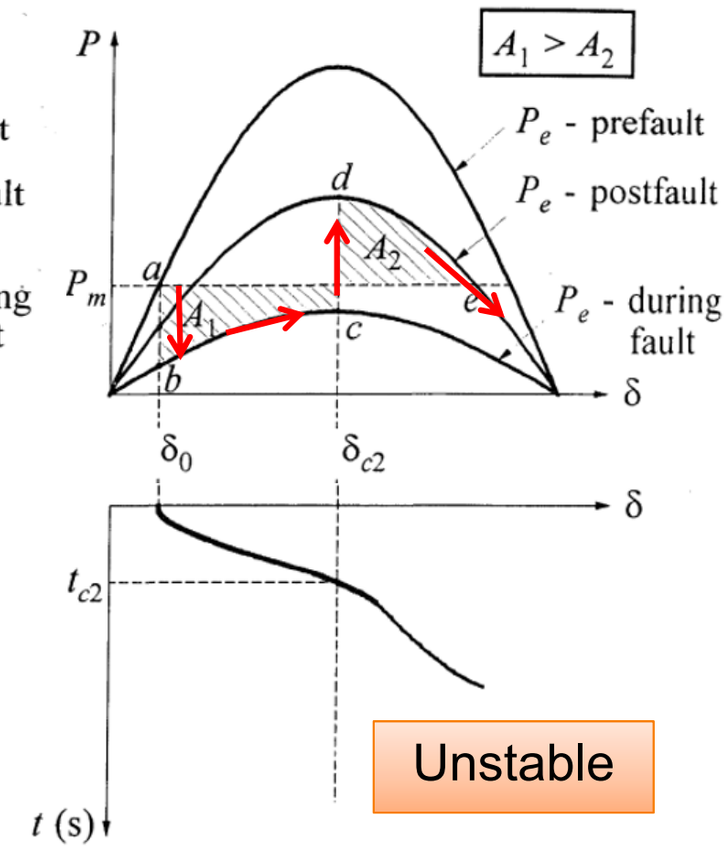








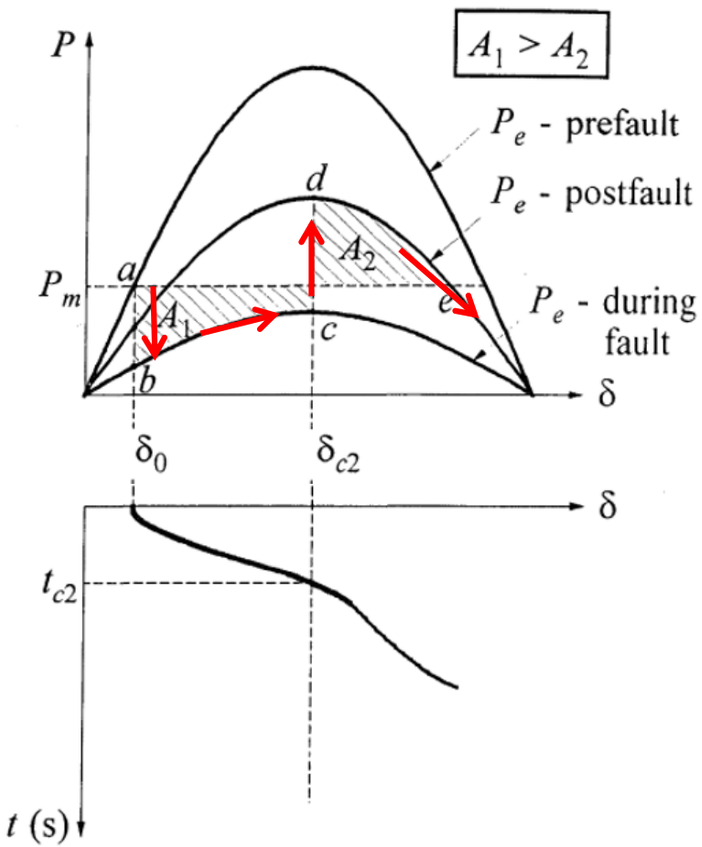
(c) Response to a fault cleared in t_{c1} seconds - stable case



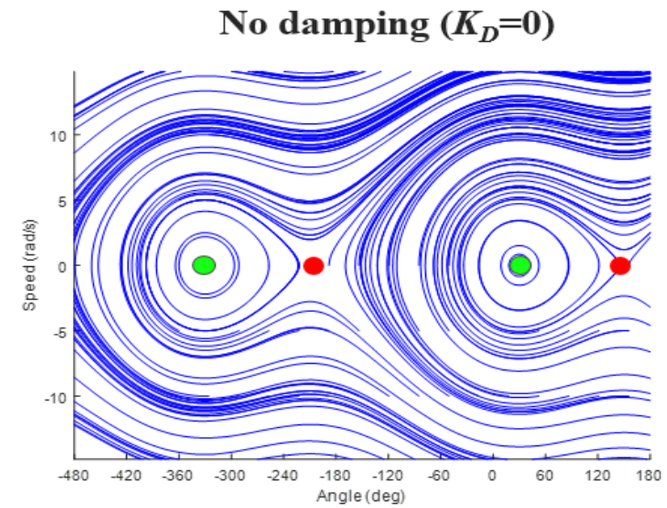
(d) Response to a fault cleared in t_{c2} seconds - unstable case

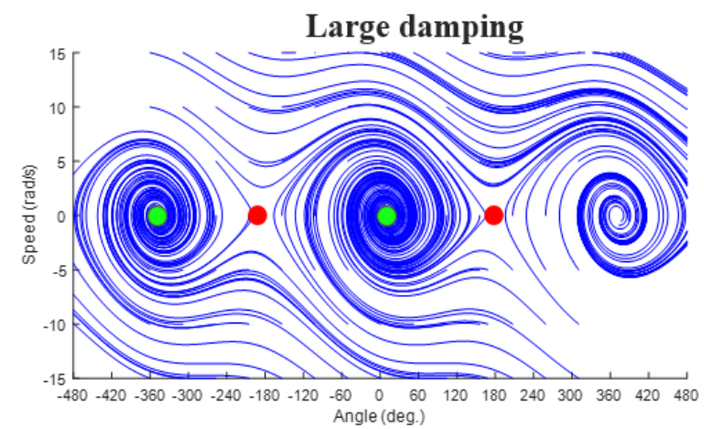
- In general, $P_{e,during\ fault}$ << $P_{e,\ post\text{-}fault}$ for any fault
- For a permanent fault cleared by tripping the fault circuit, $P_{e,post\text{-}fault}$ < $P_{e,pre\text{-}fault}$
- For a temporary fault without losing any circuit, $P_{e,\ post\text{-}fault}$ = $P_{e,pre\text{-}fault}$

Multi-Swing Stability



(d) Response to a fault cleared in t_{c2} seconds - unstable case





- For an SMIB or two-machine system (a 2-body problem), first-swing stability is usually sufficient to judge its transient stability over multiple swings.
- However, this is not enough to determine transient stability of a multi-machine system (an N-body problem in which chaos often arises).

Critical Clearing Angle (CCA) and Critical Clearing Time (CCT)

- Consider a simple case: a three-phase fault at the sending end with $P_{e,\ during\ fault}$=0 (all resistances are neglected)
- Solve Critical Clearing Angle δ_c

$$\underbrace{\int_{\delta_0}^{\delta_c} P_m d\delta}_{|A_1|} = \underbrace{\int_{\delta_c}^{\delta_{max}} (P_{max}\sin\delta - P_m) d\delta}_{|A_2|}$$

Integrating both sides:

$$P_m(\delta_c - \delta_0) = P_{max}(\cos\delta_c - \cos\delta_{max}) - P_m(\delta_{max} - \delta_c)$$

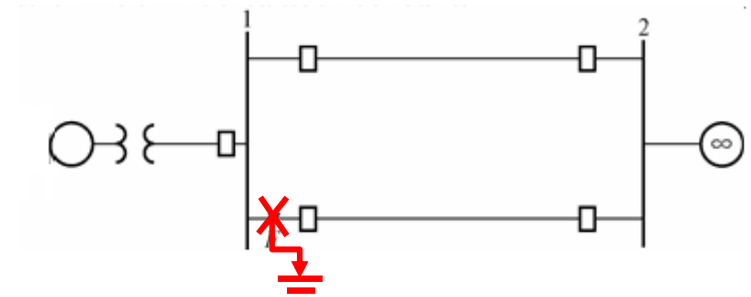
$$\cos\delta_c = \frac{P_m}{P_{max}}(\pi - 2\delta_0) - \cos\delta_0 \quad (\text{note: } \delta_{max} = \pi - \delta_0)$$

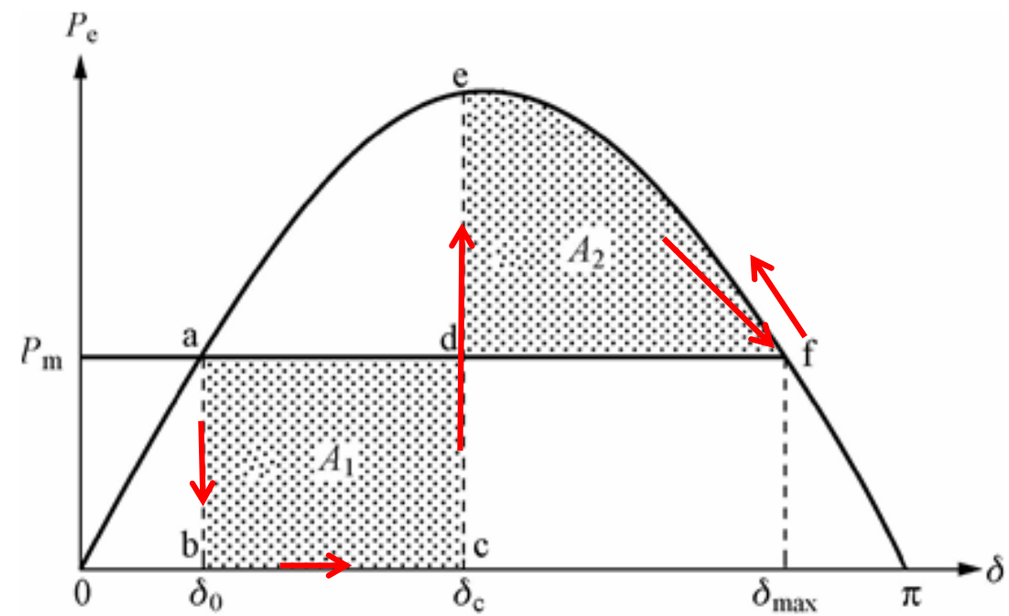
- Solve the CCT from the CCA:

During the fault ($P_{e,\ during\ fault}$=0):

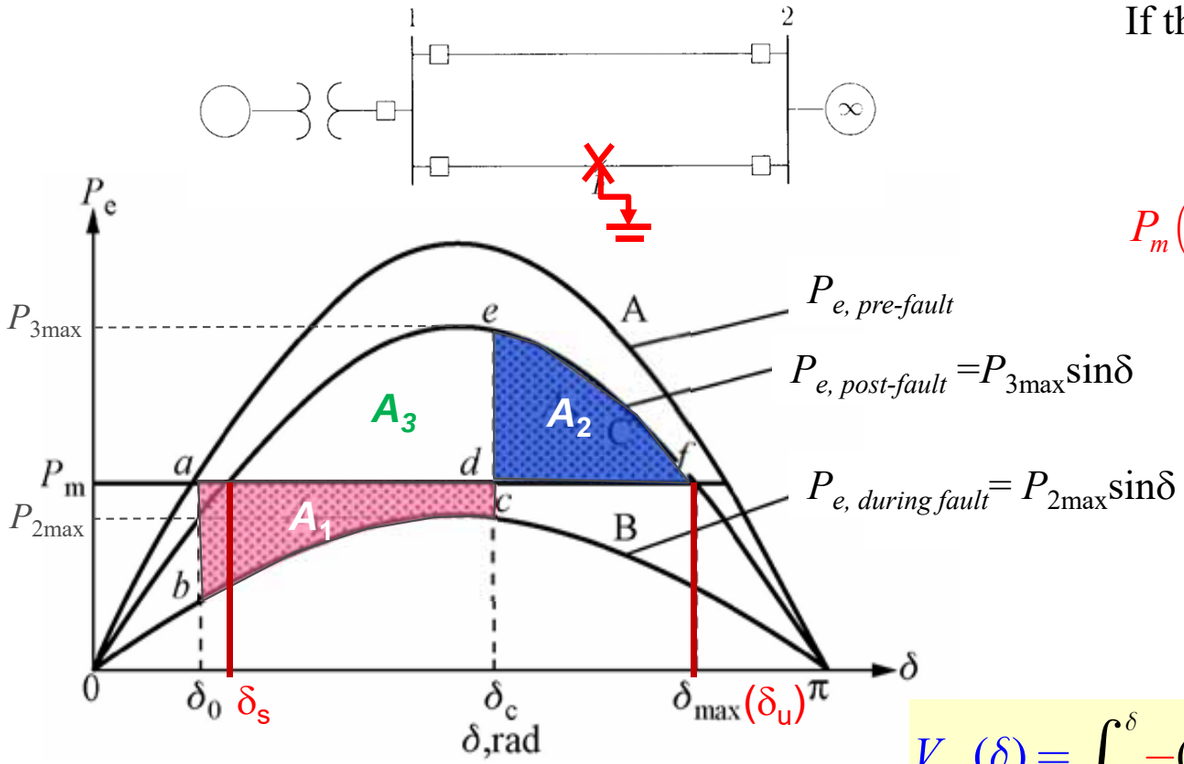
$$\frac{2H}{\omega_0}\frac{d^2\delta}{dt^2} = P_m - 0 \quad \Leftrightarrow \quad \frac{d\delta}{dt} = \frac{\omega_0}{2H}P_m\int_0^t dt = \frac{\omega_0}{2H}P_m t$$

$$\delta = \frac{\omega_0}{4H}P_m t^2 + \delta_0 \quad \Leftrightarrow \quad t_c = \sqrt{\frac{4H(\delta_c - \delta_0)}{\omega_0 P_m}}$$





A more general case with $P_{e,\ during\ fault}>0$



If the fault is cleared at the CCA δ_c

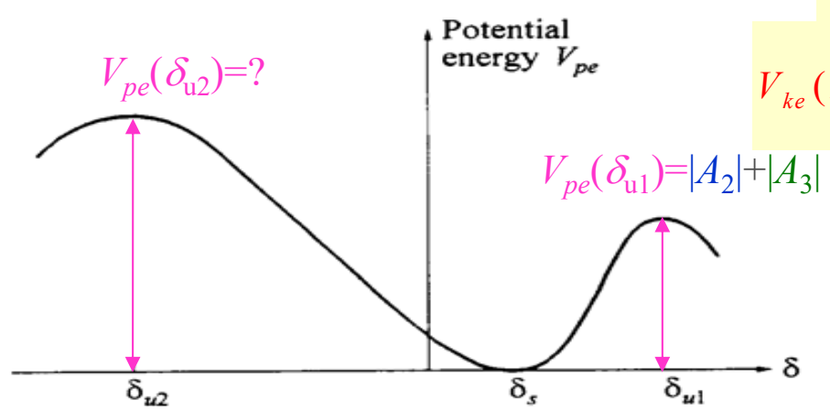
$$|A_1| \quad = \quad |A_2|$$

$$P_m(\delta_c-\delta_0)-\int_{\delta_0}^{\delta_c} P_{2max}\sin\delta d\delta=\int_{\delta_c}^{\delta_{max}} P_{3max}\sin\delta\, d\delta-P_m(\delta_{max}-\delta_c)$$

$$\cos\delta_c=\frac{P_m(\delta_{max}-\delta_0)+P_{3max}\cos\delta_{max}-P_{2max}\cos\delta_0}{P_{3max}-P_{2max}}$$

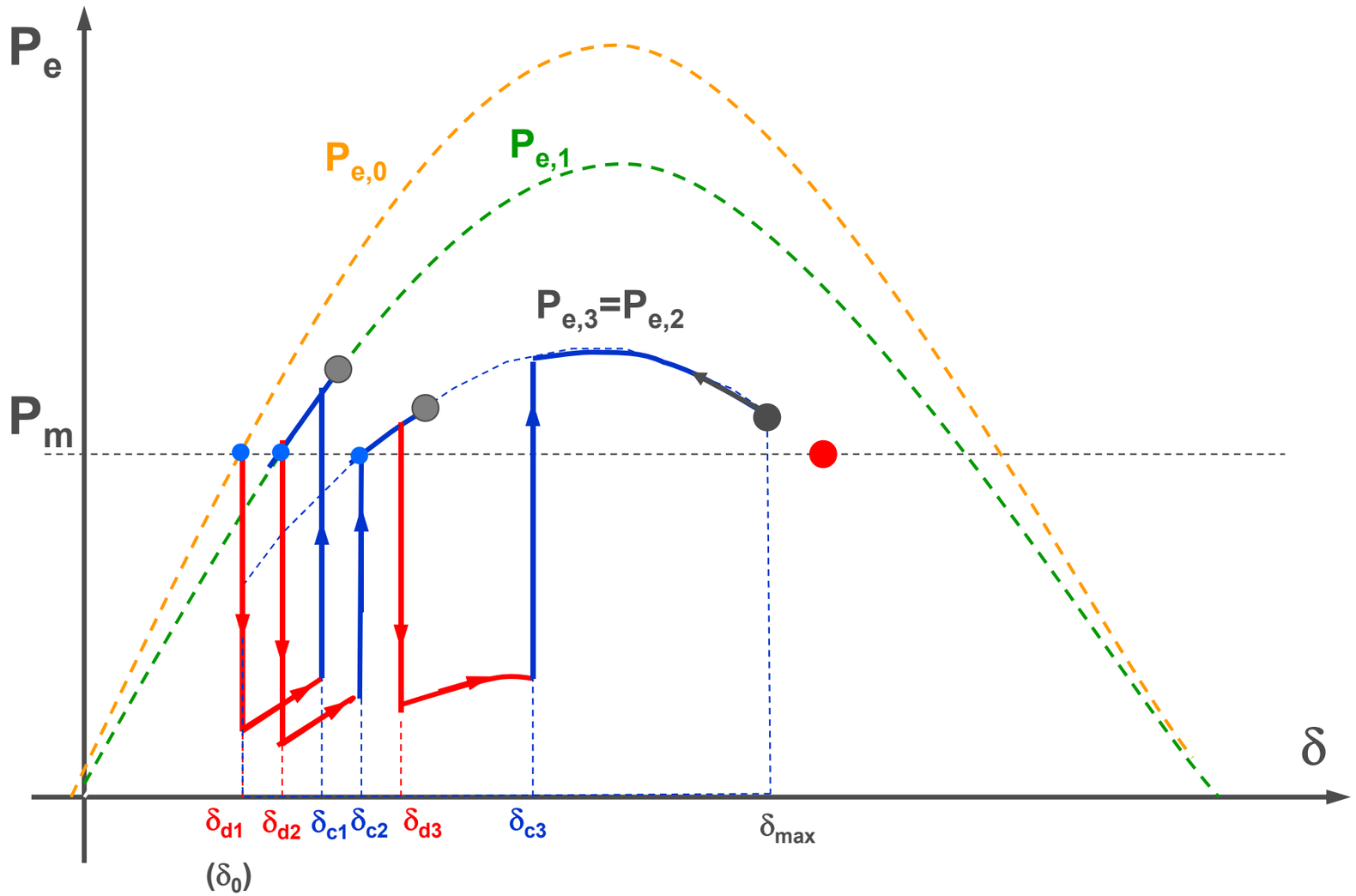
$$V_{pe}(\delta)=\int_{\delta_s}^{\delta}-(P_m-P_e)\,d\delta$$

$$V_{ke}(\Delta\omega_r)=\frac{1}{2}(2H\omega_0)\cdot\left(\frac{\Delta\omega_r}{\omega_0}\right)^2$$



- $|A_1|$ is the kinetic energy V_{ke} at δ_c
- $|A_1|+|A_3|$ is the total energy $V=V_{ke}+V_{pe}$ gained at δ_c after the fault is cleared
- $|A_2|+|A_3|=V_{pe}(\delta_u)=V_{cr}$, i.e. the maximum potential energy, or in other words, the critical energy
- The generator is stable if and only if $V(\delta_c)\le V_{cr}$ (i.e. $|A_1|+|A_3|\le|A_2|+|A_3|$ or equivalently, $|A_1|\le|A_2|$),

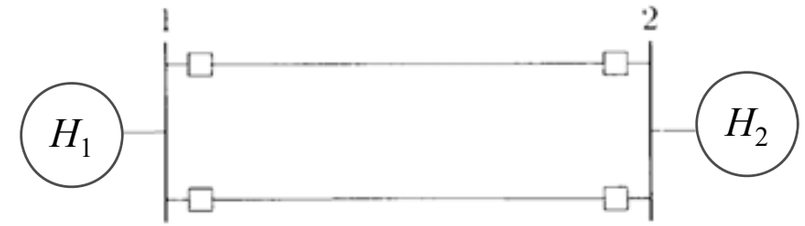
Multiple Faults



Some factors that influence transient stability of a generator

- How heavily the generator is loaded.
- The fault-clearing time
- The post-fault transmission system reactance
- The generator reactance: A lower reactance increases peak power and reduces initial rotor angle.
- The generator output during the fault: This depends on the fault location and type
- The generator inertia: The higher the inertia, the slower the rate of change in angle. This reduces the kinetic energy gained during fault; i.e. the accelerating area A_1 is reduced.
- The generator internal voltage magnitude (E'): This depends on the field excitation
- The voltage magnitude (E_B) of the bus receiving power from the generator

Applying EAC to a Two-Machine System



- Reduce two interconnected machines to an equivalent SMIB system.

$$\frac{d^2\delta_1}{dt^2} = \frac{\omega_0}{2H_1}(P_{m1} - P_{e1}) = \frac{\omega_0}{2H_1}P_{a1}$$

$$\frac{d^2\delta_2}{dt^2} = \frac{\omega_0}{2H_2}(P_{m2} - P_{e2}) = \frac{\omega_0}{2H_2}P_{a2}$$

$$\frac{d^2\delta_{12}}{dt^2} = \frac{d^2\delta_1}{dt^2} - \frac{d^2\delta_2}{dt^2} = \frac{\omega_0}{2}\left(\frac{P_{a1}}{H_1} - \frac{P_{a1}}{H_2}\right)$$

$$\frac{2}{\omega_0}\frac{H_1H_2}{H_1+H_2}\frac{d^2\delta_{12}}{dt^2} = \frac{H_2P_{a1} - H_1P_{a2}}{H_1+H_2} = \frac{H_2P_{m1} - H_1P_{m2}}{H_1+H_2} - \frac{H_2P_{e1} - H_1P_{e2}}{H_1+H_2}$$

$$\frac{2H_{12}}{\omega_0}\frac{d^2\delta_{12}}{dt^2} = P_{m,12} - P_{e,12}$$

$$\int_{\delta_{12,0}}^{\delta_{12,\max}} \frac{\omega_0}{H_{12}}(P_{\mathrm{m},12} - P_{e,12})d\delta = 0$$

H_2	H_{12}	$P_{m,12}$	$P_{e,12}$
$H_2=\infty$	H_1	$P_{m,1}$	$P_{e,1}$
$H_2=10H_1$	$0.909H_1$	$0.91P_{m,1}-0.09P_{m,2}$	$0.91P_{e,1}-0.09P_{e,2}$
$H_2=H_1$	$H_1/2$	$(P_{m,1}-P_{m,2})/2$	$(P_{e,1}-P_{e,2})/2$

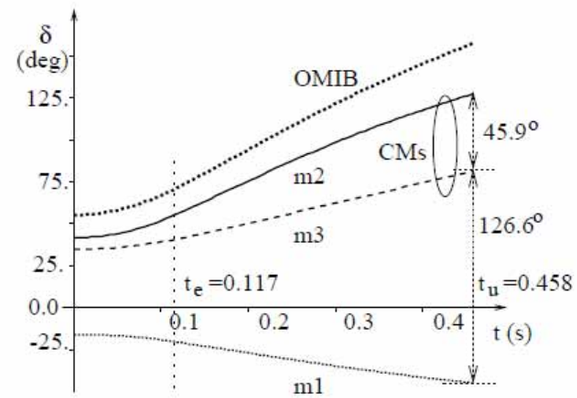
SMIB Equivalent Based Method (EEAC or SIME) for Multi-Machine Systems [1]-[3]

Main Idea:

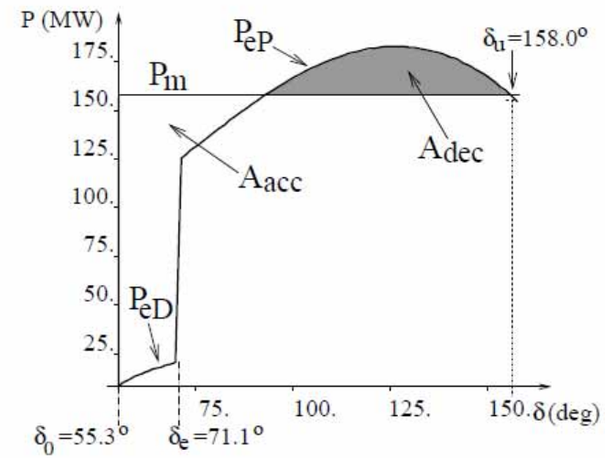
- According to rotor angle curves over a time window (e.g. being obtained from simulation), partition *m* machines into 2 groups
  - Critical machines (CMs)
  - Non-critical machines (NMs)

- Only *m*-1 ways of partitioning need to be studied.

- For each way of partitioning, construct a 2-machine equivalent and consequently an SMIB (i.e. "OMIB" in the papers) equivalent, such that conclusions of the EAC can be applied.



(a) Critical machines and OMIB trajectories
Nr of CMs: 2
Nr of NMs: 1



(b) Power-angle OMIB representation.
Resulting EAC parameters:
$\delta_e = 71.1°$; $\delta_u = 158.0° \rightarrow t_u = 0.458$ s .

(c) Notation:

P_e : electrical power ; P_m : mechanical power ;
$P_a = P_m - P_e$: accelerating power
t_e : clearing time ; t_u : time to instability
$\delta_e = \delta(t_e)$: clearing angle ; $\delta_u = \delta(t_u)$: unstable angle
Subscript D stands for "during-fault" (or fault-on) configuration
Subscript P stands for "post-fault" configuration

Figure 2.1. Swing curves and OMIB $P-\delta$ representation of the 3-machine system. Contingency Nr 2, $t_e = 117$ ms. General notation

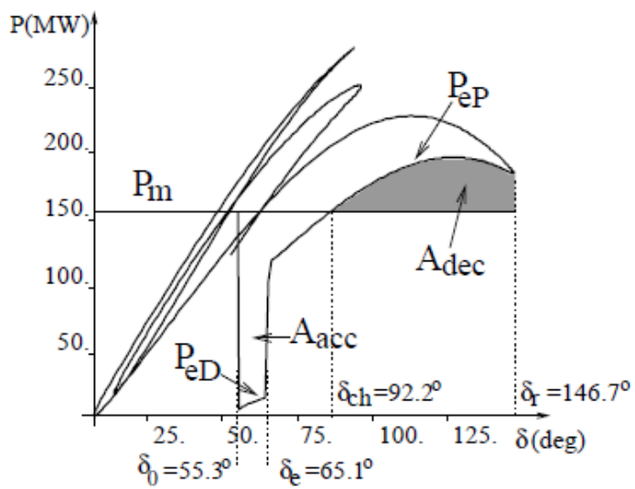
[1] Y. Xue, et al, "A Simple Direct Method for Fast Transient Stability Assessment of Large Power Systems". *IEEE Trans. PWRS*, PWRS3: 400–412, 1988.

[2] Y. Xue, et al, "Extended Equal-Area Criterion Revisited". *IEEE Trans. PWRS*, PWRS7: 10101022,1992.

[3] M. Pavella, et al, "Transient Stability of Power Systems: a Unified Approach to Assessment and Control", Kluwer, 2000.

Main Steps [3]

$$M\ddot{\delta} = P_m - P_e = P_a$$



$$\eta = A_{dec}^{\max} - A_{acc}$$

$$= -\int_{\delta_0}^{\delta_u} P_a d\delta$$

(i) Denoting by $\delta_C(t)$ the COA of the group of CMs, one writes:

$$\delta_C(t) \triangleq M_C^{-1} \sum_{k \in C} M_k \delta_k(t) \ . \tag{2.3}$$

Similarly:

$$\delta_N(t) = M_N^{-1} \sum_{j \in N} M_j \delta_j(t) \ . \tag{2.4}$$

In the above formulas:

$$M_C = \sum_{k \in C} M_k \ ; \ M_N = \sum_{j \in N} M_j \ . \tag{2.5}$$

(ii) Define the rotor angle of the corresponding OMIB by the transformation

$$\delta(t) \triangleq \delta_C(t) - \delta_N(t) \ . \tag{2.6}$$

The corresponding OMIB rotor speed is expressed by

$$\omega(t) = \omega_C(t) - \omega_N(t) \tag{2.7}$$

where

$$\omega_C(t) = M_C^{-1} \sum_{k \in C} M_k \omega_k(t) \ ; \ \omega_N(t) = M_N^{-1} \sum_{j \in N} M_j \omega_j(t) \ . \tag{2.8}$$

(iii) Define the equivalent OMIB mechanical power by

$$P_m(t) = M \left(M_C^{-1} \sum_{k \in C} P_{mk}(t) - M_N^{-1} \sum_{j \in N} P_{mj}(t) \right) \ , \tag{2.9}$$

the equivalent OMIB electric power by

$$P_e(t) = M \left(M_C^{-1} \sum_{k \in C} P_{ek}(t) - M_N^{-1} \sum_{j \in N} P_{ej}(t) \right) \ , \tag{2.10}$$

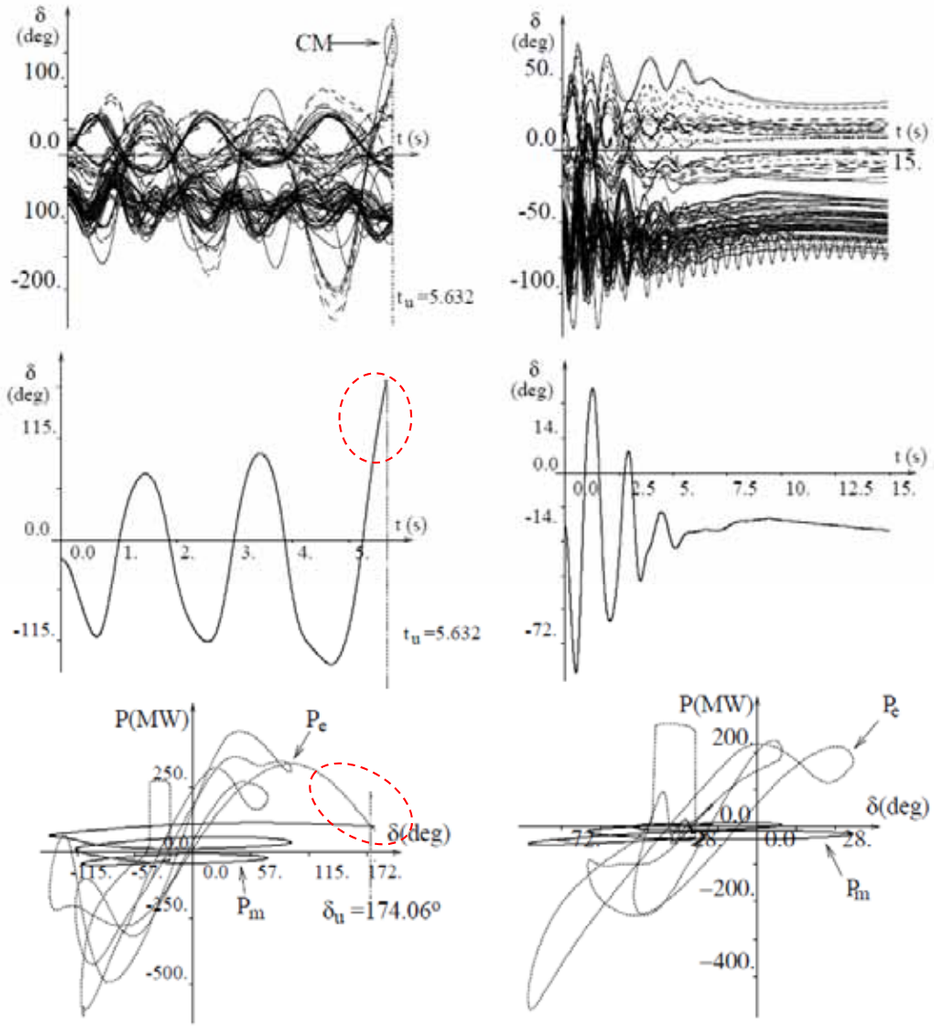
and the resulting OMIB accelerating power by

$$P_a(t) = P_m(t) - P_e(t) \ . \tag{2.11}$$

In the above expressions, M denotes the equivalent OMIB inertia coefficient

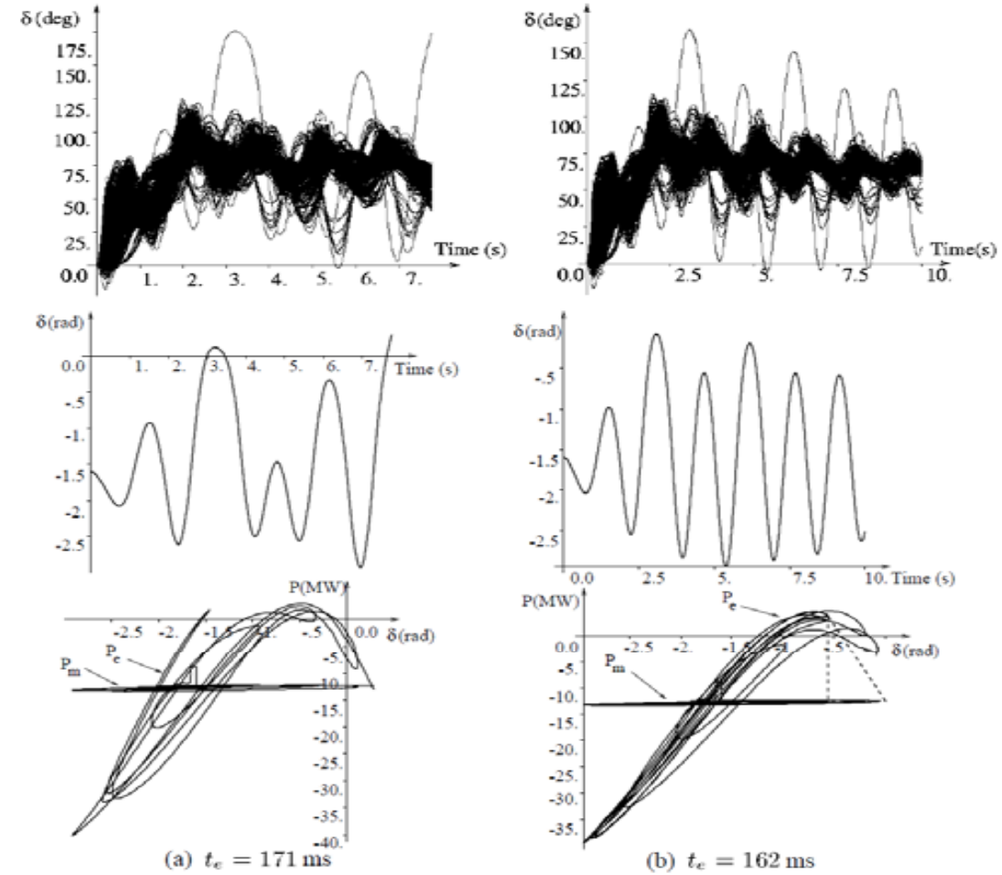
$$M = \frac{M_C M_N}{M_C + M_N} \ . \tag{2.12}$$

Applications to real systems [3]



(a) $t_e = 198$ ms (b) $t_e = 175$ ms

Figure 2.9. Illustration of back- and multi-swing phenomena on the Hydro-Québec system: swing and $P - \delta$ curves. CCT(SIME) = 175.5 ms ; CCT(ST-600) = 176 ms .



(a) $t_e = 171$ ms (b) $t_e = 162$ ms

Figure 4.3. Multimachine and OMIB curves for ctg Nr 9 on the 627-machine system.

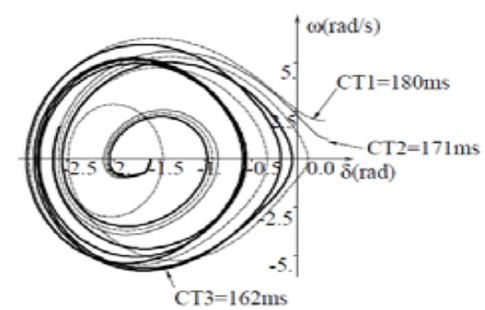
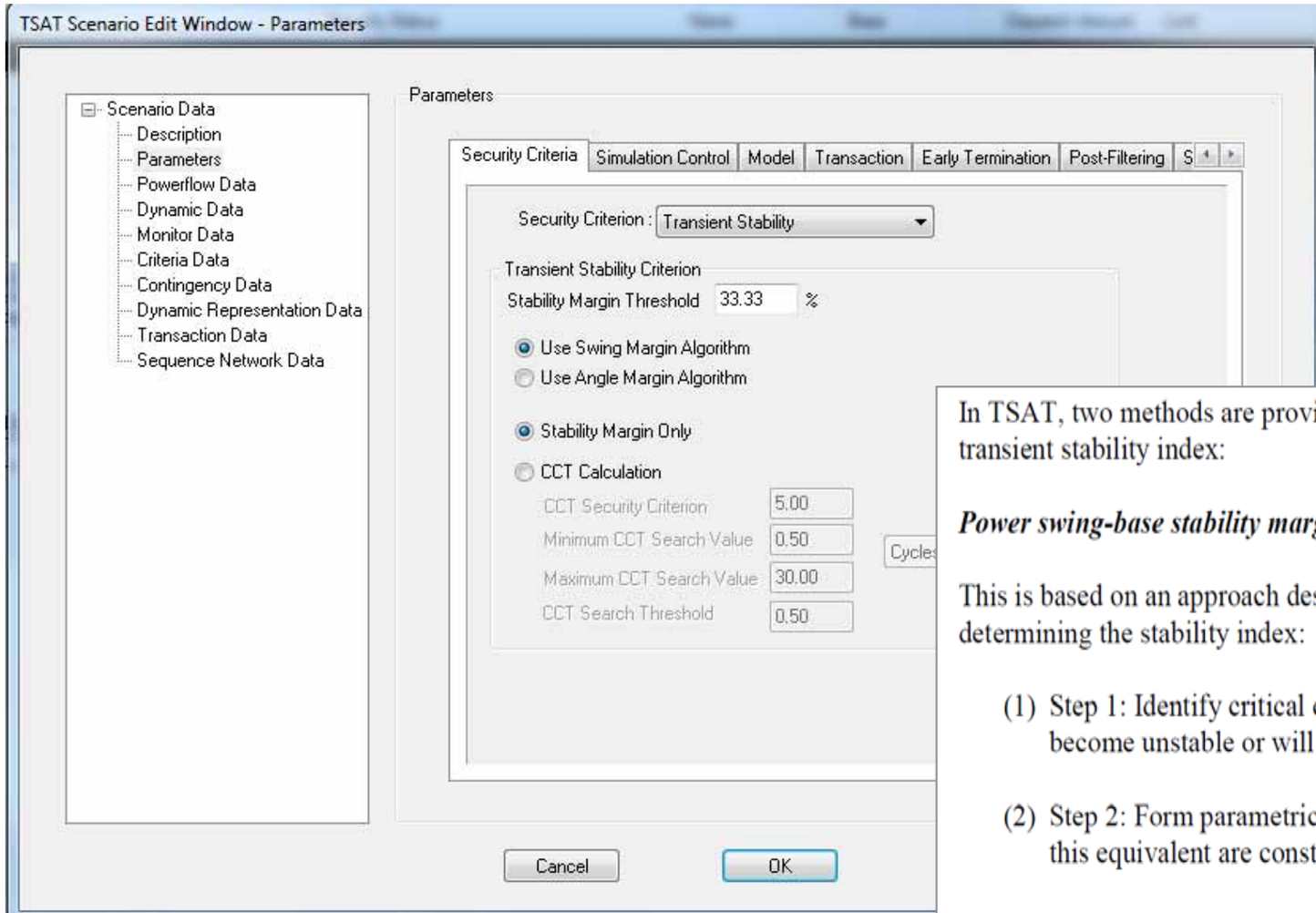


Figure 4.4. Phase plane representation of simulations of Table 4.3. 627-machine system. CCT(SIME) = 168 ms ; CCT(ETMSP) = 168 ms

Application in Commercialized Software



From TSAT v.14 User Manual by Powertech Labs

In TSAT, two methods are provided to assess the severity of a contingency, each of which gives a transient stability index:

Power swing-base stability margin or index (SM)

This is based on an approach described in reference [2]. This method consists of three steps in determining the stability index:

(1) Step 1: Identify critical cluster of generators (CCG). This is the group of generators that become unstable or will likely become unstable at the more stressed system condition.

(2) Step 2: Form parametric one-machine-infinite-bus (OMIB) equivalent. The parameters of this equivalent are constantly updated using simulation results of the full system.

(3) Step 3: Determine stability of the system and compute stability margin (index).

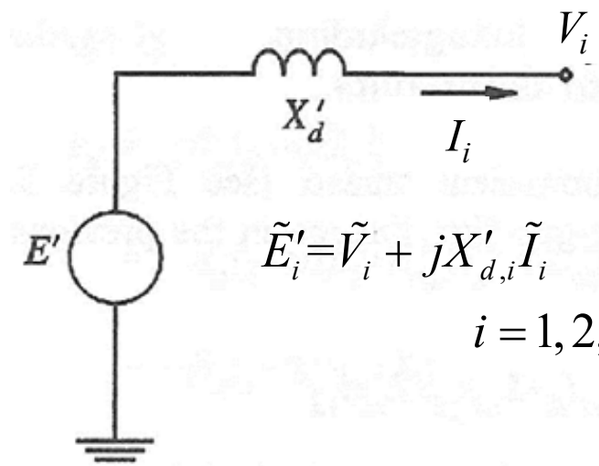
The full derivation and description of this method is contained in [2] and the associated literature on the subject.

Multi-Machine System in a Simplified Model

- Assumptions:
  - Each group of coherent machines, which swing together, are represented by one equivalent machine with damping neglected:

$$\frac{2H_i}{\omega_0}\frac{d^2\delta_i}{dt^2} = P_{mi} - P_{ei} \qquad i = 1, 2, \cdots, m$$

  - Then, each machine is represented by a voltage source E' behind X'_d (neglecting armature resistance R_a, the effect of saliency and the changes in flux linkages); The mechanical rotor angle of each machine coincides with the angle of E'.



$$\tilde{E}'_i = \tilde{V}_i + jX'_{d,i}\tilde{I}_i$$

$$i = 1, 2, \cdots, m$$

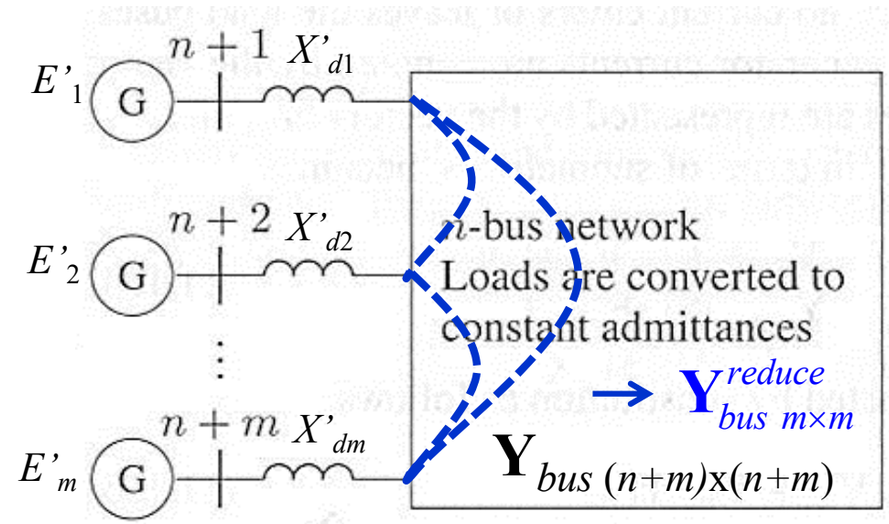
  - P_{mi} is assumed to remain constant during the entire period of simulation (neglecting governor control).

  - Using the pre-fault bus voltages, all loads are converted to equivalent admittances to ground. These admittances are assumed to remain constant, i.e. constant impedance load models.

- Add m internal generator nodes (i.e. E'_i behind X'_{di}) to the n-bus network to form an $(n+m)$-bus network.
- Then node voltage equations with ground as the reference

$$\mathbf{I}_{bus}=\mathbf{Y}_{bus}\,\mathbf{V}_{bus}$$

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \\ \hline I_{n+1} \\ \vdots \\ I_{n+m} \end{bmatrix} = \left[\begin{array}{ccc|ccc} Y_{11} & \cdots & Y_{1n} & Y_{1(n+1)} & \cdots & Y_{1(n+m)} \\ Y_{21} & \cdots & Y_{2n} & Y_{2(n+1)} & \cdots & Y_{2(n+m)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{n1} & \cdots & Y_{nn} & Y_{n(n+1)} & \cdots & Y_{n(n+m)} \\ \hline Y_{(n+1)1} & \cdots & Y_{(n+1)n} & Y_{(n+1)(n+1)} & \cdots & Y_{(n+1)(n+m)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{(n+m)1} & \cdots & Y_{(n+m)n} & Y_{(n+m)(n+1)} & \cdots & Y_{(n+m)(n+m)} \end{array}\right] \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ \hline E'_{n+1} \\ \vdots \\ E'_{n+m} \end{bmatrix}$$



$\mathbf{I}_{bus}$ is the vector of the injected bus currents

$\mathbf{V}_{bus}$ is the vector of bus voltages measured from the reference node

$\mathbf{Y}_{bus}$ is the bus admittance matrix of $(n+m)\times(n+m)$:

Y_{ii} (diagonal element) is the sum of admittances connected to bus i

Y_{ij} (off-diagonal element) equals the negative of the admittance between buses i and j

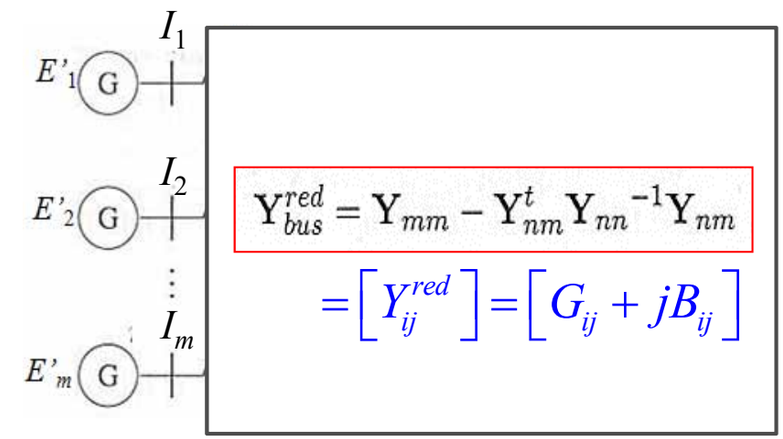
Compared to the $\mathbf{Y}_{bus}$ for power flow analysis, additional m internal generator nodes are added and Y_{ii} $(i\leq n)$ is modified to include the load admittance at node i

- To simplify the analysis, all nodes other than the generator internal nodes are eliminated as follows

$$\begin{bmatrix} 0 \\ \mathbf{I}_m \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{nn} & \mathbf{Y}_{nm} \\ \mathbf{Y}_{nm}^t & \mathbf{Y}_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{V}_n \\ \mathbf{E}'_m \end{bmatrix}$$

$$0 = \mathbf{Y}_{nn}\mathbf{V}_n + \mathbf{Y}_{nm}\mathbf{E}'_m \quad \text{(a)}$$

$$\mathbf{I}_m = \mathbf{Y}_{nm}^t\mathbf{V}_n + \mathbf{Y}_{mm}\mathbf{E}'_m \quad \text{(b)}$$



From (a):

$$\mathbf{V}_n = -\mathbf{Y}_{nn}^{-1}\mathbf{Y}_{nm}\mathbf{E}'_m$$

(a) → (b):

$$\mathbf{I}_m = [\mathbf{Y}_{mm} - \mathbf{Y}_{nm}^t\mathbf{Y}_{nn}^{-1}\mathbf{Y}_{nm}]\mathbf{E}'_m$$
$$= \mathbf{Y}_{bus}^{red}\mathbf{E}'_m$$

- The electrical power of each machine:

$$P_{ei} = \mathrm{Re}(\tilde{E}'_i\tilde{I}_i^*) = \mathrm{Re}(\tilde{E}'_i\sum_{j=1}^{m} Y_{ij}^{red*}\tilde{E}'^*_j) = E_i'^2 G_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} E'_iE'_j\,|Y_{ij}^{red}|\cos\left(\theta_{ij} - \delta_{ij}\right)$$

$$= E_i'^2 G_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{n} E'_iE'_j\left(B_{ij}\sin\delta_{ij} + G_{ij}\cos\delta_{ij}\right)$$

$$\delta_{ij} = \delta_i - \delta_j$$

θ_{ij} is the angle of Y_{ij}^{red}

- The power system model:

$$\frac{2H_i}{\omega_0}\ddot{\delta}_i = P_{mi} - E_i'^2 G_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} E'_iE'_j\left(B_{ij}\sin\delta_{ij} + G_{ij}\cos\delta_{ij}\right) = P_{mi} - E_i'^2 G_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(C_{ij}\sin\delta_{ij} + D_{ij}\cos\delta_{ij}\right)$$

$$= P'_{mi} \qquad - P'_{ei}$$

TEF Method for a Multi-machine Power System

- Consider the simplified system model (classical-model generators without damping + constant impedance loads).

$$\frac{2H_i}{\omega_0}\ddot{\delta}_i = P_{mi} - E_i'^2 G_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n}\left(C_{ij}\sin\delta_{ij} + D_{ij}\cos\delta_{ij}\right) = P_{mi}' - P_{ei}'$$

- Define the center of inertia (COI) and the motion of the COI about all generators:

$$\delta_{COI} \overset{\text{def}}{=} \frac{\sum_{i=1}^{m} H_i\delta_i}{\sum_{i=1}^{n} H_i} = \frac{\sum_{i=1}^{m} H_i\delta_i}{H_\Sigma} \qquad \Delta\omega_{COI} \overset{\text{def}}{=} \frac{\dot{\delta}_{COT}}{\omega_0} \qquad P_{COI} \overset{\text{def}}{=} \sum_{i=1}^{m}(P_{mi}' - P_{ei}') = \frac{2H_\Sigma}{\omega_0}\ddot{\delta}_{COT} = 2H_\Sigma\Delta\dot{\omega}_{COI}$$

- Consider the motion of each generator w.r.t. the COI

$$\theta_i = \delta_i - \delta_{COI} \quad \text{rad}$$

$$\omega_i = \frac{\dot{\theta}_i}{\omega_0} = \left(\frac{\dot{\delta}_i}{\omega_0} - \Delta\omega_{COI}\right) \quad \text{pu}$$

$$2H_i\dot{\omega}_i = P_{mi}' - P_{ei}' - \frac{H_i}{H_\Sigma}P_{COI}$$

$$\dot{\theta}_i = \omega_i\omega_0$$

Easy to prove:

$$\sum_{i=1}^{m} H_i\theta_i = 0$$

$$\sum_{i=1}^{m} H_i\omega_i = 0$$

$$\sum_{i=1}^{m}\frac{H_i}{H_\Sigma}P_{COI}\frac{d\theta_i}{dt} = \sum_{i=1}^{m}\frac{H_i}{H_\Sigma}P_{COI}\omega_i\omega_0 = \frac{P_{COI}\omega_0}{H_\Sigma}\sum_{i=1}^{m} H_i\omega_i = 0$$

Defining the post-disturbance TEF

θ_i^s = angle of bus i at the postdisturbance SEP
$J_i = 2H_i\omega_0$ = per unit moment of inertia of the ith generator

$$P'_{ei} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(C_{ij}\sin\delta_{ij} + D_{ij}\cos\delta_{ij} \right)$$

$$V \overset{\text{def}}{=} \frac{1}{2}\sum_{i=1}^{m} J_i\omega_i^2 + \sum_{i=1}^{m}\int_{\theta_i^S}^{\theta_i} -(P'_{mi} - P'_{ei} - \frac{H_i}{H_T}P_{COI})d\theta_i$$

Assume a linear integration path [4]

$$d(\theta_i+\theta_j) \approx k\cdot d\theta_{ij} = \frac{(\theta_i^{cl}+\theta_j^{cl}) - (\theta_i^S+\theta_j^S)}{\theta_{ij}^{cl} - \theta_{ij}^S} d\theta_{ij}$$

$$V = \frac{1}{2}\sum_{i=1}^{m} J_i\omega_i^2 - \sum_{i=1}^{m} P'_{mi}(\theta_i - \theta_i^s) - \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left[C_{ij}(\cos\theta_{ij} - \cos\theta_{ij}^s) - \int_{\theta_i^s+\theta_j^s}^{\theta_i+\theta_j} D_{ij}\cos\theta_{ij}\, d(\theta_i+\theta_j) \right] \quad +0$$

$$= \sum_i V_{ke,i}(\dot{\theta}_i) \quad = V_{ke}$$

$$+\sum_i V_{pe,i}(\theta_i) \quad +\sum_{i,j} V_{magnetic,ij}(\theta_{ij}) \quad +\sum_{i,j} V_{dissipated,ij}(\text{trajectory of } \theta_i + \theta_j) \quad \approx V_{pe}$$

- Procedure of the TEF method:
  1. Run time-domain simulation up to the instant of fault clearing (t_{cl}) to obtain angles and speeds of all generators, which are used to calculate $V(\mathbf{x}_{cl})$
  2. Calculate the critical energy V_{cr} for the post-disturbance system (this is the most difficult step for a large-scale systems; V_{cr} may be defined as the maximum V_{pe} at the closest UEP or controlling UEP)
  3. Check V_{cr}-$V(\mathbf{x}_{cl})$

[4] J.N. Qiang, Clarifications on the Integration Path of Transient Energy Function, IEEE Trans Power Systems, May 2005

Time-domain transient stability simulation

Simulating a Multi-Machine System in a Simplified Model

- Solve the initial power flow and determine the initial bus voltage phasors V_i.

$$I_i = \frac{S_i^*}{V_i^*} = \frac{P_i - jQ_i}{V_i^*}$$

- Terminal currents I_i of m generators prior to disturbance are calculated by their terminal voltages V_i and power outputs S_i, and then used to calculate E'_i

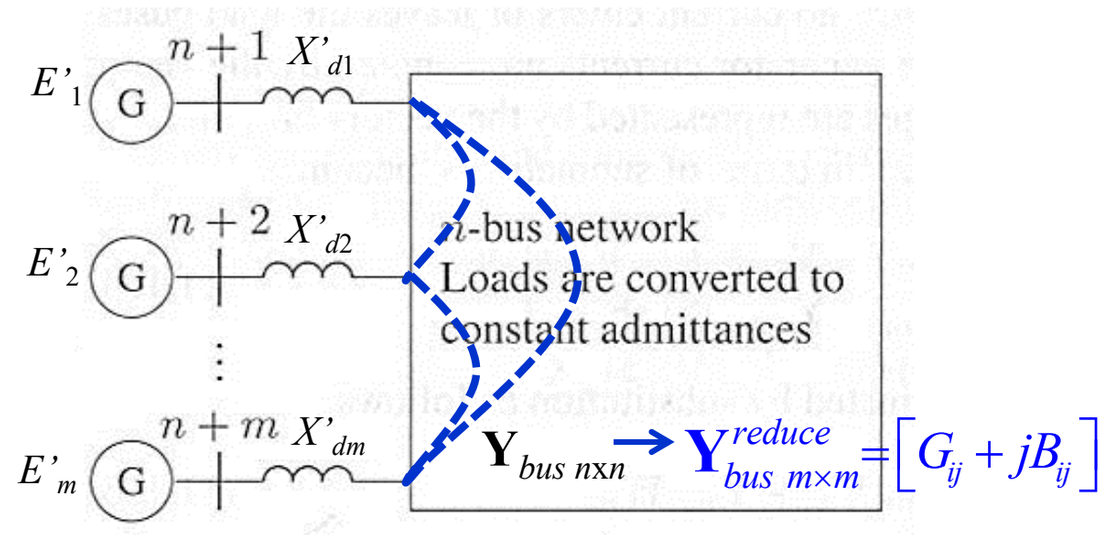
$$E_i' = V_i + jX_{di}' I_i \qquad i = 1, 2, \ldots, m$$

- All loads are converted to equivalent admittances:

$$y_{i0} = \frac{S_i^*}{|V_i|^2} = \frac{P_i - jQ_i}{|V_i|^2}$$

- To include voltages behind X'_{di}, add m internal generator buses to the n-bus power system network to form an $n+m$ bus network (ground as the reference for voltages):

$$\frac{2H_i}{\omega_0}\ddot{\delta}_i = P_{mi} - E_i'^2 G_{ii} - \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i' E_j' \left(B_{ij}\sin\delta_{ij} + G_{ij}\cos\delta_{ij} \right)$$



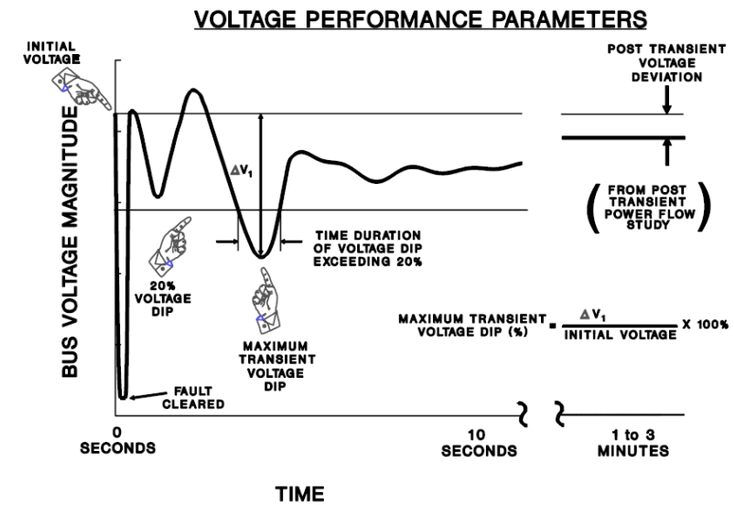
Today's Bulk Power System Simulation

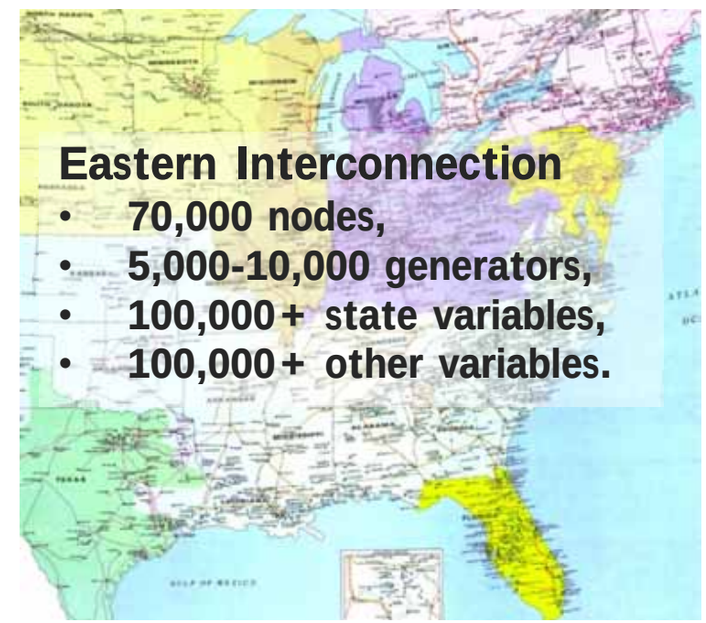
- For transient stability simulation of a bulk power system, its phasor model is usually adopted, which includes a set of nonlinear differential-algebraic equations validated by industry using event data.

$$\dot{x} = f(x, V) \qquad \text{DE}$$

$$I(x, V) = Y_N V \qquad \text{AE}$$

- Simulation solves its initial value problems (IVPs) on contingencies using numerical solvers (explicit or implicit methods)
- Commercial software (PSS/E or PSLF) takes 1-5 min to simulate the 70K-node Eastern Interconnection model (e.g. NERC MMWG model) for each wall-clock second (it can be speeded up to 4-5 sec/sec on a supercomputer with 512 processors by a parallel-in-time algorithm [5])
- Best industry practices in North America simulate 1-3K critical contingencies every 10-15 min on a reduced model (~10K nodes & 2K generators), while most of power companies run off-line simulations.





[5] D. Osipov, N. Duan, S. Allu, S. Simunovic, A. Dimitrovski, K. Sun, "Distributed Parareal in Time with Adaptive Coarse Solver for Large Scale Power System Simulations," 2019 IEEE PES General Meeting in Atlanta, GA

Numerical Integration Methods

- The differential equations to be solved for transient stability analysis are nonlinear ordinary differential equations with known initial values $\mathbf{x}=\mathbf{x_0}$ and $t=t_0$

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x},t) \qquad \rightarrow \qquad \Delta\mathbf{x} \approx f(\mathbf{x},t)\Delta t$$

where $\mathbf{x}$ is the state vector of n dependent variables and t is the independent variable (time). The objective is to solve $\mathbf{x}$ as a function of t

- **Explicit Methods**
  - computing the value of $\mathbf{x}$ at any time t using the values of $\mathbf{x}$ from only the previous time steps, e.g. Euler and R-K methods

(Forward) Euler method: $\mathbf{x}_{i+1} = \mathbf{x}_i + f(\mathbf{x}_i, t_i)\Delta t$

- **Implicit Methods**
  - Using interpolation functions involving future time steps for the expression under the integral, e.g. the Backward Euler and Trapezoidal Rule methods

Backward Euler method: $\mathbf{x}_{i+1} = \mathbf{x}_i + f(\mathbf{x}_{i+1}, t_{i+1})\Delta t$

(Forward) Euler Method

- The Euler method is equivalent to using the first two terms of the Taylor series about x around the point (x_0, t_0), referred to as a ***first-order* method** (whose error is on the order of Δt^2)
  - Approximate the curve at x=x_0 and t=t_0 by its tangent

$$\left.\frac{dx}{dt}\right|_{x_0} = f(x_0, t_0) \qquad \Delta x \approx \left.\frac{dx}{dt}\right|_{x_0} \Delta t$$

$$x_1 = x_0 + \Delta x = x_0 + \left.\frac{dx}{dt}\right|_{x_0} \Delta t = x_0 + f(x_0, t_0)\Delta t$$

  - At step i+1:

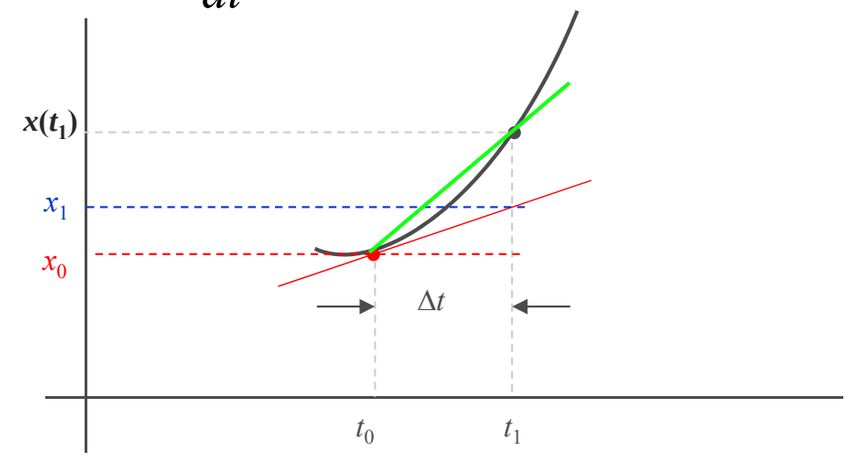
$$x_{i+1} = x_i + f(x_i, t_i)\Delta t$$

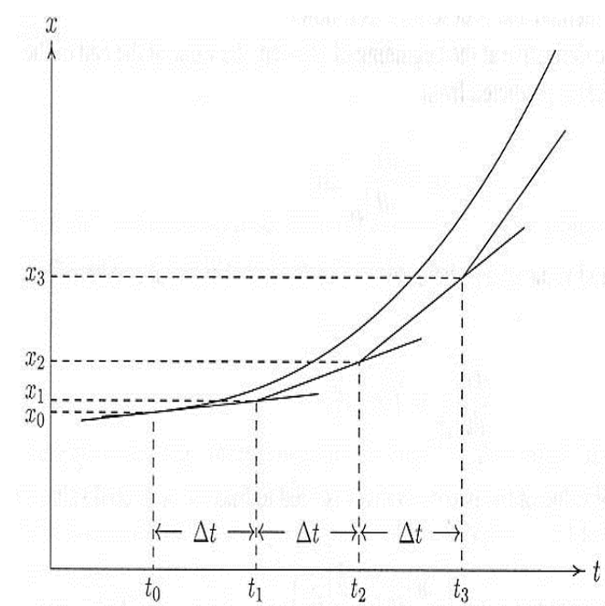
- It is explicit compared to the Backward Euler Method (implicit)

$$x_{i+1} = x_i + f(x_{i+1}, t_{i+1})\Delta t$$

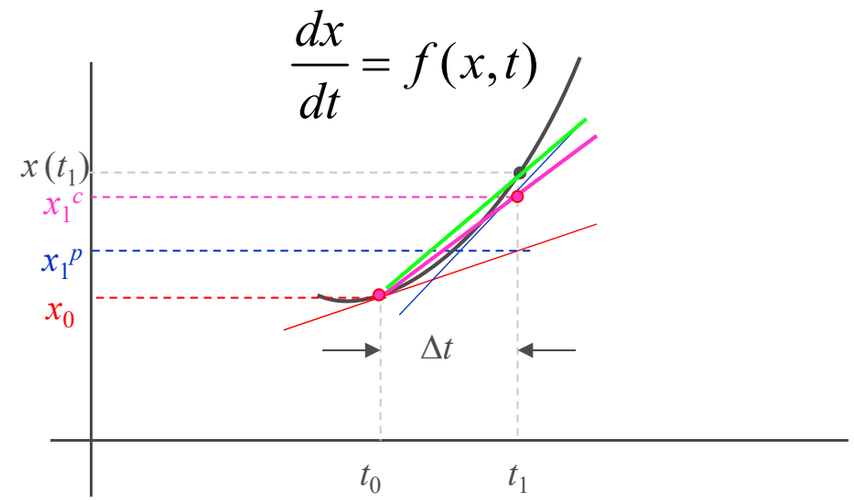
- The forward Euler method results in inaccuracy and a propagation of the truncation error because it uses the derivative only from the beginning of each interval throughout the entire interval.

$$\frac{dx}{dt} = f(x,t)$$





Modified Euler (ME) Method



- Modified Euler method consists of two steps:

  (a) Predictor step:

$$x_1^p = x_0 + \left.\frac{dx}{dt}\right|_{x_0} \Delta t = x_0 + f(x_0, t_0)\Delta t$$

Slope at the beginning of Δt

The derivative at the end of the Δt is estimated using x_1^p

$$\left.\frac{dx}{dt}\right|_{x_1^p} \simeq f(x_1^p, t_1)$$ Estimated slope at the end of Δt

  (b) Corrector step:

$$x_1^c = x_0 + \frac{\left.\frac{dx}{dt}\right|_{x_0} + \left.\frac{dx}{dt}\right|_{x_1^p}}{2}\Delta t = x_0 + \frac{f(x_0,t_0) + f(x_1^p, t_1)}{2}\Delta t \qquad x_{i+1}^c = x_i + \frac{f(x_i,t_i) + f(x_{i+1}^p, t_{i+1})}{2}\Delta t$$

- It is a ***second-order* method** (error is on the order of Δt^3)
- Step size Δt must be small enough to obtain a reasonably accurate solution, but at the same time, large enough to avoid the numerical instability with the computer.

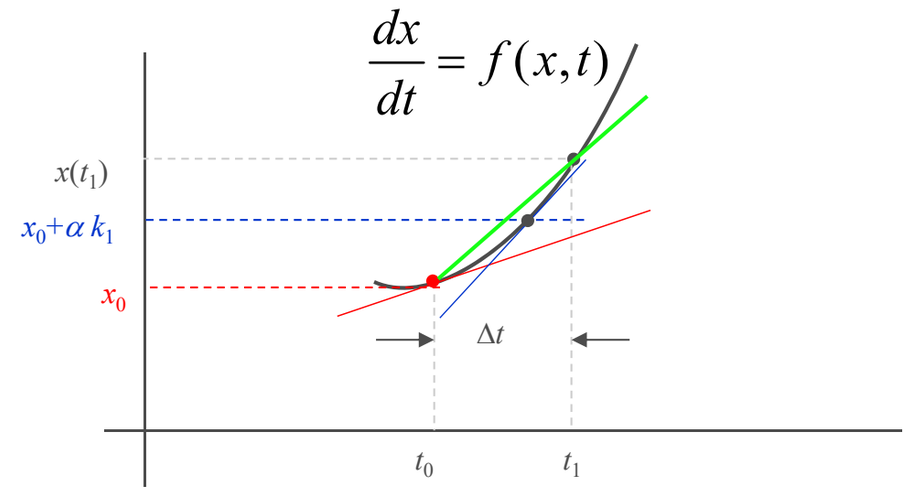
Runge-Kutta (R-K) Methods

- General formula of the 2nd order R-K method (RK2): (error is on the order of Δt^3)

$$k_1 = f(x_0, t_0)$$
$$k_2 = f(x_0 + \alpha k_1, t_0 + \beta \Delta t)$$
$$x_1 = x_0 + (a_1 k_1 + a_2 k_2)\Delta t$$

At Step i+1:

$$k_1 = f(x_i, t_i)$$
$$k_2 = f(x_i + \alpha k_1, t_i + \beta \Delta t) \qquad \sim O(\Delta t)$$
$$x_{i+1} = x_i + (a_1 k_1 + a_2 k_2)\Delta t \qquad \sim O(\Delta t^2)$$



When a_1=a_2=1/2, α=β=1, the RK2 method becomes the ME method (i.e. a special case of RK2)

- General formula of the 4th order R-K method: (error is on the order of Δt^5)

$$x_{i+1} = x_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\Delta t \qquad \sim O(\Delta t^4)$$

$$k_1 = f(x_i, t_i)$$
$$k_2 = f(x_i + \frac{k_1}{2}, t_i + \frac{\Delta t}{2}) \qquad \sim O(\Delta t)$$
$$k_3 = f(x_i + \frac{k_2}{2}, t_i + \frac{\Delta t}{2}) \qquad \sim O(\Delta t^2)$$
$$k_4 = f(x_i + k_3, t_i + \Delta t) \qquad \sim O(\Delta t^3)$$

Numerical Stability of Explicit Integration Methods

- Numerical stability is related to the **stiffness** of the set of differential equations representing the system.

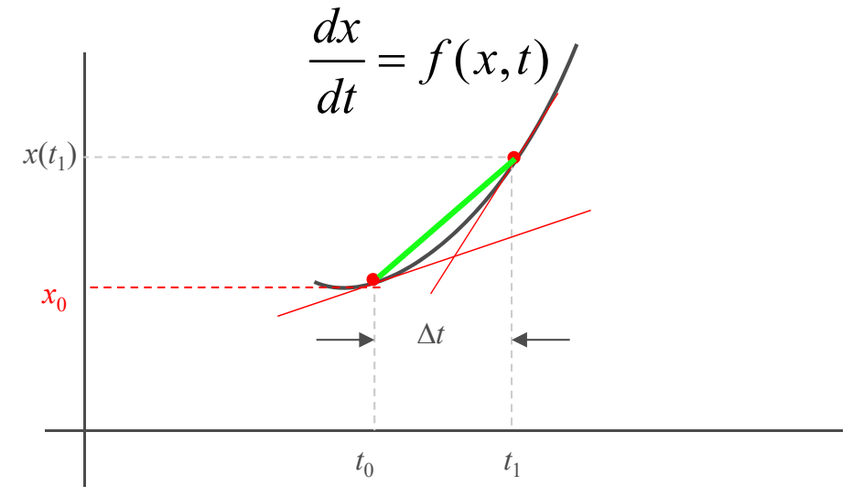
- The **stiffness** is measured by the ratio of the largest to smallest time constant, or more precisely by $|\lambda_{max}/\lambda_{min}|$ of the linearized system.

- **Stiffness** in a transient stability simulation increases with more details (more smaller time constants) being modeled.

- Explicit integration methods have weak stability numerically; with stiff systems, the solution "blows up" unless a small step size is used. Even after the fast modes die out, small time steps continue to be required to maintain numerical stability.

Implicit Methods

- Implicit methods use interpolation functions for the expression under the integral. "Interpolation" implies the function must pass through the yet unknown points at t_1.
- A widely used implicit integration method is the Trapezoidal Rule method. It uses linear interpolation.
- The stiffness of the system being analyzed affects accuracy but not numerical stability. With larger time steps, high frequency modes and fast transients are filtered out, and the solutions for the slower modes is still accurate. For example, for the Trapezoidal rule, only dynamic modes faster than $f(x_n,t_n)$ and $f(x_{n+1},t_{n+1})$ are neglected.



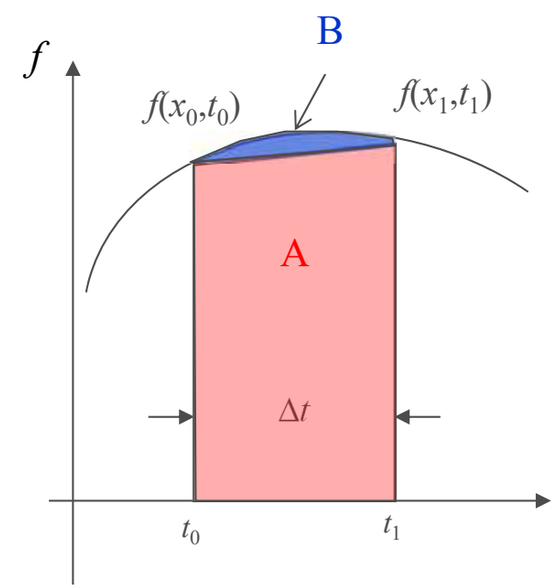
$$x_1 = x_0 + \int_{t_0}^{t_1} f(x,t)dt = x_0 + |A| + |B| \approx x_0 + |A|$$

$$x_1 = x_0 + \frac{\Delta t}{2}\left[f(x_0,t_0) + f(x_1,t_1)\right]$$

$$x_{n+1} = x_n + \frac{\Delta t}{2}\left[f(x_n,t_n) + f(x_{n+1},t_{n+1})\right]$$

Compared to ME method:

$$x_1 = x_0 + \frac{\Delta t}{2}\left[f(x_0,t_0) + f(x_1^p,t_1)\right]$$



Comparison of Explicit and Implicit Methods

$$\dot{x} = f(x,t) \approx \lambda_{\max} x$$

Euler Method (explicit)

$$x_i = x_{i-1} + f(x_{i-1}, t_{i-1})\Delta t$$

$$\approx x_{i-1} + \lambda_{\max} x_{i-1} \Delta t$$

$$= x_{i-1}(1 + \lambda_{\max}\Delta t)$$

$$x_i = x_0 (1 + \lambda_{\max}\Delta t)^i$$

The method is numerically stabile if

$$|1 + \lambda_{\max}\Delta t| < 1$$

$\Rightarrow \lambda_{\max}$ has a negative real part and

$$\Delta t < \frac{2}{|\lambda_{\max}|}$$

Backward Euler Method (implicit)

$$x_i = x_{i-1} + f(x_i, t_i)\Delta t$$

$$\approx x_{i-1} + \lambda_{\max} x_i \Delta t$$

$$x_i = x_{i-1} \frac{1}{1 - \lambda_{\max}\Delta t}$$

$$x_i = x_0 \left(\frac{1}{1 - \lambda_{\max}\Delta t} \right)^i$$

Δt can be arbitrarily large as long as $\lambda_{\max}$ has a negative real part (this method has **A-Stability**)

Simulating a General Multi-Machine System

- Overall system equations are expressed in the general form comprising a set of 1st-order DEs (dynamic devices) and a set of algebraic equations (devices and network)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{V}) \qquad \text{DE}$$

$$\mathbf{I}(\mathbf{x}, \mathbf{V}) = \mathbf{Y}_N \mathbf{V} \qquad \text{AE}$$

where

$\mathbf{x}$	state vector of the system
$\mathbf{V}$	bus voltage vector
$\mathbf{I}$	current injection vector
$\mathbf{Y}_N$	node admittance matrix. It is constant except for changes introduced by network-switching operations; it is symmetrical except for the dissymmetry introduced by phase-shifting transformers.

- Schemes for the solution of **DE** and **AE** are characterized by these factors
  - **The integration method used to solve the DE**, either an implicit method or an explicit method.
  - **The method used to solve the AE** (power flow analysis), e.g. the Newton-Raphson method.
  - **The interfacing** between the DE and AE: either a partitioned approach or a simultaneous approach may be used
- Most commercialized power system simulation programs provide the Modified Euler, 2nd order R-K, 4th order R-K and Trapezoidal Rule methods

Partitioned (alternating) scheme

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{V}) \qquad \text{DE}$$

$$\mathbf{I}(\mathbf{x}, \mathbf{V}) = \mathbf{Y}_\mathbf{N}\mathbf{V} \qquad \text{AE}$$

- At each time step t_n:

$$\mathbf{I}(\mathbf{x}_n, \mathbf{V}_n) - \mathbf{Y}_\mathbf{N}\mathbf{V}_n = 0$$

$\mathbf{x}_n$ ↑ Alternating ↓ $\mathbf{V}_n$

$$\dot{\mathbf{x}}_n = \mathbf{f}(\mathbf{x}_n, \mathbf{V}_n)$$

- To solve **V**, **x** may not be known accurately.
- Similarly, to solve **x**, **V** may be only approximately available.
- This can lead to interface errors, the elimination of which would require extrapolation (i.e. $\mathbf{V}_n^k \leftarrow \mathbf{V}_n^{k-1}$, $\mathbf{V}_n^{k-2}$, …) or iteration of the solution process at each time step until

$$|\mathbf{V}_n^k - \mathbf{V}_n^{k-1}| < \varepsilon$$

- **Partitioned approach with explicit integration**:
  1. At t=0⁻, the system is in the steady state, initial values of **x**, **V** and **I** are known, and $\mathbf{f}(\mathbf{x}_0, \mathbf{V}_0)=0$.
  2. Following a disturbance (e.g. a fault), **x** cannot change instantly. Solve **AE** for **V** and **I**, the corresponding power flows and other non-state variables of interest at $t=0^+$. Then, compute $\mathbf{f}(\mathbf{x}, \mathbf{V})$, i.e. $d\mathbf{x}/dt$.
  3. Perform explicit integration by, e.g., the RK2 method to solve **DE** for each time step of Δt: (say step t_{n+1})
     i. Compute $\mathbf{k}_1=\mathbf{f}(\mathbf{x}_n, \mathbf{V}_n)\Delta t$ and $\mathbf{k}_2=\mathbf{f}(\mathbf{x}_n+\mathbf{k}_1, \mathbf{V}_n)\Delta t$ at t_n.
     ii. Compute $\mathbf{x}_{n+1}=\mathbf{x}_n+(\mathbf{k}_1+\mathbf{k}_2)/2$
  4. Using $\mathbf{x}_{n+1}$, solve **AE** by, e.g. N-R method, to give $\mathbf{V}_{n+1}$ and $\mathbf{I}_{n+1}$. Thus all values for t_{n+1} can be obtained. If the system has a switching operation, the network variables change instantly but not the state variables
- **Advantage**: Programming flexibility, simplicity, reliability and robustness. Since the solution of DE requires values only from the previous step, the DE associated with each device may be solved independently
- **Disadvantage**: Susceptibility to numerical instability. For a stiff system, a small time step is required throughout the solution period.

Partitioned scheme with current-injection models for all devices

$$\mathbf{I}(\mathbf{x}_n, \mathbf{V}_n) - \mathbf{Y}_\mathbf{N}\mathbf{V}_n = 0$$

$\mathbf{x}_n$ ↑ Alternating ↓ $\mathbf{V}_n$

$$\dot{\mathbf{x}}_n = \mathbf{f}(\mathbf{x}_n, \mathbf{V}_n)$$

Generator DEs:

$$\frac{d\delta}{dt} = \Delta\omega_r$$

$$\frac{2H}{\omega_0}\frac{d\Delta\omega_r}{dt} = P_m - P_e - K_D\frac{\Delta\omega_r}{\omega_0}$$

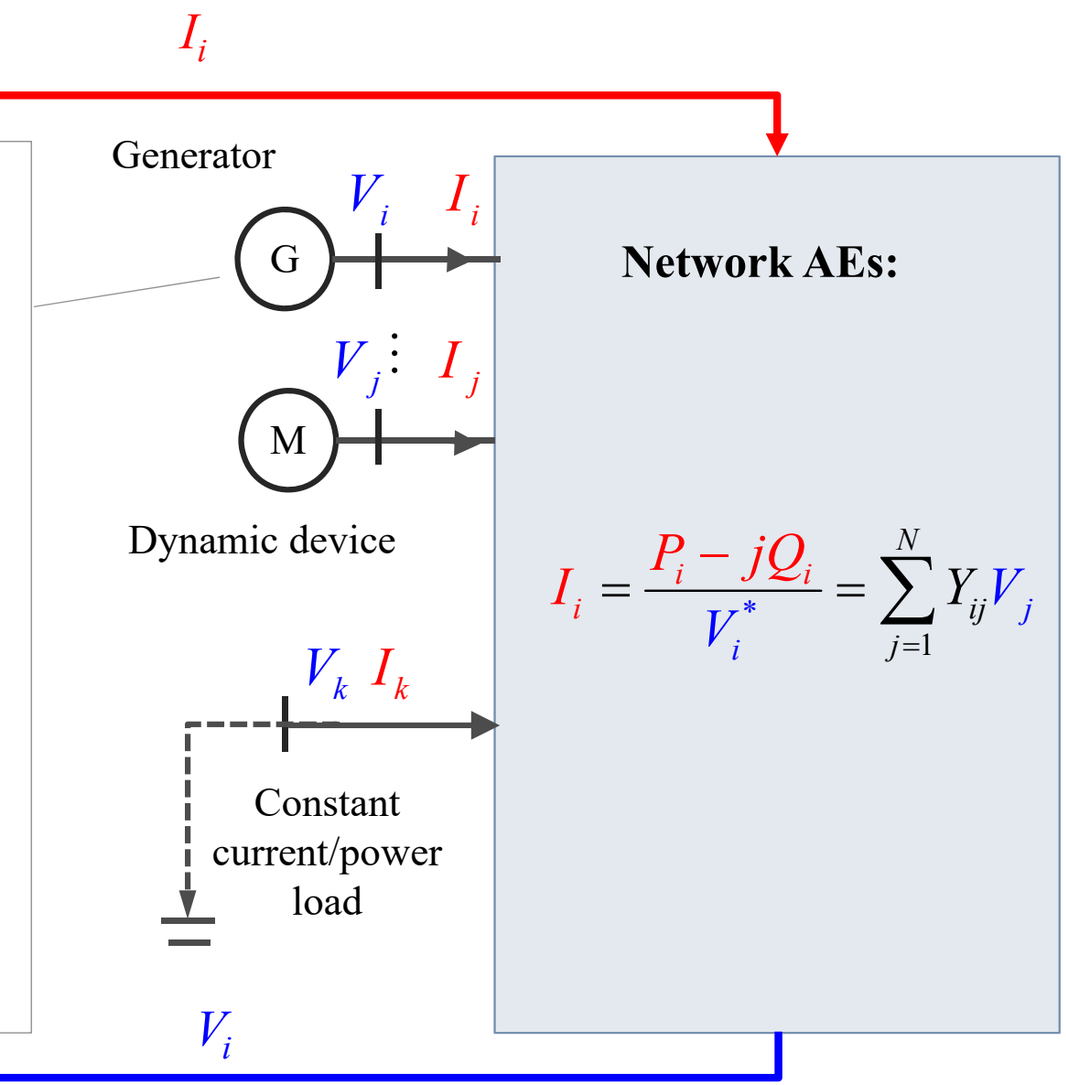
Generator AEs:

$$E_t = e^{-j(\delta-\pi/2)}V_i$$

$$E_t = E' - (R_a + jX'_d)I_t$$

$$P_e = \mathrm{Re}(E_t I_t^*)$$

$$I_i = e^{j(\delta-\pi/2)}I_t$$



Simultaneous Solution with Implicit Integration

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{V})$$

$$0 = \mathbf{I}(\mathbf{x}, \mathbf{V}) - \mathbf{Y}_N \mathbf{V}$$

Trapezoidal rule

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{\Delta t}{2}[\mathbf{f}(\mathbf{x}_{n+1}, \mathbf{V}_{n+1}) + \mathbf{f}(\mathbf{x}_n, \mathbf{V}_n)]$$

$$\mathbf{I}(\mathbf{x}_{n+1}, \mathbf{V}_{n+1}) = \mathbf{Y}_N \mathbf{V}_{n+1}$$

$$\mathbf{F}(\mathbf{x}_{n+1}, \mathbf{V}_{n+1}) = \mathbf{x}_{n+1} - \mathbf{x}_n - \frac{\Delta t}{2}[\mathbf{f}(\mathbf{x}_{n+1}, \mathbf{V}_{n+1}) + \mathbf{f}(\mathbf{x}_n, \mathbf{V}_n)] = 0$$

$$\mathbf{G}(\mathbf{x}_{n+1}, \mathbf{V}_{n+1}) = \mathbf{Y}_N \mathbf{V}_{n+1} - \mathbf{I}(\mathbf{x}_{n+1}, \mathbf{V}_{n+1}) = 0$$

Solved by N-R method:

$$\begin{bmatrix} -\mathbf{F}(\mathbf{x}^k_{n+1}, \mathbf{V}^k_{n+1}) \\ -\mathbf{G}(\mathbf{x}^k_{n+1}, \mathbf{V}^k_{n+1}) \end{bmatrix} = \mathbf{J} \begin{bmatrix} \Delta \mathbf{x}^k_{n+1} \\ \Delta \mathbf{V}^k_{n+1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}^{k+1}_{n+1} \\ \mathbf{V}^{k+1}_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^k_{n+1} \\ \mathbf{V}^k_{n+1} \end{bmatrix} + \begin{bmatrix} \Delta \mathbf{x}^k_{n+1} \\ \Delta \mathbf{V}^k_{n+1} \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F}}{\partial \mathbf{V}} \\ \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & \frac{\partial \mathbf{G}}{\partial \mathbf{V}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_D & \mathbf{B}_D \\ \mathbf{C}_D & (\mathbf{Y}_N + \mathbf{Y}_D) \end{bmatrix}$$