ECE 522 - Power Systems Analysis II
Spring 2021

Small-Signal Stability

Spring 2021

Instructor: Kai Sun

Content

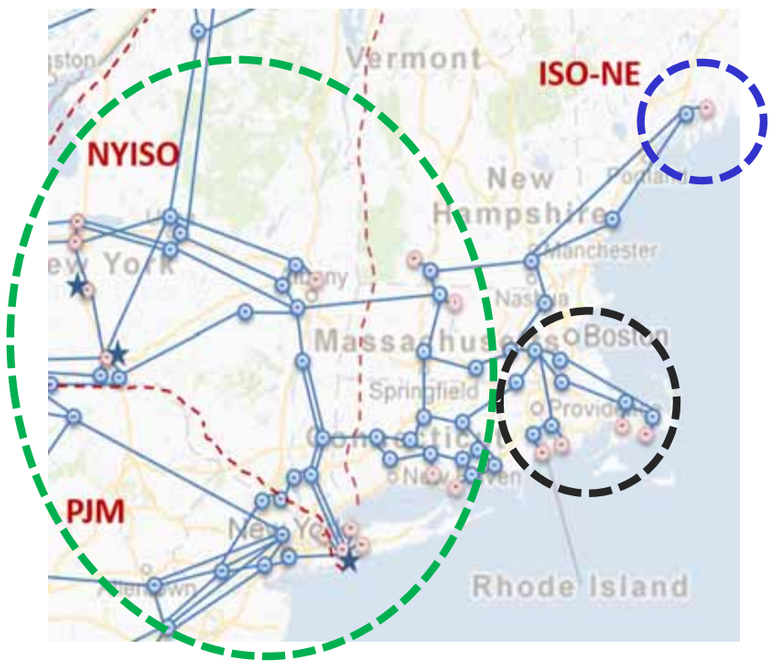
- Small-signal stability analysis methods
- Small-signal stability enhancement

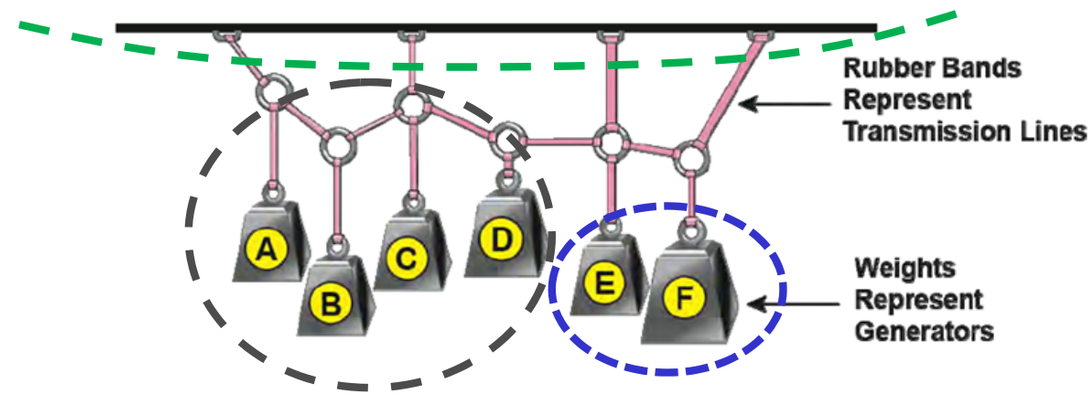
References:

- EPRI Dynamic Tutorial
- Chapters 12 and 17 of Kundur's "Power System Stability and Control"
- Chapter 3 of Anderson's "Power System Control and Stability"
- Joe H. Chow, "Power System Coherency and Model Reduction," Springer, 2013

Power Oscillations

- The power system naturally enters periods of oscillation as it continually adjusts to new operating conditions or experiences other disturbances.
- Typically, oscillations have a small amplitude and do not last long.
- When the oscillation amplitude becomes large or the oscillations are sustained, a response is required:
  - A system operator may have the opportunity to respond and eliminate harmful oscillations or,
  - less desirably, protective relays may activate to trip system elements.





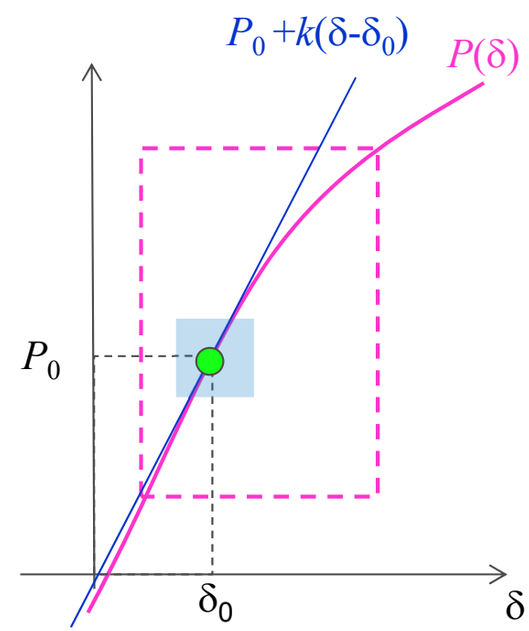
Small Signal Stability

Small signal stability (also referred to as small-disturbance stability) is the ability of a power system to maintain synchronism when subjected to small disturbances

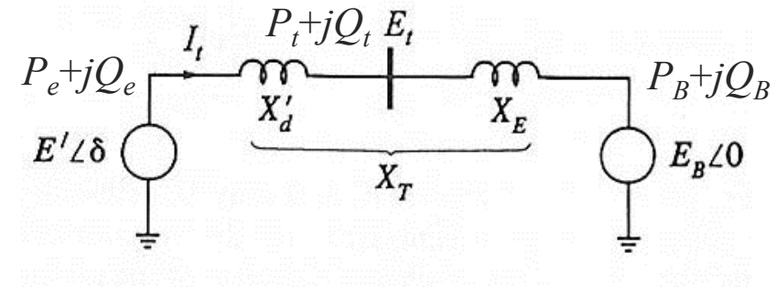
- In this context, a disturbance is considered to be small if the equations that describe the resulting response of the system may be linearized for the purpose of analysis.

- It is convenient to assume that the disturbances causing the changes already disappear and details on the disturbance are unimportant

- The system is stable only if it returns to its original state, i.e. a stable equilibrium point (SEP). Thus, only the behaviors in a small neighborhood of the SEP are concerned and can be analyzed using the linear control theory.



Classic-Model SMIB System



With all resistances neglected:

$$\frac{d\delta}{dt} = \omega_r - \omega_0$$

$$\frac{2H}{\omega_0}\frac{d\omega_r}{dt} = T_m - \frac{K_D}{\omega_0}(\omega_r - \omega_0) - T_e$$

$$T_e = P_e = P_{\max}\sin\delta = \frac{E'E_B}{X_T}\sin\delta$$

$\Delta\delta = \delta - \delta_0$

$\Delta\omega_r = \omega_r - \omega_0$

$$\frac{d\Delta\delta}{dt} = \omega_0\Delta\omega_r$$

$$2H\frac{d\Delta\omega_r}{dt} = T_m - K_D\Delta\omega_r - T_e$$

$$= \Delta T_m - K_D\Delta\omega_r - \Delta T_e$$

$$\approx \Delta T_m - K_D\Delta\omega_r - K_S\Delta\delta$$

$\omega_0 = 2\pi\times 60 \approx 377$ rad/s

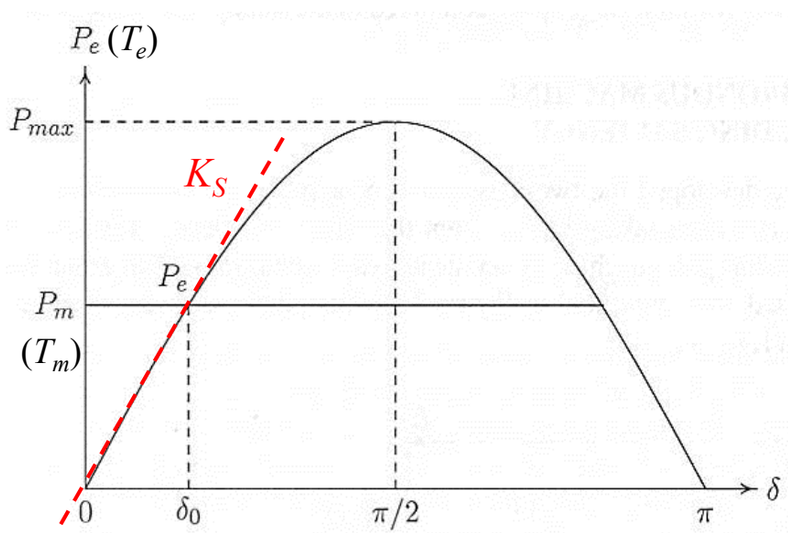
H in s

ω_r in rad/s

δ in rad

K_D in p.u

Linearize swing equations at $\delta=\delta_0$:



$$\Delta T_e \approx \frac{\partial T_e}{\partial\delta}\Delta\delta = K_S\Delta\delta$$

$$K_S = P_{\max}\cos\delta_0 = \frac{E'E_B}{X_T}\cos\delta_0$$

Synchronizing torque coefficient

$$\frac{d\Delta\delta}{dt} = \omega_0\Delta\omega_r$$

$$2H\frac{d\Delta\omega_r}{dt} = \Delta T_m - K_D\Delta\omega_r - K_S\Delta\delta$$

State-space representation

$$\frac{d}{dt}\begin{bmatrix}\Delta\delta \\ \Delta\omega_r\end{bmatrix} = \begin{bmatrix}0 & \omega_0 \\ -\frac{K_S}{2H} & -\frac{K_D}{2H}\end{bmatrix}\begin{bmatrix}\Delta\delta \\ \Delta\omega_r\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\frac{\Delta T_m}{2H}$$

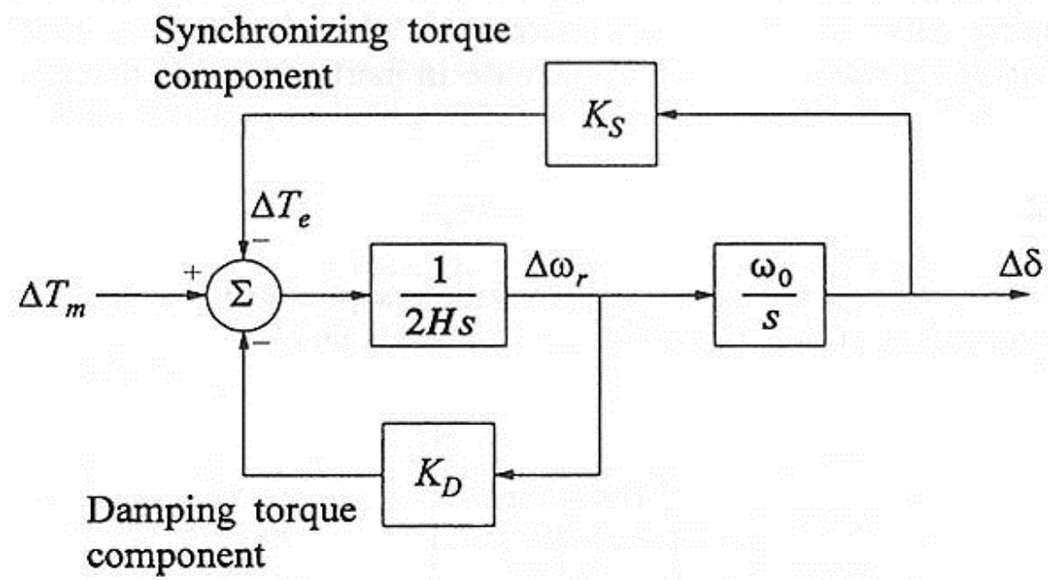
$$\Delta\ddot{\delta} + \frac{K_D}{2H}\Delta\dot{\delta} + \frac{K_s\omega_0}{2H}\Delta\delta = \frac{\omega_0\Delta T_m}{2H}$$

- Apply Laplace Transform:

$$\Delta\delta = \frac{1}{s^2 + \frac{K_D}{2H}s + \frac{K_s\omega_0}{2H}} \times \frac{\omega_0\Delta T_m}{2H}$$

- Characteristic equation:

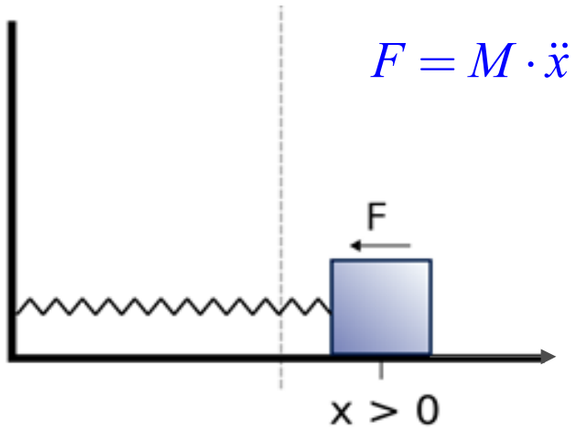
$$s^2 + \frac{K_D}{2H}s + \frac{K_s\omega_0}{2H} = 0$$



K_S = synchronizing torque coefficient in pu torque/rad
K_D = damping torque coefficient in pu torque/pu speed deviation
H = inertia constant in MW·s/MVA
$\Delta\omega_r$ = speed deviation in pu = $(\omega_r - \omega_0)/\omega_0$
$\Delta\delta$ = rotor angle deviation in elec. rad
s = Laplace operator
ω_0 = rated speed in elec. rad/s = $2\pi f_0$
= 377 for a 60 Hz system

Figure 12.5 Block diagram of a single-machine infinite bus system with classical generator model

Harmonic oscillator



$$F = M \cdot \ddot{x} = -K \cdot x - D \cdot \dot{x}$$

$$s^2 + \frac{D}{M} s + \frac{K}{M} = 0$$

$$s^2 + \frac{K_D}{2H} s + \frac{K_s \omega_0}{2H} = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s - s_1)(s - s_2) = 0$$

ζ – Damping ratio

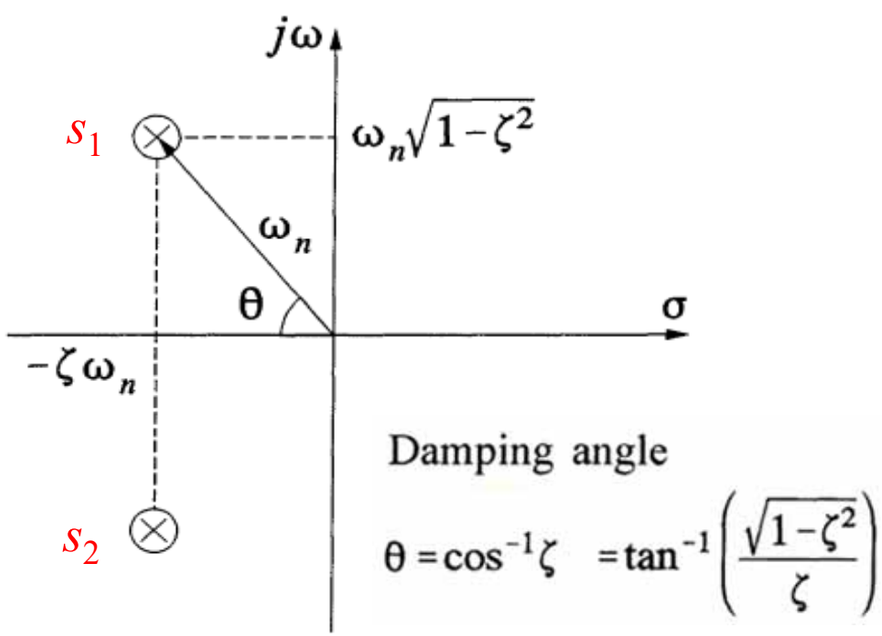
ω_n – Natural frequency

- It has two conjugate complex roots and its zero-input response is a damped sinusoidal oscillation:

$$s_1, s_2 = \sigma \pm j\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Observed oscillation frequency

$$x(t) = Ae^{\sigma t}\sin(\omega t + \varphi)$$
$$= Ae^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\, t + \varphi)$$



Time to decay to 1/*e*=36.8%: $\tau = -1/\sigma = \dfrac{1}{\zeta\omega_n}$

Oscillation Frequency and Damping Ratio of an SMIB System

$$s^2 + 2\zeta\omega_n s + \omega_n{}^2 = 0 \qquad s_1, s_2 = \sigma \pm j\omega = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$s^2 + \frac{K_D}{2H}s + \frac{K_s\omega_0}{2H} = 0 \qquad \left(K_S = \frac{E'E_B}{X_T}\cos\delta_0 = P_{\max}\cos\delta_0\right)$$

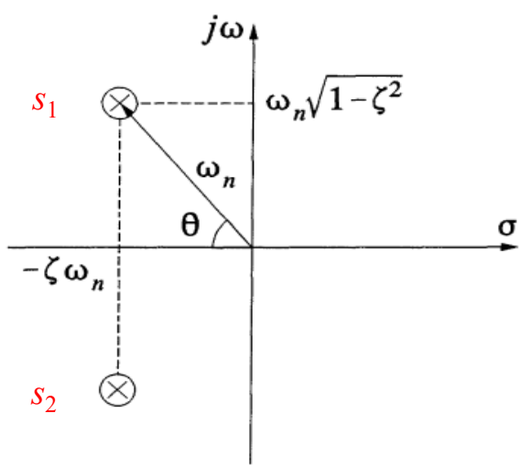
Note the units:

$\Delta\omega_r$ is in p.u.

$\Delta\delta$ is in rad.

K_D is in p.u

K_S is in p.u/rad



Observed oscillation frequency

$$\omega = \omega_n\sqrt{1-\zeta^2} = \sqrt{K_S\frac{\omega_0}{2H} - \frac{K_D^2}{16H^2}}$$

"Damping" in general

$$-\sigma = \zeta\omega_n = \frac{K_D}{4H}$$

Natural frequency

$$\omega_n = \sqrt{\sigma^2 + \omega^2} = \sqrt{K_S\frac{\omega_0}{2H}} = \sqrt{\frac{\omega_0 E'E_B\cos\delta_0}{2HX_T}}$$

Damping ratio frequency

$$\zeta = \frac{-\sigma}{\omega_n} = \frac{1}{2}\frac{K_D}{\sqrt{K_S 2H\omega_0}} = K_D\sqrt{\frac{X_T}{8\omega_0 HE'E_B\cos\delta_0}}$$

- How do ω_n and ζ change with the following?
  - if $H\downarrow$ (lower inertia)
  - if $X_T\downarrow$ (stronger transmission)
  - if $\delta_0\downarrow$ (lower loading)

System Response after a Small Disturbance

$$\frac{d}{dt}\begin{bmatrix}\Delta\delta \\ \Delta\omega_r\end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\frac{K_S}{2H} & -\frac{K_D}{2H}\end{bmatrix}\begin{bmatrix}\Delta\delta \\ \Delta\omega_r\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\frac{\Delta T_m}{2H}$$

$$x_1 = \Delta\delta \qquad x_2 = \Delta\omega_r = \Delta\dot{\delta}/\omega_0$$

$$\Delta u = \frac{\Delta T_m}{2H}$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\ -\omega_n^{\ 2}/\omega_0 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\Delta u$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\Delta u(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

(Assuming the angle and speed to be directly measured)

Apply Laplace transform:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\Delta U(s)$$

$$\Delta U(s) = \frac{\Delta u}{s}$$

$$\mathbf{Y}(s) = \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\left[\underbrace{\mathbf{x}(0)}_{\text{Zero-input}} + \underbrace{\mathbf{B}\Delta U(s)}_{\text{Zero-state}}\right]$$

$$\mathbf{X}(s) = \frac{\begin{bmatrix} s+2\zeta\omega_n & \omega_0 \\ -\omega_n^{\ 2}/\omega_0 & s\end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}\left[\mathbf{x}(0) + \mathbf{B}\Delta U(s)\right]$$

$$\begin{bmatrix}\Delta\delta(s) \\ \Delta\omega_r(s)\end{bmatrix} = \frac{\begin{bmatrix} s+2\zeta\omega_n & \omega_0 \\ -\omega_n^{\ 2}/\omega_0 & s\end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}\left(\underbrace{\begin{bmatrix}\Delta\delta(0) \\ \Delta\omega_r(0)\end{bmatrix}}_{\text{Zero-input}} + \underbrace{\begin{bmatrix}0 \\ \frac{\Delta u}{s}\end{bmatrix}}_{\text{Zero-state}}\right)$$

$$\begin{bmatrix}\Delta\delta(s)\\ \Delta\omega_r(s)\end{bmatrix}=\frac{\begin{bmatrix}s+2\zeta\omega_n & \omega_0\\ -\omega_n^{\,2}/\omega_0 & s\end{bmatrix}}{s^2+2\zeta\omega_n s+\omega_n^{\,2}}\left(\begin{bmatrix}\Delta\delta(0)\\ \Delta\omega_r(0)\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{\Delta u}{s}\end{bmatrix}\right)$$

$$\Delta\omega_r=\Delta\dot{\delta}/\omega_0=(\omega_r-\omega_0)/\omega_0 \quad \text{in pu}$$

$$\Delta u=\frac{\Delta T_m}{2H}\ \text{pu}$$

Zero-input response

- E.g. when the rotor is suddenly perturbed by a small angle $\Delta\delta(0)\neq 0$ and assume $\Delta\omega_r(0)=0$

$$\Delta\delta(s)=\frac{(s+2\zeta\omega_n)\Delta\delta(0)}{s^2+2\zeta\omega_n s+\omega_n^{\,2}}$$

$$\Delta\omega_r(s)=-\frac{\omega_n^2\Delta\delta(0)/\omega_0}{s^2+2\zeta\omega_n s+\omega_n^{\,2}}$$

Zero-state response

- E.g. when there is a small increase in mechanical torque ΔT_m (= ΔP_m in pu)

$$\Delta\delta(s)=\frac{\omega_0\Delta u}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$\Delta\omega_r(s)=\frac{\Delta u}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Inverse Laplace transform

$$\Delta\delta \text{ in rad}=\frac{\Delta\delta(0)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega t+\theta)$$

$$\Delta\omega_r \text{ in rad/s}=-\frac{\omega_n\Delta\delta(0)}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega t$$

$$\Delta\delta \text{ in rad}=\frac{\omega_0\Delta T_m}{2H\omega_n^2}\left[1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega t+\theta)\right]$$

$$\Delta\omega_r \text{ in rad/s}=\frac{\omega_0\Delta T_m}{2H\omega_n\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega t$$

Damping angle: $\theta=\cos^{-1}\zeta$

Saadat's Examples 11.2 & 11.3

- H=9.94s, K_D=0.138pu, T_m=0.6 pu with PF=0.8. Find the responses of the rotor angle and frequency under these disturbances
  - (1) $\Delta\delta(0)=10^o=0.1745$ rad
  - (2) $\Delta P_m=0.2$pu

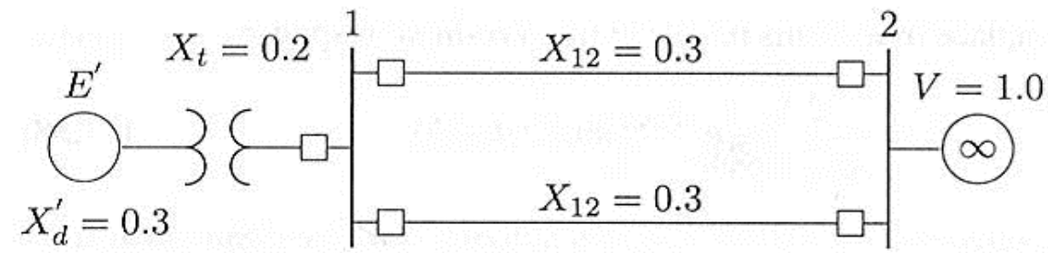
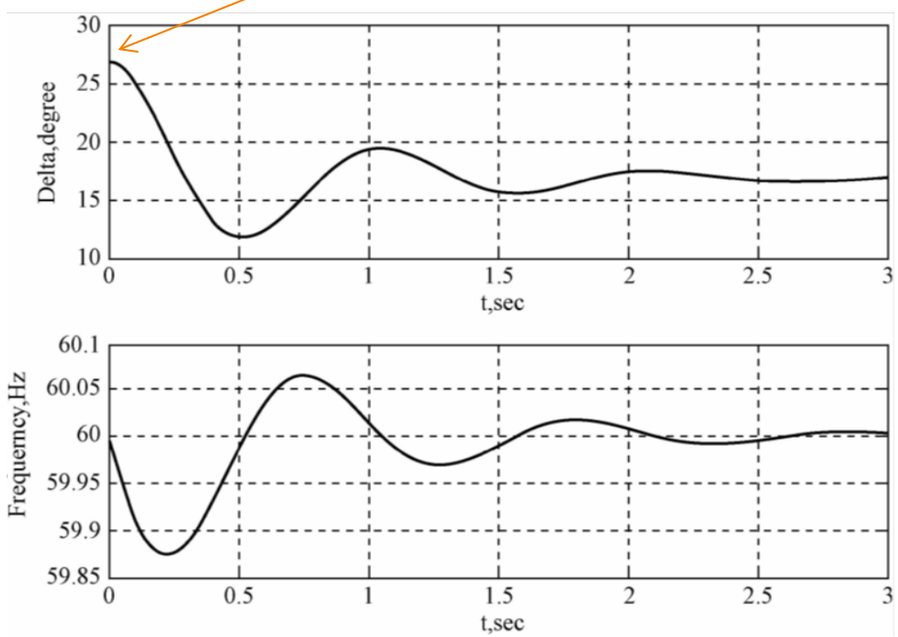


FIGURE 11.7
One-line diagram for Example 11.2.

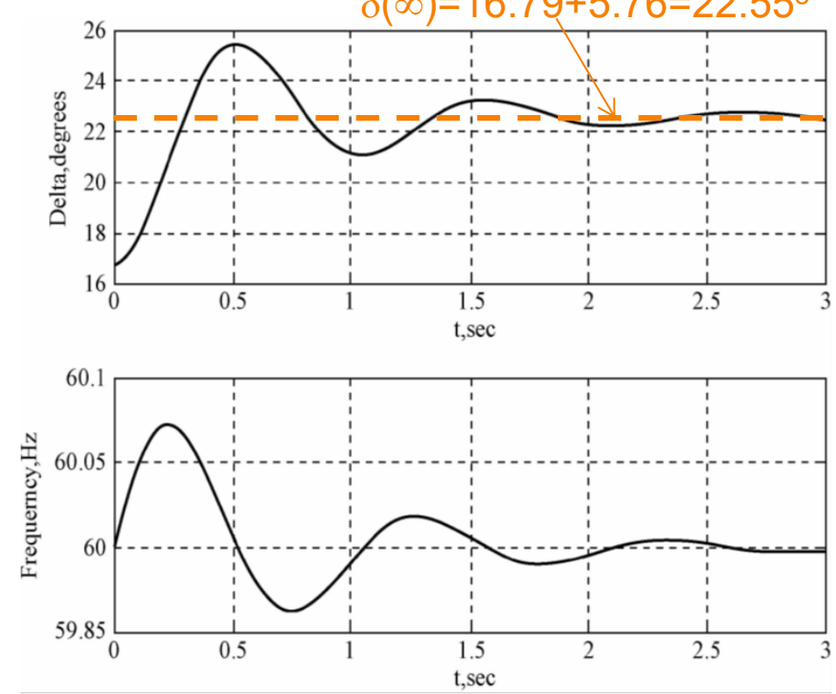
Zero-input response: $\Delta\delta(0)=10^o$

$\delta(0)=16.79+10=26.79^o$



Zero-state response: $\Delta P_m=0.2$pu

$\delta(\infty)=16.79+5.76=22.55^o$



Small-Signal Stability of a Multi-machine System

Inter-area or intra-area modes (0.1-0.7Hz): **machines in one part of the system swing against machines in other parts**	**Inter-area model** (0.1-0.3Hz): involving all the generators in the system; the system is essentially split into two parts, with generators in one part swinging against machines in the other parts.
	Intra-area mode (0.4-0.7Hz): involving subgroups of generators swinging against each other.
Local modes (0.7-2Hz): **oscillations involve a small part of the system**	**Local plant modes**: associated with rotor angle oscillations of a single generator or a single plant against the rest of the system; similar to the single-machine-infinite bus system
	Inter-machine or interplant modes: associated with oscillations between the rotors of a few generators close to each other
Control or torsional modes (2Hz –)	Due to inadequate tuning of the control systems, e.g. generator excitation systems, HVDC converters and SVCs, or torsional interaction (sub-synchronous resonance) with power system control

High & Low Frequency Oscillations

- Whenever power flows, I^2R losses occur. These energy losses help to reduce the amplitude of the oscillation.

- High frequency (>1.0 HZ) oscillations are damped more rapidly than low frequency (<1.0 HZ) oscillations. The higher the frequency of the oscillation, the faster it is damped.

- Power system operators do not want any oscillations. However, when oscillations occur, it is better to have high frequency oscillations than low frequency oscillations.

- The power system can naturally dampen high frequency oscillations. Low frequency oscillations are more damaging to the power system, which may exist for a long time, become sustained (undamped) oscillations, and even trigger protective relays to trip elements

Oscillation Modes of a Multi-machine System in the Classic Model

$$\frac{2H_i}{\omega_0}\frac{d^2\delta_i}{dt^2} = P_{mi} - P_{ei} \qquad i = 1, 2, \cdots, n \qquad \text{(Ignoring damping)}$$

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} P_{ij} = E_i'^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i' E_j' \,|\, Y_{ij}^{red} \,|\cos\left(\theta_{ij} - \delta_{ij}\right) = E_i'^2 G_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i' E_j' \left(B_{ij}\sin\delta_{ij} + G_{ij}\cos\delta_{ij}\right)$$

$$\delta_{ij} = \delta_i - \delta_j, \quad Y_{ij}^{red} = |\, Y_{ij}^{red} \,| \angle\, \theta_{ij} = G_{ij} + jB_{ij}$$

Linearization at δ_{ij0}:

$$\delta_{ij} = \delta_{ij0} + \delta_{ij\Delta}$$

$$\sin\delta_{ij} \approx \sin\delta_{ij0} + \delta_{ij\Delta}\cos\delta_{ij0}$$

$$\cos\delta_{ij} \approx \cos\delta_{ij0} - \delta_{ij\Delta}\sin\delta_{ij0}$$

$$\frac{2H_i}{\omega_0}\frac{d^2\delta_{i\Delta}}{dt^2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} K_{sij}\delta_{ij\Delta} = 0 \quad i = 1, 2, \cdots, n \quad \Leftrightarrow$$

$$\frac{2H_i}{\omega_0}\frac{d^2\delta_{i\Delta}}{dt^2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} E_i' E_j' \left(B_{ij}\cos\delta_{ij0} - G_{ij}\sin\delta_{ij0}\right)\delta_{ij\Delta} = 0$$

Synchronizing power coefficient

$$K_{sij} = \left.\frac{\partial P_{ij}}{\partial \delta_{ij}}\right|_{\delta_{ij0}} = E_i' E_j' \left(B_{ij}\cos\delta_{ij0} - G_{ij}\sin\delta_{ij0}\right)$$

compared to $K_S = \dfrac{E'E_B}{X_T}\cos\delta_0$

$$\frac{2H_i}{\omega_0}\frac{d^2\delta_{i\Delta}}{dt^2}+\sum_{\substack{j=1\\ j\neq i}}^{n}K_{sij}\delta_{ij\Delta}=0 \quad i=1,2,\cdots,n$$

Note: There are only (n−1) independent equations because $\sum\delta_{ij}$=0, so we need to formulate the (n−1) independent relative rotor angle equations with one reference machine, e.g., the n-th machine.

$$\frac{d^2\delta_{i\Delta}}{dt^2}-\frac{d^2\delta_{n\Delta}}{dt^2}+\frac{\omega_0}{2H_i}\sum_{\substack{j=1\\ j\neq i}}^{n}K_{sij}\delta_{ij\Delta}-\frac{\omega_0}{2H_n}\sum_{j=1}^{n-1}K_{snj}\delta_{nj\Delta}=0, \quad i=1,\cdots,n-1$$

Consider each $\delta_{in\Delta}=\delta_{i\Delta}-\delta_{n\Delta}$

$$\frac{d^2\delta_{in\Delta}}{dt^2}+\left(\frac{\omega_0}{2H_i}\sum_{\substack{j=1\\ j\neq i}}^{n}K_{sij}+\frac{\omega_0}{2H_n}K_{sni}\right)\delta_{in\Delta}+\sum_{\substack{j=1\\ j\neq i}}^{n-1}\left(\frac{\omega_0}{2H_n}K_{snj}-\frac{\omega_0}{2H_i}K_{sij}\right)\delta_{jn\Delta}=0, \quad i=1,\cdots,n-1$$

$$\frac{d^2\delta_{in\Delta}}{dt^2}+\sum_{j=1}^{n-1}\alpha_{ij}\delta_{jn\Delta}=0 \qquad i=1,2,\cdots,n-1$$

$$\alpha_{ii}=\frac{\omega_0}{2H_i}\sum_{\substack{j=1\\ j\neq i}}^{n}K_{sij}+\frac{\omega_0}{2H_n}K_{sni} \qquad \alpha_{ij}=\frac{\omega_0}{2H_n}K_{snj}-\frac{\omega_0}{2H_i}K_{sij}$$

State-space representation

Let $x_1, x_2, \cdots, x_{n-1} = \delta_{1n\Delta}, \delta_{2n\Delta}, \cdots, \delta_{(n-1)n\Delta}$ and $x_n, x_{n+1}, \cdots, x_{2n-2} = \dot{\delta}_{1n\Delta}, \dot{\delta}_{2n\Delta}, \cdots, \dot{\delta}_{(n-1)n\Delta}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \\ \dot{x}_{n+1} \\ \vdots \\ \dot{x}_{2n-2} \end{bmatrix} = \begin{bmatrix} & & & & 1 & 0 & \cdots & 0 \\ & & & & 0 & 1 & \cdots & 0 \\ & & \mathbf{0} & & \cdots & \cdots & \cdots & \cdots \\ & & & & 0 & 0 & \cdots & 1 \\ -\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1(n-1)} & & & & \\ -\alpha_{21} & -\alpha_{22} & \cdots & -\alpha_{2(n-1)} & & & \mathbf{0} & \\ \cdots & \cdots & \cdots & \cdots & & & & \\ -\alpha_{(n-1)1} & -\alpha_{(n-1)2} & \cdots & -\alpha_{(n-1)(n-1)} & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \\ x_{2n-2} \end{bmatrix} \qquad \begin{bmatrix} \dot{\mathbf{X}}_1 \\ \dot{\mathbf{X}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$$

- Its characteristic equation $|\lambda^2\mathbf{I}\text{-}\mathbf{A}|=0$ has $2(n\text{-}1)$ imaginary roots, which occur in $(n\text{-}1)$ complex conjugate pairs
- **An *n*-machine system has (*n*-1) modes**

Read Anderson's Examples 3.2 and 3.3 on linearization and eigen-analysis of the IEEE 9-bus system.

Formulation of General Multi-machine State Equations

- The linearized model of each dynamic device:

$$\dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\Delta\mathbf{v}$$
$$\Delta\mathbf{i}_i = \mathbf{C}_i\mathbf{x}_i - \mathbf{Y}_i\Delta\mathbf{v}$$

$\mathbf{x}_i$ – Perturbed values of state variables

$\mathbf{i}_i$ – Current injection into network from device i

$\Delta\mathbf{v}$ – Vector of network bus voltages

$\mathbf{B}_i$ and $\mathbf{Y}_i$ have non-zero element corresponding only to the terminal voltage of the device and any remote bus voltages used to control the device

$\Delta\mathbf{i}_i$ and $\Delta\mathbf{v}$ both have real and imaginary components

- Such state equations for all the dynamic devices in the system may be combined into the form:

$$\dot{\mathbf{x}} = \mathbf{A}_D\mathbf{x} + \mathbf{B}_D\Delta\mathbf{v}$$
$$\Delta\mathbf{i} = \mathbf{C}_D\mathbf{x} - \mathbf{Y}_D\Delta\mathbf{v}$$

$\mathbf{x}$ is the vector of state variables of the complete system

$\mathbf{A}_D$ and $\mathbf{C}_D$ are block diagonal matrices composed of $\mathbf{A}_i$ and $\mathbf{C}_i$ associated with the individual devices

- Node equation of the transmission network:

$$\Delta\mathbf{i} = \mathbf{Y}_N\Delta\mathbf{v}$$

- The overall system state equation:

$$\dot{\mathbf{x}} = \mathbf{A}_D\mathbf{x} + \mathbf{B}_D(\mathbf{Y}_N + \mathbf{Y}_D)^{-1}\mathbf{C}_D\mathbf{x} = \mathbf{A}\mathbf{x}$$
$$\mathbf{A} = \mathbf{A}_D + \mathbf{B}_D(\mathbf{Y}_N + \mathbf{Y}_D)^{-1}\mathbf{C}_D$$

- Read Kundur's sec. 12.7 for other related information, e.g. load model linearization and selection of a reference rotor angle

Modal analysis (eigen-analysis) on an n-dimensional nonlinear system

A power system can be described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u}) \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_r) \qquad i = 1, 2, \ldots, n$$

Equilibrium $\mathbf{x}_0$ (with $\mathbf{u}_0$):

$$\dot{\mathbf{x}}_0 = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{0}$$

$$\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x}$$

$$\mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u}$$

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = \mathbf{f}\big((\mathbf{x}_0 + \Delta\mathbf{x}), (\mathbf{u}_0 + \Delta\mathbf{u})\big)$$

Linearization at the equilibrium $\mathbf{x}_0$: consider a perturbation at $\mathbf{x}_0$ and $\mathbf{u}_0$

$$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u}$$

A is the Jacobin matrix of **f**

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_r} \end{bmatrix}$$

$$\Leftrightarrow$$

$$s\Delta\mathbf{X}(s) - \Delta\mathbf{X}(0) = \mathbf{A}\Delta\mathbf{X}(s) + \mathbf{B}\Delta\mathbf{U}(s)$$

$$\Delta\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\left[\underbrace{\Delta\mathbf{X}(0)}_{Zero-Input} + \underbrace{\mathbf{B}\Delta\mathbf{U}(s)}_{Zero-State}\right]$$

$$= \frac{\mathrm{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}[\Delta\mathbf{X}(0) + \mathbf{B}\Delta\mathbf{U}(s)]$$

Characteristic equation of A

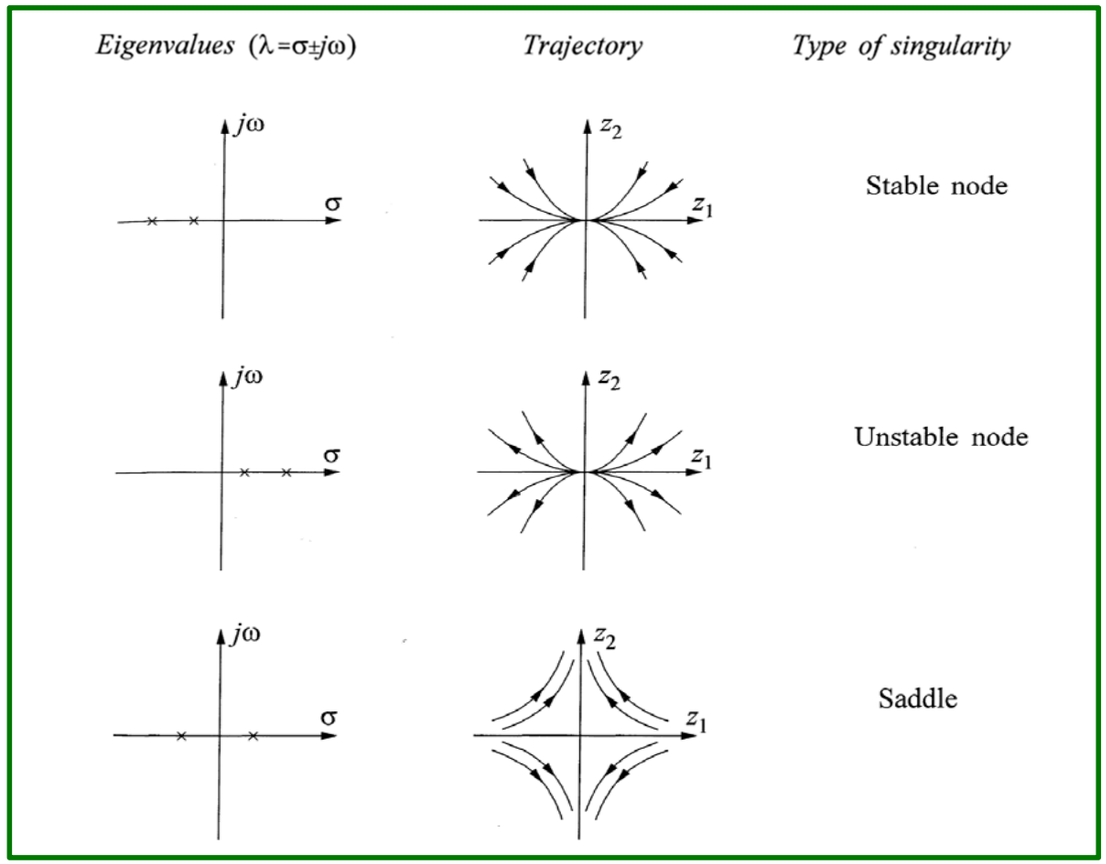
$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

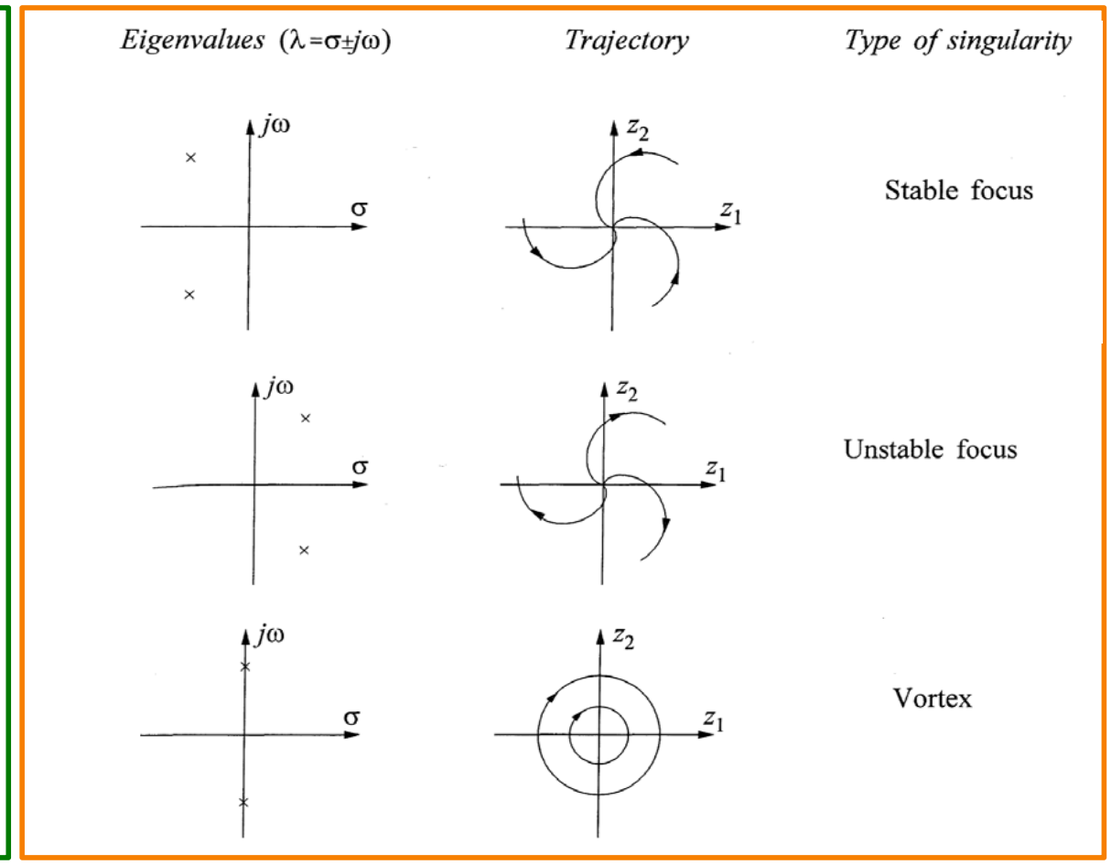
Poles of $\Delta\mathbf{X}(s)$

$\rightarrow$ Eigenvalues of **A**, i.e. $\lambda = \lambda_1 \cdots \lambda_n$

Eigenvalues: $\lambda = \sigma \pm j\omega$

- Each real eigenvalue (ω=0) corresponds to **one non-oscillatory mode**:
  a decaying mode has σ<0; a mode with σ>0 has aperiodic instability.
- Each conjugate pair of complex eigenvalues ($\omega \neq 0$) corresponds to **one oscillatory mode**:
  - Frequency of oscillation in Hz: $f = \omega/2\pi$
  - Damping ratio (rate of decay) of the oscillation amplitude $\xi = \dfrac{-\sigma}{\sqrt{\sigma^2 + \omega^2}}$





Eigen-vectors

- For any λ_i, a column vector $\boldsymbol{\phi}_i$ satisfying $\mathbf{A}\boldsymbol{\phi}_i = \lambda_i \boldsymbol{\phi}_i$ is called a right eigenvector of $\mathbf{A}$ associated with λ_i.
- Similarly, a row vector $\boldsymbol{\psi}_i$ satisfying $\boldsymbol{\psi}_i \mathbf{A} = \lambda_i \boldsymbol{\psi}_i$ is called a left eigenvector of $\mathbf{A}$ associated with λ_i.
- If $\mathbf{A}$ has distinct eigenvalues (true for a system with no resonance), it has n right eigenvectors and n left eigenvectors:

Modal matrix $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_n]$

$\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda} \quad \boldsymbol{\Lambda} = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$\boldsymbol{\Phi}^{-1}\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Lambda}$ i.e. $\mathbf{A}$ is diagonalized by $\boldsymbol{\Phi}$

Define $\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \vdots \\ \boldsymbol{\psi}_n \end{bmatrix}$. There is $\boldsymbol{\psi}\mathbf{A} = \boldsymbol{\Lambda}\boldsymbol{\psi}$

or $\boldsymbol{\psi}\mathbf{A}\boldsymbol{\psi}^{-1} = \boldsymbol{\Lambda}$ i.e. $\mathbf{A}$ is diagonalized by $\boldsymbol{\psi}^{-1}$.

- Obviously, row vectors of $\boldsymbol{\Phi}^{-1}$ are left eigenvectors of $\mathbf{A}$, or in other words, $\boldsymbol{\Psi} = \mathbf{C}\,\boldsymbol{\Phi}^{-1}$ where $\mathbf{C}$ is a diagonal matrix or simply equal to $\mathbf{I}$ if normalized.
- The left and right eigenvectors corresponding to different eigenvalues are orthogonal:

$$\boldsymbol{\Psi}\boldsymbol{\Phi} = \mathbf{I} \quad \Leftrightarrow \quad \boldsymbol{\psi}_j \boldsymbol{\phi}_i = 0 \text{ if } i \neq j, \quad \text{or} \quad \boldsymbol{\psi}_i \boldsymbol{\phi}_i = 1$$

Free (zero-input) response and stability

$\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x}$ (Linearized power system without external forcing)

- Consider a new state vector **z** (defining a mode) that eliminates cross-coupling between state variables:

$$\dot{z}_i = \lambda_i z_i \rightarrow z_i(t) = z_i(0)e^{\lambda_i t}$$

$$\dot{\mathbf{z}} = \mathbf{\Lambda z} = \mathbf{\Phi}^{-1}\mathbf{A\Phi z}$$

$$\Leftrightarrow \mathbf{\Phi}\dot{\mathbf{z}} = \mathbf{A\Phi z} \qquad \Rightarrow \Delta\mathbf{x}(t) = \mathbf{\Phi z}(t) = \begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_n\end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t)\end{bmatrix} = \sum_{i=1}^{n}\phi_i z_i(0)e^{\lambda_i t}$$

$$\mathbf{z}(t) = \mathbf{\Phi}^{-1}\Delta\mathbf{x}(t) = \mathbf{\Psi}\Delta\mathbf{x}(t)$$

$$z_i(0) = \psi_i \Delta\mathbf{x}(0)$$

This is the magnitude of the excitation on the ith mode

$$\Delta\mathbf{x}(t) = \sum_{i=1}^{n}\phi_i\psi_i\Delta\mathbf{x}(0)e^{\lambda_i t} = \sum_{i=1}^{n}\phi_i z_i(0)e^{\lambda_i t}$$

$$\Delta x_k(t) = \sum_{i=1}^{n}\phi_{ki}z_i(0)e^{\lambda_i t} = \phi_{k1}z_1(0)e^{\lambda_1 t} + \ldots + \phi_{kn}z_n(0)e^{\lambda_n t}$$

Free response is a linear combination of n modes.

Mode Shape and Mode Composition

- Variables $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ are deviations of original state variables x_k away from the SEP.
- Variables $z_1, z_2, \ldots, z_n$ are transformed state variables each associated with only one mode, or on other words, are representations of the modes.

$$\Delta\mathbf{x}(t) = \mathbf{\Phi}\mathbf{z}(t) = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_n \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \mathbf{z}(t) = \mathbf{\Psi}\Delta\mathbf{x}(t) = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \vdots \\ \boldsymbol{\psi}_n \end{bmatrix} \Delta\mathbf{x}(t)$$

$$\Delta x_k(t) = \sum_{i=1}^{n} \phi_{ki} z_i(t)$$

$$z_i(t) = \sum_{k=1}^{n} \psi_{ik} \Delta x_k(t)$$

$$\begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \vdots \\ \Delta x_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{1n} & \cdots & \phi_{nn} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix}$$

$$\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \\ \vdots \\ \Delta x_n(t) \end{bmatrix}$$

A right eigenvector $\boldsymbol{\phi}_i$ gives the **mode shape** of the ith mode, i.e. relative activities of original state variables when the ith mode is excited: ϕ_{ki}, the kth element of $\boldsymbol{\phi}_i$, measures the activity of x_k in the ith mode.

A left eigenvector $\boldsymbol{\psi}_i$ gives the **mode composition** of the ith mode, i.e. contributions of original state variables to the mode: ψ_{ik}, the kth element of $\boldsymbol{\psi}_i$, weights the contribution of x_k's activity to the ith mode

Participation factor

- **Mode shape** ϕ_{ki} measures the activity of x_k in the ith mode
- **Mode composition** ψ_{ik} weights the contribution of the activity of x_k to the ith mode
- **Participation factor** $p_{ki} = \psi_{ik}\phi_{ki}$ measures the bi-directional participation of the kth state variable x_k in the ith mode.
- p_{ki} is dimensionless and hence invariant under changes of scale on the variables

$$\mathbf{\Psi\Phi} = \mathbf{I} \rightarrow \left(\mathbf{\Psi\Phi}\right)_{ii} = \sum_{k=1}^{n} \psi_{ik}\phi_{ki} \triangleq \sum_{k=1}^{n} p_{ki} = 1 \qquad \bar{p}_{ki} = \frac{p_{ki}}{max\left\{|\,p_{1i}\,|,\cdots,|\,p_{ni}\,|\right\}},\ 0 \le \left|\bar{p}_{ki}\right| \le 1$$

Normalized participation factor based on the largest one

- Question: considering $\mathbf{\Psi}=\mathbf{\Phi}^{-1}$, can we obtain mode compositions and participation factors from mode shapes in practice?

Learn Kundur's Example 12.2 on an SMIB system

Example: modal analysis of a 3-machine system

$$\frac{2H_i}{\omega_0}\ddot{\delta}_i = P_{mi} - E_i'^2 G_{ii} - D_i\frac{\dot{\delta}_i}{\omega_0} - \sum_{\substack{j=1\\ j\neq i}}^{n} E_i'E_j'\left(B_{ij}\sin\delta_{ij} + G_{ij}\cos\delta_{ij}\right)$$

$$\delta_{i\Delta} = \delta_i - \delta_{i0}, \delta_{ij\Delta} = \delta_{i\Delta} - \delta_{j\Delta}$$

Linearization at δ_{i0}:

$$\begin{cases} \dot{\delta}_{i\Delta} = \omega_{ri\Delta} \\ \dfrac{2H_i}{\omega_0}\dot{\omega}_{ri\Delta} = -\dfrac{D_i}{\omega_0}\omega_{ri\Delta} - \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{n} K_{sij}(\delta_{i\Delta} - \delta_{j\Delta}) = 0 \quad i = 1,2,\cdots,n \end{cases}$$

$$\frac{2H_1}{\omega_0}\ddot{\delta}_{1\Delta} + \frac{D_1}{\omega_0}\dot{\delta}_{1\Delta} + (K_{s12}+K_{s13})\delta_{1\Delta} - K_{s12}\delta_{2\Delta} - K_{s13}\delta_{3\Delta} = 0$$

$$\frac{2H_2}{\omega_0}\ddot{\delta}_{2\Delta} + \frac{D_2}{\omega_0}\dot{\delta}_{2\Delta} - K_{s21}\delta_{1\Delta} + (K_{s21}+K_{s23})\delta_{2\Delta} - K_{s23}\delta_{3\Delta} = 0$$

$$\frac{2H_3}{\omega_0}\ddot{\delta}_{3\Delta} + \frac{D_3}{\omega_0}\dot{\delta}_{3\Delta} - K_{s31}\delta_{1\Delta} - K_{s32}\delta_{2\Delta} + (K_{s31}+K_{s32})\delta_{3\Delta} = 0$$

$$K_{sij} = \left.\frac{\partial P_{ij}}{\partial \delta_{ij}}\right|_{\delta_{ij0}} = E_i'E_j'\left(B_{ij}\cos\delta_{ij0} - G_{ij}\sin\delta_{ij0}\right)$$

$$\begin{cases} \dot{\delta}_{i\Delta} = \omega_{ri\Delta} \\ \dot{\omega}_{ri\Delta} = -\beta_i\omega_{ri\Delta} - \displaystyle\sum_{j=1}^{3}\alpha_{ij}\delta_{j\Delta} \qquad i = 1,2,3 \end{cases}$$

$$\alpha_{ii} = \frac{\omega_0}{2H_i}\sum_{\substack{j=1\\ j\neq i}}^{3} K_{sij} \qquad \alpha_{ij} = -\frac{\omega_0}{2H_i}K_{sij} \qquad \beta_i = \frac{D_i}{2H_i}$$

$$\frac{d}{dt}\begin{bmatrix} \delta_{1\Delta} \\ \delta_{2\Delta} \\ \delta_{3\Delta} \\ \omega_{1\Delta} \\ \omega_{2\Delta} \\ \omega_{3\Delta} \end{bmatrix} = \begin{bmatrix} & & & 1 & 0 & 0 \\ & \mathbf{0} & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ -\alpha_{11} & -\alpha_{12} & -\alpha_{13} & -\beta_1 & 0 & 0 \\ -\alpha_{21} & -\alpha_{22} & -\alpha_{23} & 0 & -\beta_2 & 0 \\ -\alpha_{31} & -\alpha_{32} & -\alpha_{33} & 0 & 0 & -\beta_3 \end{bmatrix}\begin{bmatrix} \delta_{1\Delta} \\ \delta_{2\Delta} \\ \delta_{3\Delta} \\ \omega_{1\Delta} \\ \omega_{2\Delta} \\ \omega_{3\Delta} \end{bmatrix}$$

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$$

Modal Analysis on the IEEE 9-Bus System

- Linearized classic generator model: 6 state variables:

$$\left(\Delta\omega_{r1}, \Delta\delta_1, \Delta\omega_{r2}, \Delta\delta_2, \Delta\omega_{r3}, \Delta\delta_3\right)$$

- Two loading conditions:
  - Light load (LL)
  - Heavy load (HL)

- Two fault scenarios:
  - N-0
  - N-1 (line 5-7 is tripped)

- Cases summary

Case 1	LL N-0
Case 2	LL N-1
Case 3	HL N-0
Case 4	HL N-1

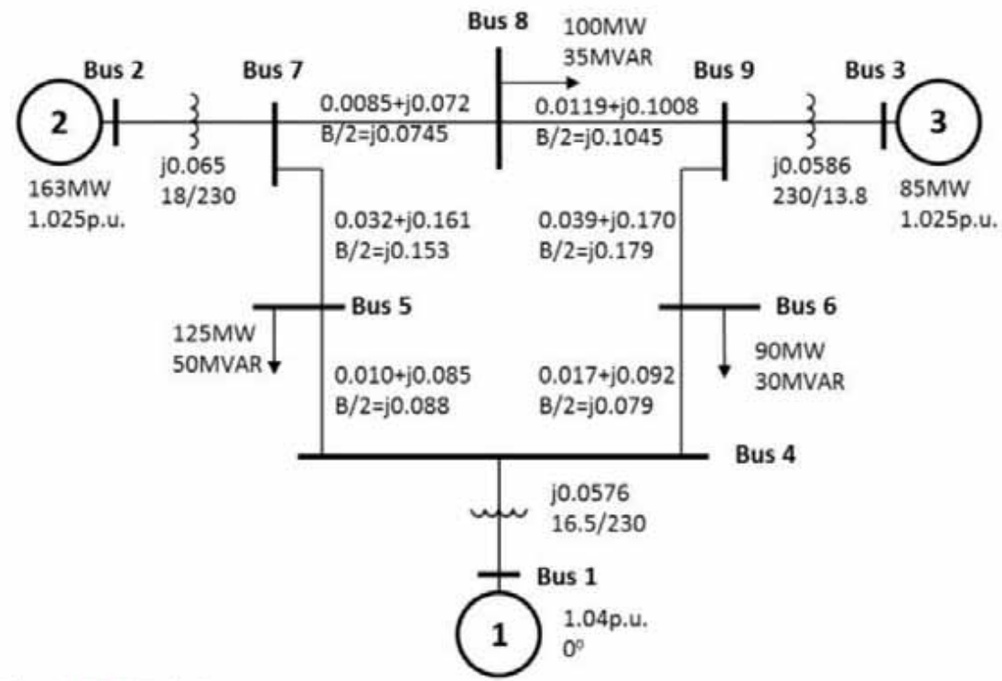


Fig.1. IEEE 9-bus system

TABLE I. PARAMETERS AND INITIAL CONDITIONS FOR THE SYSTEM THE CLASSICAL MODEL

	Generator 1	Generator 2	Generator 3
H	23.64	6.40	3.01
D	23.64	6.40	3.01
R_a	0	0	0
X'_d	0.0608	0.1198	0.1813

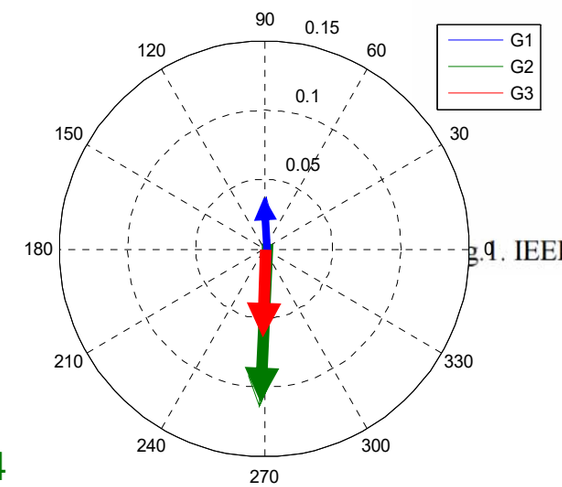
Case 1: Light-load, N-0

- **Mode 1:** $\lambda_{1,2}=\sigma\pm j\omega= -0.2500 \pm j8.7969$

Damp. ratio (ζ)= 2.84 %, Freq. ($\omega/2\pi$)= 1.4001 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k1}\rvert$	$\angle\phi_{k1}$	$\lvert\psi_{1k}\rvert$	$\angle\psi_{1k}$	$\lvert p_{1k}\rvert$	$\angle p_{1k}$
$\Delta\omega_{r1}$	0.326	180.00	1.261	180.00	0.411	0.00
$\Delta\delta_1$	0.037	88.37	11.094	-91.63	0.411	-3.26
$\Delta\omega_{r2}$	**1.000**	**0.00**	**1.000**	**0.00**	**1.000**	**0.00**
$\Delta\delta_2$	0.114	-91.63	8.800	88.37	1.000	-3.26
$\Delta\omega_{r3}$	0.556	0.00	0.261	0.00	0.145	0.00
$\Delta\delta_3$	0.063	-91.63	2.294	88.37	0.145	-3.26

Mode shapes
(ϕ_{k1} and ϕ_{k2} on rotor angles)



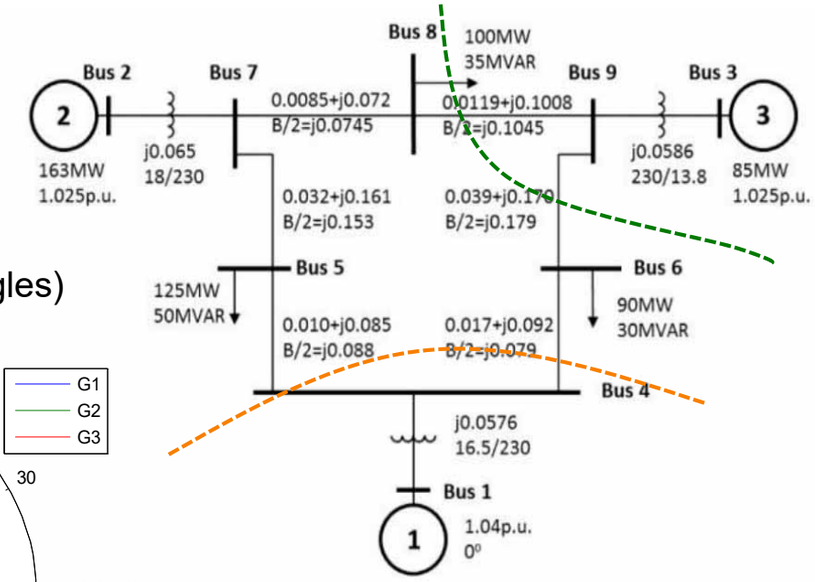
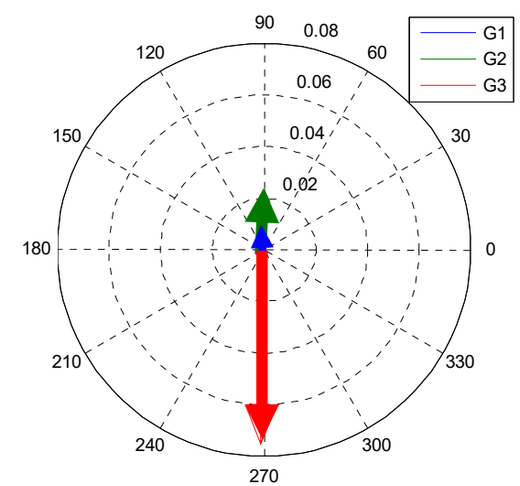


Fig. 9. IEEE 9-bus system

- **Mode 2:** $\lambda_{3,4}=\sigma\pm j\omega= -0.2500 \pm j13.3564$

Damp. ratio (ζ)= 1.87 %, Freq. ($\omega/2\pi$)= 2.1257 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k2}\rvert$	$\angle\phi_{k2}$	$\lvert\psi_{2k}\rvert$	$\angle\psi_{2k}$	$\lvert p_{2k}\rvert$	$\angle p_{2k}$
$\Delta\omega_{r1}$	0.042	180.00	0.335	-180.00	0.014	0.00
$\Delta\delta_1$	0.003	88.93	4.476	-91.07	0.014	-2.14
$\Delta\omega_{r2}$	0.313	0.00	0.665	180.00	0.208	0.00
$\Delta\delta_2$	0.023	88.93	8.883	-91.07	0.208	-2.14
$\Delta\omega_{r3}$	**1.000**	**0.00**	**1.000**	**0.00**	**1.000**	**0.00**
$\Delta\delta_3$	0.075	-91.07	13.359	88.93	1.000	-2.14



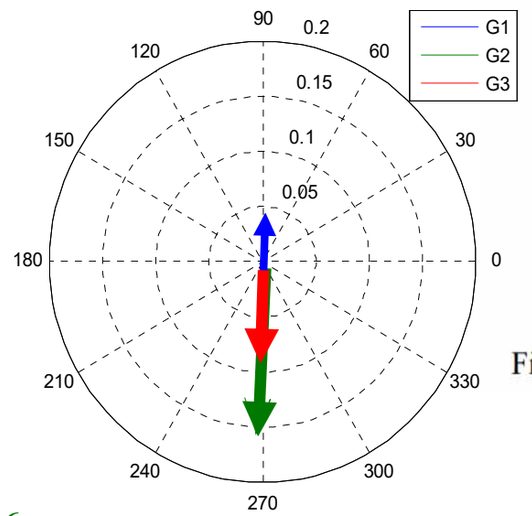
Case 2: Light-load, N-1

- **Mode 1:** $\lambda_{1,2}=\sigma\pm j\omega=-0.25\pm j6.4294$

Damp. ratio (ζ)= 3.89%, Freq. ($\omega/2\pi$)= 1.0233 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k1}\rvert$	$\angle\phi_{k1}$	$\lvert\psi_{1k}\rvert$	$\angle\psi_{1k}$	$\lvert p_{1k}\rvert$	$\angle p_{1k}$
$\Delta\omega_{r1}$	0.277	-180.00	1.243	-175.55	0.344	4.45
$\Delta\delta_1$	0.043	87.77	7.999	-87.77	0.344	0.00
$\Delta\omega_{r2}$	**1.000**	**0.00**	**1.000**	**4.45**	**1.000**	**4.45**
$\Delta\delta_2$	0.155	-92.23	6.434	92.23	1.000	0.00
$\Delta\omega_{r3}$	0.533	0.00	0.243	4.45	0.130	4.45
$\Delta\delta_3$	0.083	-92.23	1.565	92.23	0.130	0.00

Mode shapes
(ϕ_{k1} and ϕ_{k2} on rotor angles)



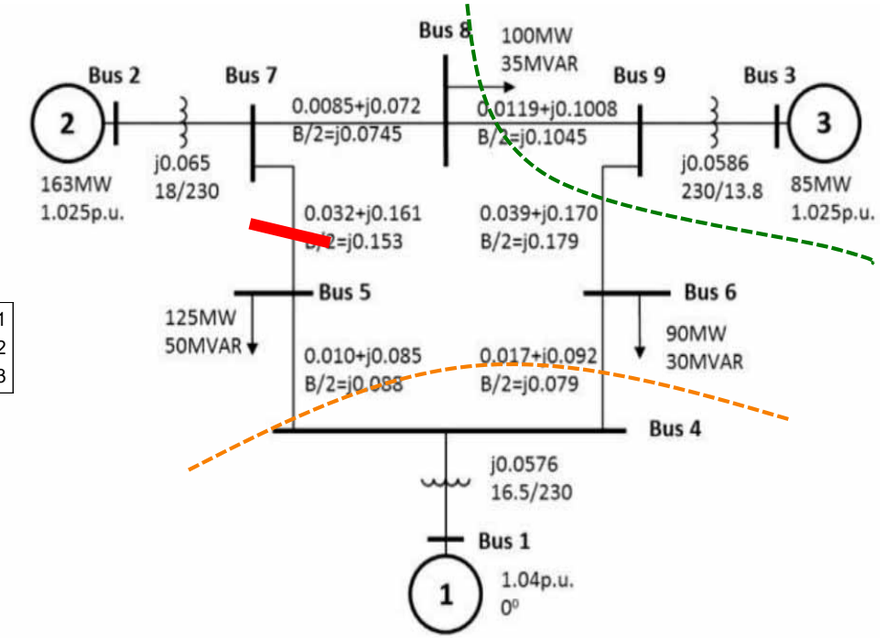
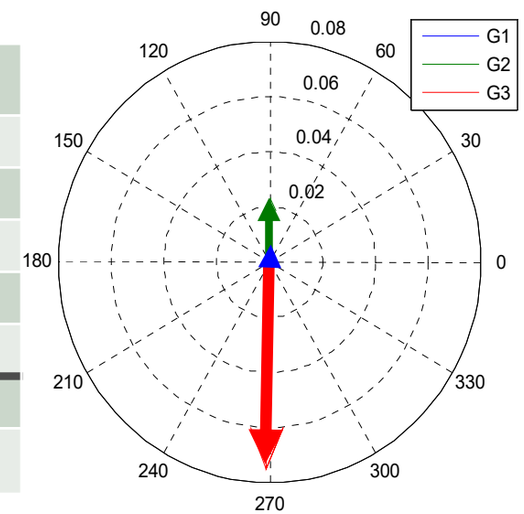


Fig.1. IEEE 9-bus system

- **Mode 2:** $\lambda_{3,4}=\sigma\pm j\omega=-0.25\pm j13.3026$

Damp. ratio (ζ)= 1.88%, Freq. ($\omega/2\pi$)= 2.1172 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k2}\rvert$	$\angle\phi_{k2}$	$\lvert\psi_{2k}\rvert$	$\angle\psi_{2k}$	$\lvert p_{2k}\rvert$	$\angle p_{2k}$
$\Delta\omega_{r1}$	0.042	180.00	0.335	180.00	0.014	0.00
$\Delta\delta_1$	0.003	88.93	4.476	-91.07	0.014	-2.14
$\Delta\omega_{r2}$	0.313	180.00	0.665	180.00	0.208	0.00
$\Delta\delta_2$	0.023	88.93	8.883	-91.07	0.208	-2.14
$\Delta\omega_{r3}$	**1.000**	**0.00**	**1.000**	**0.00**	**1.000**	**0.00**
$\Delta\delta_3$	0.075	-91.07	13.359	88.93	1.000	-2.14



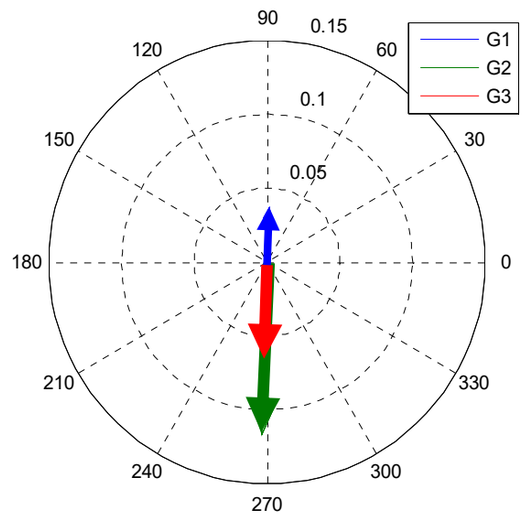
Case 3: Heavy-load, N-0

- **Mode 1:** $\lambda_{1,2}=\sigma\pm j\omega= -0.25\pm j8.5816$

Damp. ratio (ζ)= 2.91 %, Freq. ($\omega/2\pi$)= 1.3658 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k1}\rvert$	$\angle\phi_{k1}$	$\lvert\psi_{1k}\rvert$	$\angle\psi_{1k}$	$\lvert p_{1k}\rvert$	$\angle p_{1k}$
$\Delta\omega_{r1}$	0.313	180.00	1.244	-180.00	0.389	0.00
$\Delta\delta_1$	0.037	88.33	10.676	-91.67	0.389	-3.34
$\Delta\omega_{r2}$	**1.000**	**0.00**	**1.000**	**0.00**	**1.000**	**0.00**
$\Delta\delta_2$	0.117	-91.67	8.585	88.33	1.000	-3.34
$\Delta\omega_{r3}$	0.522	0.00	0.244	0.00	0.127	0.00
$\Delta\delta_3$	0.061	-91.67	2.091	88.33	0.127	-3.34

Mode shapes
(ϕ_{k1} and ϕ_{k2} on rotor angles)



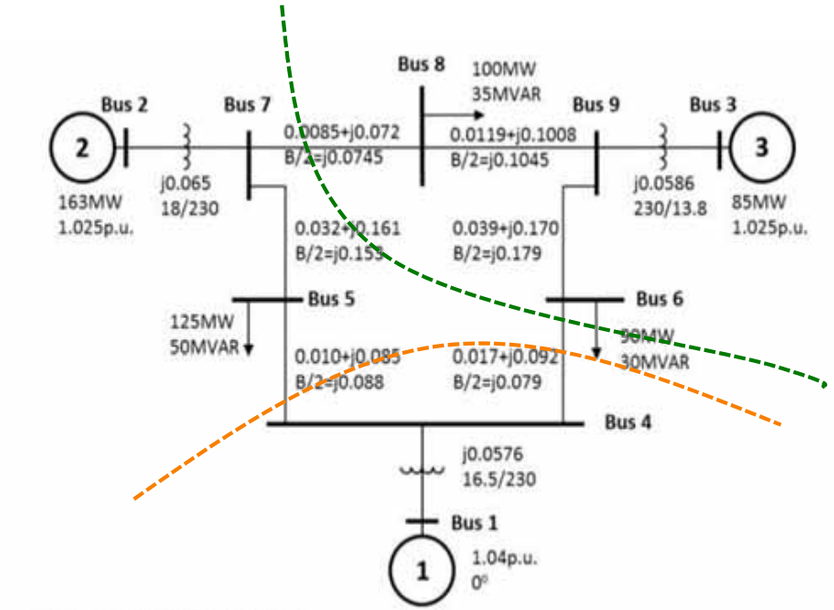
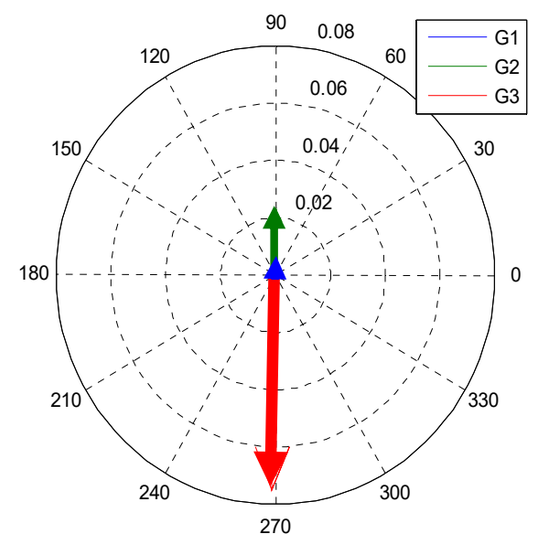


Fig.1. IEEE 9-bus system

- **Mode 2:** $\lambda_{3,4}=\sigma\pm j\omega= -0.25\pm j13.3185$

Damp. ratio (ζ)= 1.88 %, Freq. ($\omega/2\pi$)= 2.1197 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k2}\rvert$	$\angle\phi_{k2}$	$\lvert\psi_{2k}\rvert$	$\angle\psi_{2k}$	$\lvert p_{2k}\rvert$	$\angle p_{2k}$
$\Delta\omega_{r1}$	0.045	180.00	0.364	180.00	0.016	0.00
$\Delta\delta_1$	0.003	88.92	4.847	-91.08	0.016	-2.15
$\Delta\omega_{r2}$	0.300	180.00	0.636	180.00	0.191	0.00
$\Delta\delta_2$	0.023	88.92	8.474	-91.08	0.191	-2.15
$\Delta\omega_{r3}$	**1.000**	**0.00**	**1.000**	**0.00**	**1.000**	**0.00**
$\Delta\delta_3$	0.075	-91.08	13.321	88.92	1.000	-2.15



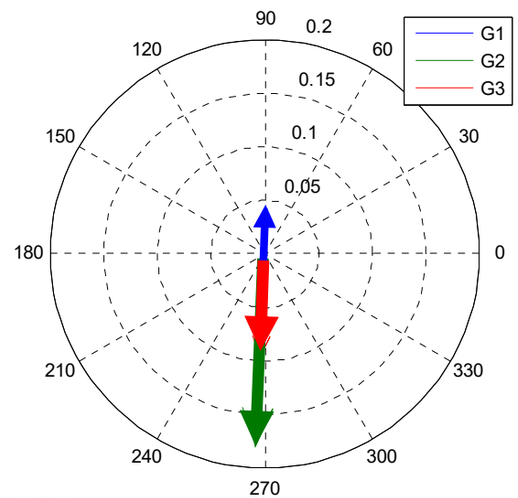
Case 4: Heavy-load, N-1

- **Mode 1:** $\lambda_{1,2}=\sigma\pm j\omega= -0.25\pm j5.5509$

Damp. ratio (ζ)= 4.50%, Freq. ($\omega/2\pi$)= 0.8835 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k1}\rvert$	$\angle\phi_{k1}$	$\lvert\psi_{1k}\rvert$	$\angle\psi_{1k}$	$\lvert p_{1k}\rvert$	$\angle p_{1k}$
$\Delta\omega_{r1}$	0.226	180.00	1.235	-174.84	0.279	5.16
$\Delta\delta_1$	0.041	87.42	6.863	-87.42	0.279	0.00
$\Delta\omega_{r2}$	**1.000**	**0.00**	**1.000**	**5.16**	**1.000**	**5.16**
$\Delta\delta_2$	0.180	-92.58	5.557	92.58	1.000	0.00
$\Delta\omega_{r3}$	0.531	0.00	0.235	5.16	0.125	5.16
$\Delta\delta_3$	0.096	-92.58	1.305	92.58	0.125	0.00

Mode shapes
(ϕ_{k1} and ϕ_{k2} on rotor angles)



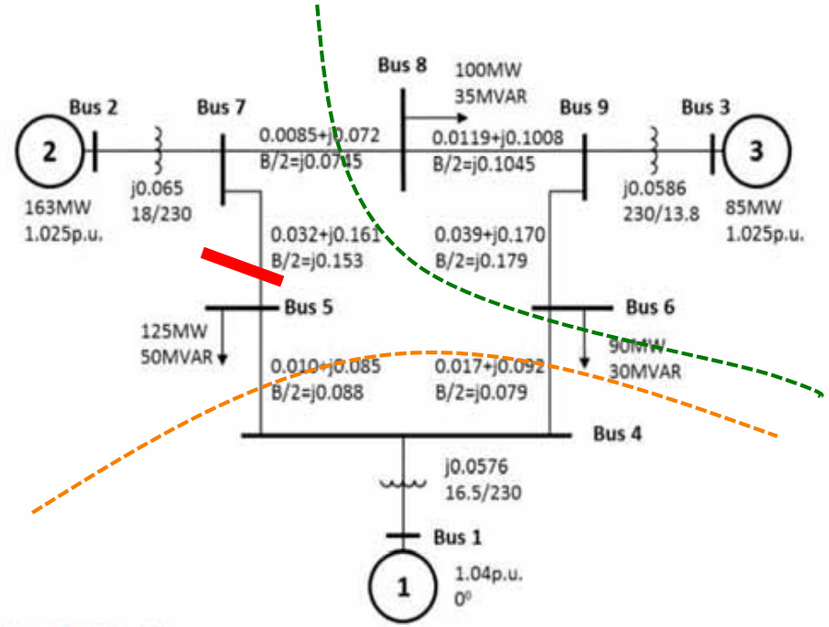
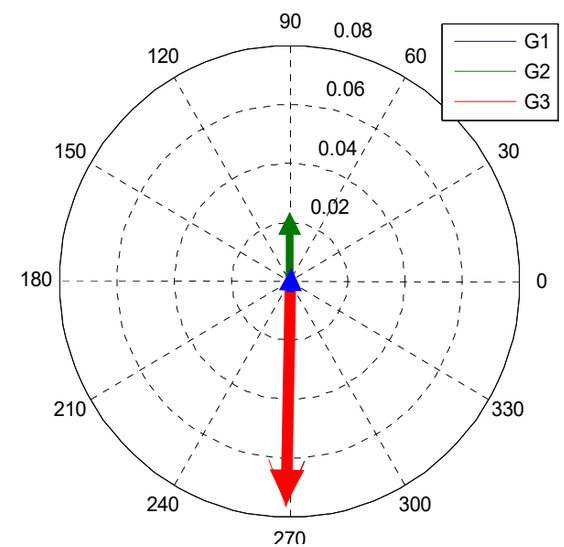


Fig.1. IEEE 9-bus system

- **Mode 2:** $\lambda_{3,4}=\sigma\pm j\omega= -0.25\pm j13.1477$

Damp. ratio (ζ)= 1.90%, Freq. ($\omega/2\pi$)= 2.0925 Hz

State	Mode shape		Mode Comp.		Part. Factor	
	$\lvert\phi_{k2}\rvert$	$\angle\phi_{k2}$	$\lvert\psi_{2k}\rvert$	$\angle\psi_{2k}$	$\lvert p_{2k}\rvert$	$\angle p_{2k}$
$\Delta\omega_{r1}$	0.044	180.00	0.382	180.00	0.017	0.00
$\Delta\delta_1$	0.003	88.91	5.024	-91.09	0.017	-2.18
$\Delta\omega_{r2}$	0.289	180.00	0.618	180.00	0.179	0.00
$\Delta\delta_2$	0.022	88.91	8.120	-91.09	0.179	-2.18
$\Delta\omega_{r3}$	**1.000**	**0.00**	**1.000**	**0.00**	**1.000**	**0.00**
$\Delta\delta_3$	0.076	-91.09	13.150	88.91	1.000	-2.18



Overview of Small-Signal Stability Enhancement

- The problem of small-signal stability is usually associated with insufficient damping of system oscillations.
  - The use of power system stabilizers (PSS) to control generator excitation systems is the most cost-effective method, whose idea is to modulate the generator excitation so as to develop a component of electrical torque in phase with rotor speed deviation, i.e. a positive damping torque component.
  - Additionally, supplemental stabilizing signals may be used to modulate HVDC converters and FACTS devices, e.g. SVCs, to enhance damping.
- The controls used for small-signal stability enhancement should also perform satisfactorily under severe disturbances. Therefore, while the controls are designed using linear system techniques, their overall performance should be assessed by considering both small and large signal responses.

- Read Kundur's chapter 17.2 "Small- Signal Stability Enhancement"

Controllability and Observability of a Mode

- Generate the mode-decoupled normal form (n modes, r inputs and m outputs) on $\mathbf{z}$:

$$\begin{aligned}\Delta\dot{\mathbf{x}} &= \mathbf{A}_{n\times n}\Delta\mathbf{x}+\mathbf{B}_{n\times r}\Delta\mathbf{u}\\ \Delta\mathbf{y} &= \mathbf{C}_{m\times n}\Delta\mathbf{x}+\mathbf{D}_{m\times r}\Delta\mathbf{u}\end{aligned} \quad \xleftrightarrow{\Delta\mathbf{x}=\boldsymbol{\Phi}\mathbf{z}} \quad \begin{aligned}\dot{\mathbf{z}} &= \boldsymbol{\Phi}^{-1}\mathbf{A}_{n\times n}\boldsymbol{\Phi}\mathbf{z}+\boldsymbol{\Phi}^{-1}\mathbf{B}_{n\times r}\Delta\mathbf{u} = \boldsymbol{\Lambda}_{n\times n}\mathbf{z}+\mathbf{B}'_{n\times r}\Delta\mathbf{u}\\ \Delta\mathbf{y} &= \mathbf{C}_{m\times n}\boldsymbol{\Phi}\mathbf{z}+\mathbf{D}_{m\times r}\Delta\mathbf{u} = \mathbf{C}'_{m\times n}\mathbf{z}+\mathbf{D}_{m\times r}\Delta\mathbf{u}\end{aligned}$$

Mode controllability matrix (~ mode compositions): $\mathbf{B}'_{n\times r} = \boldsymbol{\Phi}^{-1}\mathbf{B}_{n\times r} = \boldsymbol{\Psi}\mathbf{B}_{n\times r}$

– If i-th row of $\mathbf{B}'$ is zero, inputs in $\Delta\mathbf{u}$ have no effect on z_i (mode i)

Mode observability matrix (~ mode shapes): $\mathbf{C}'_{m\times n} = \mathbf{C}_{m\times n}\boldsymbol{\Phi}$

– If i-th column of $\mathbf{C}'$ is zero, z_i (mode i) is unobservable from $\Delta\mathbf{y}$

- Transfer function (single input-single output)

$$\begin{aligned}\dot{\mathbf{z}} &= \boldsymbol{\Lambda}\mathbf{z}+\boldsymbol{\Psi}\mathbf{b}\Delta u\\ \Delta y &= \mathbf{c}^T\boldsymbol{\Phi}\mathbf{z}\end{aligned} \quad \Longleftrightarrow \quad G(s)=\frac{\Delta y(s)}{\Delta u(s)} = \mathbf{c}^T\boldsymbol{\Phi}\left[s\mathbf{I}-\boldsymbol{\Lambda}\right]^{-1}\boldsymbol{\Psi}\mathbf{b} = \sum_{i=1}^{n}\frac{R_i}{s-\lambda_i}$$

Residual: $R_i = \underbrace{\mathbf{c}^T\boldsymbol{\phi}_i}_{\text{Observability}}\underbrace{\boldsymbol{\psi}_i\mathbf{b}}_{\text{Controllability}}$

Types of PSS (by selection of the input signal)

1. Delta-omega PSS (Shaft speed)

- Successful on hydraulic units since the mid-1960s; it needs to minimize the noise from shaft run-out (not rotating exactly in line with the axis);
- For thermal units, a torsional filter is needed and has to be customized for each type of generators against oscillations or instability of torsional modes (typically, >10Hz).

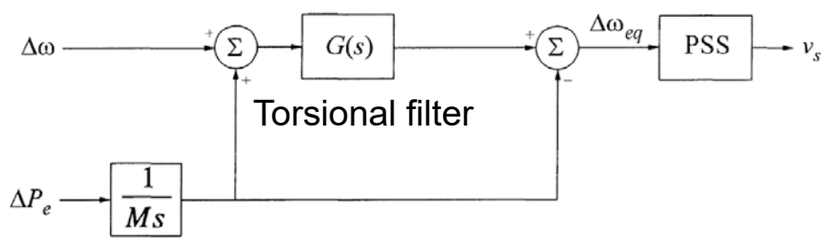
2. Delta-P-omega PSS

$$\Delta\omega_{eq}(s) = \frac{\Delta P_m(s)}{Ms} - \frac{\Delta P_e(s)}{Ms} = G(s)\left[\frac{\Delta P_e(s)}{Ms} + \Delta\omega(s)\right] - \frac{\Delta P_e(s)}{Ms}$$

No torsional modes

With torsional modes

No torsional modes



ω_0 = base frequency, rad/s
M = inertia coefficient = $2H$
s = d/dt

Figure 17.10 Block diagram realization of delta-P-omega stabilizer

3. Frequency based PSS:

- Either directly uses terminal frequency or derives approximate rotor speed from terminal voltage and current
- Frequency signal is more sensitive to inter-area oscillations, so it helps damp an inter-area mode better than the speed input signal.
- Shortcomings include
  - During a rapid transient, the terminal frequency signal will undergo a sudden phase shift and may result in a spike in the filed voltage.
  - The frequency signal often contains power system noise caused by large industrial loads such as furnaces.
  - Torsional filtering is still required since it has the same basic limitation as the delta-omega PSS

4. Digital PSS:

- Software program in the digital excitation control system.

Supplementary Control of SVCs

- An SVC can contribute to the enhancement of the power system dynamic performance by rapidly controlling the voltage and reactive power:
  - Normally, voltage regulation is the primary mode of control to improve voltage stability and transient stability.
  - Its supplementary control has the effectiveness in enhancing small-signal stability depending on the location, input signals and controller design.

- Placement of an SVC for small-signal stability enhancement:
  - Usually, an SVC may be placed on the dominant oscillation path near the center of oscillation (at the middle of the interconnection between two areas, where voltage swings are the greatest).
  - For a large complex system, the SVC should be placed at the bus where voltage swing is most sensitive to susceptance change (with the highest voltage participation factor).

Dominant Oscillation Path and Center of an Inter-area Mode

- About a specific inter-area mode at ω, the path and center of oscillation are important for monitoring and control purposes
  - Dominant oscillation path: the path where changes of line current magnitudes are the largest about the mode in terms of power spectral densities (PSDs).
  - Center of oscillation: the bus on the dominant oscillation path that has the largest change in voltage magnitude and the smallest change in voltage angle about the mode.

For $x(t)$ ($t \in [0, T]$) with Fourier transform $X(\omega)$

$$\text{PSD:}\ \ S_x(\omega) = \lim_{T \to \infty} \frac{1}{T} \mathbf{E}\left[X^*(\omega) X\ (\omega) \right]$$

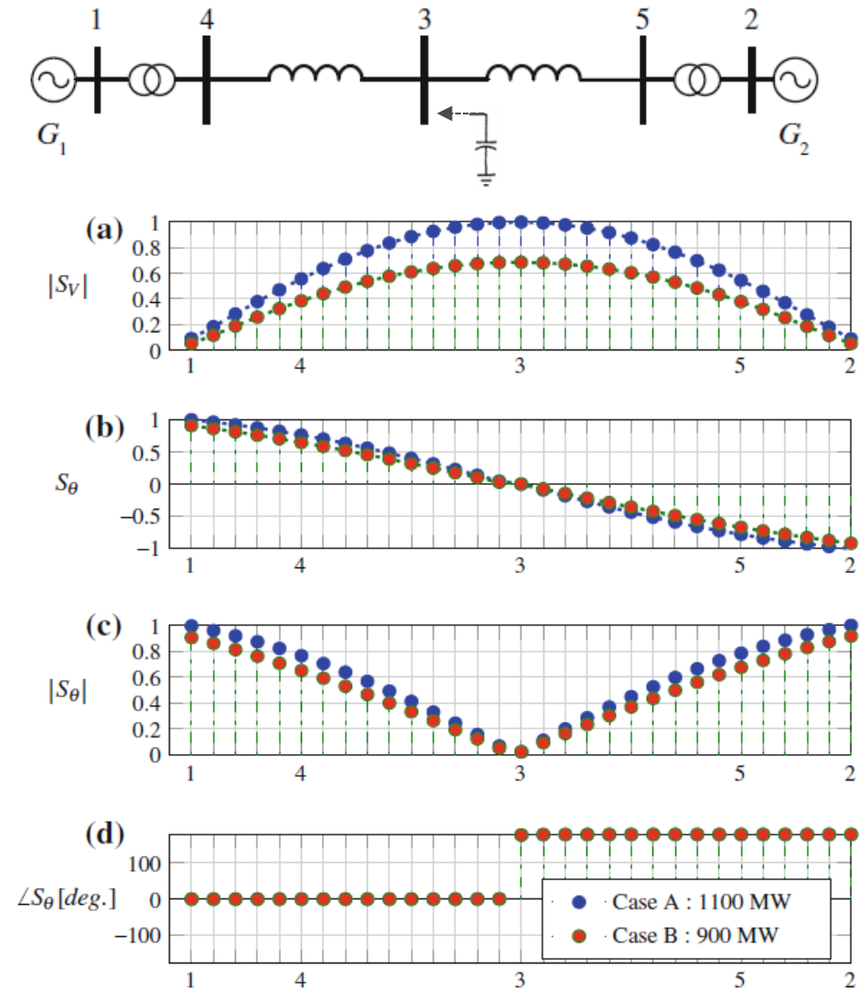
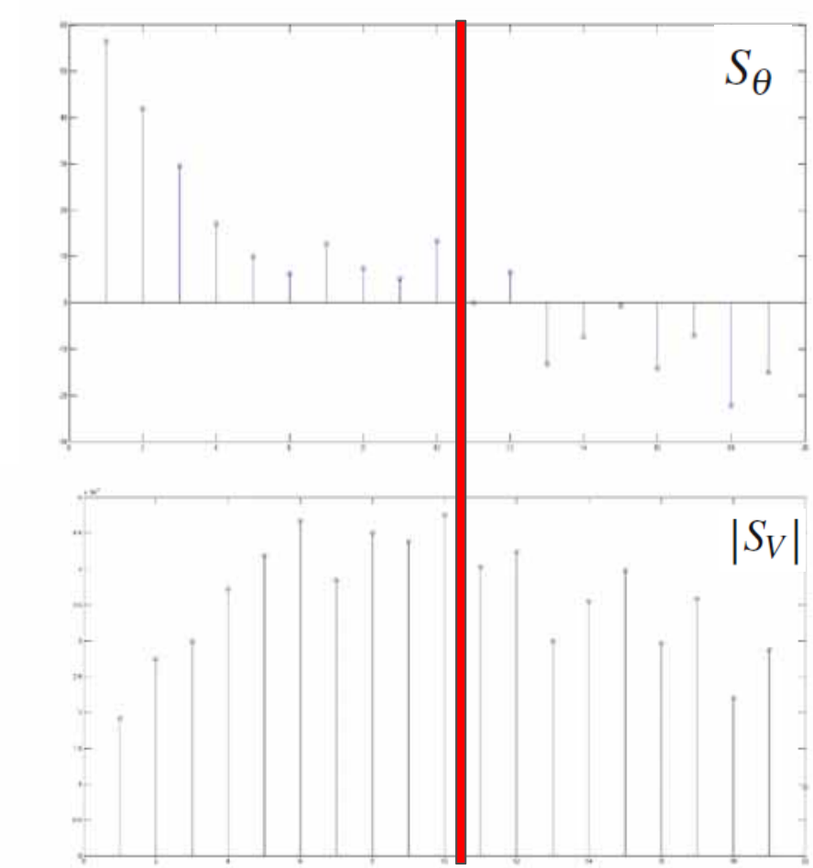


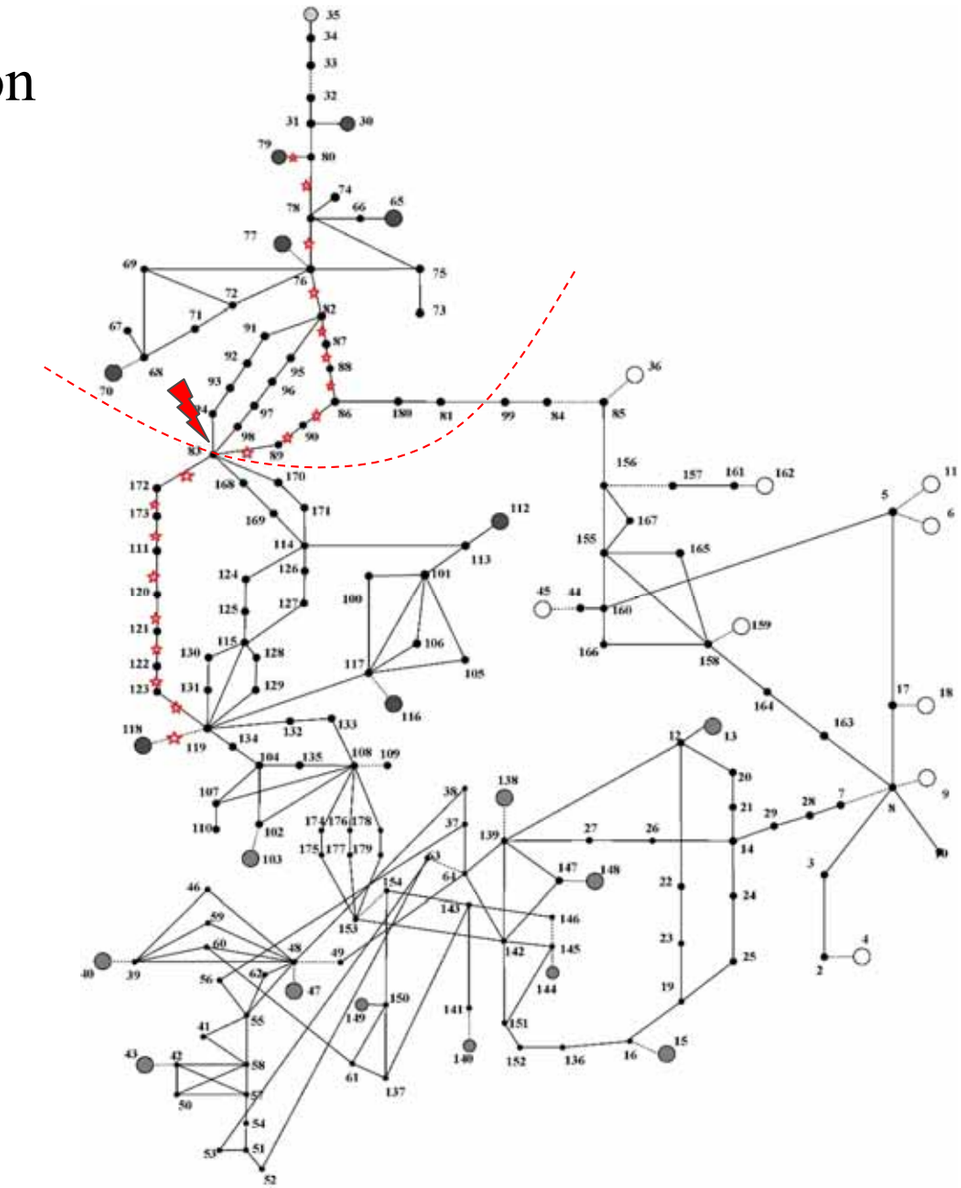
Fig. 10.15 Voltage magnitude and angle modeshapes of the dominant path in the two-area system. **a** Voltage magnitude modeshape, **b** Voltage angle modeshape, **c** Magnitude of voltage angle modeshape, **d** Phase of voltage angle modeshape

For details, see Chapter 10 (by Vanfretti, Chompoobutrgool and Chow) of Chow's book

Example on the WECC System

- Identify the oscillation path and center on the 0.25Hz mode





Effect of SVC on an Inter-area mode

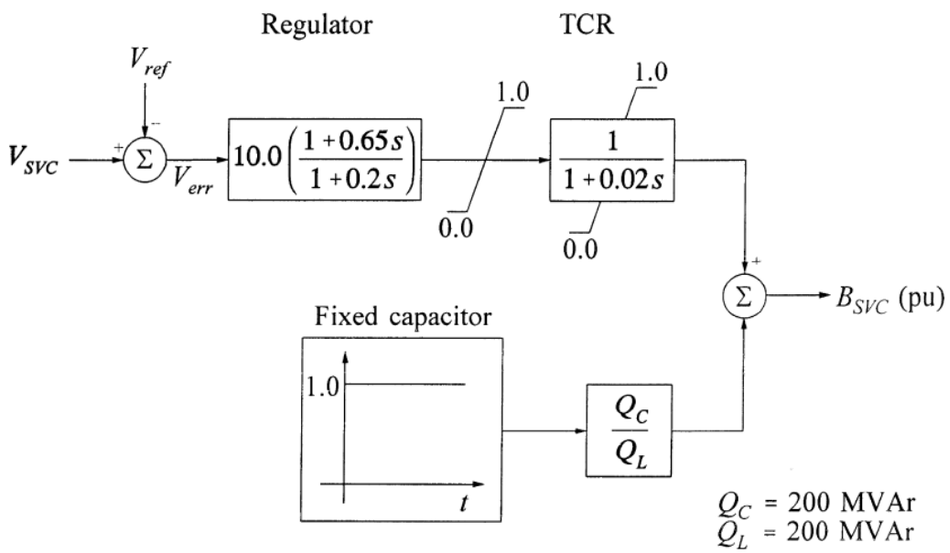


Figure 17.17 Block diagram of SVC and voltage regulator

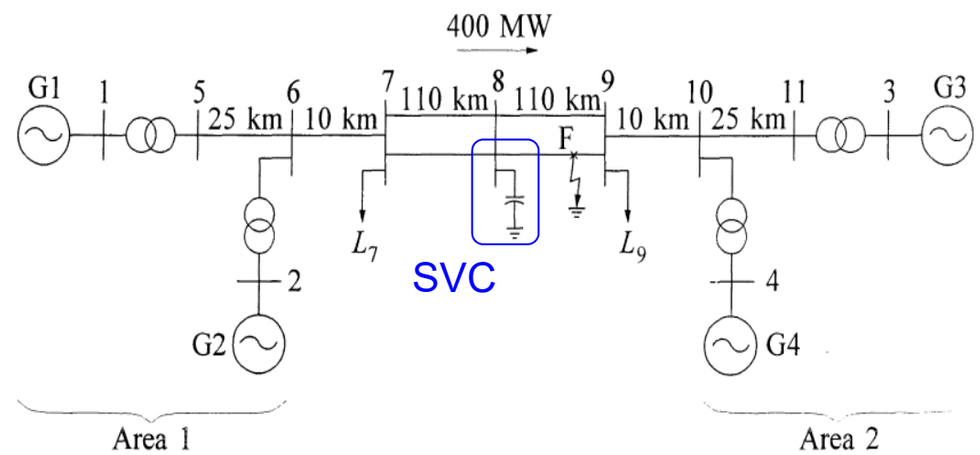


Figure 17.15 A simple two-area system

Table 17.6 Effect of SVC on interarea mode (frequency and damping ratio)

System Condition	No SVC		With SVC	
	Frequency	ζ	Frequency	ζ
Prefault	0.540 Hz	0.0064	0.547 Hz	0.0096
Postfault	0.417 Hz	−0.0228	0.476 Hz	0.0154

Selecting the supplementary control signal

$$G(s)=\frac{\Delta y(s)}{V_{err}(s)}=\sum_{i=1}^{n}\frac{R_i}{s-\lambda_i}=\sum_{i=1}^{n}\frac{\mathbf{c}^T\boldsymbol{\phi}_i\boldsymbol{\psi}_i\mathbf{b}}{s-\lambda_i}$$

- SVC (providing Δu of the system to eliminate V_{err}) should be placed at the bus with good mode controllability, i.e. having a large value of $\boldsymbol{\psi}_i\mathbf{b}$
- Its supplementary control signal (i.e. Δy of the system) should have good mode observability, i.e. having a large value of $\mathbf{c}^T\boldsymbol{\phi}_I$

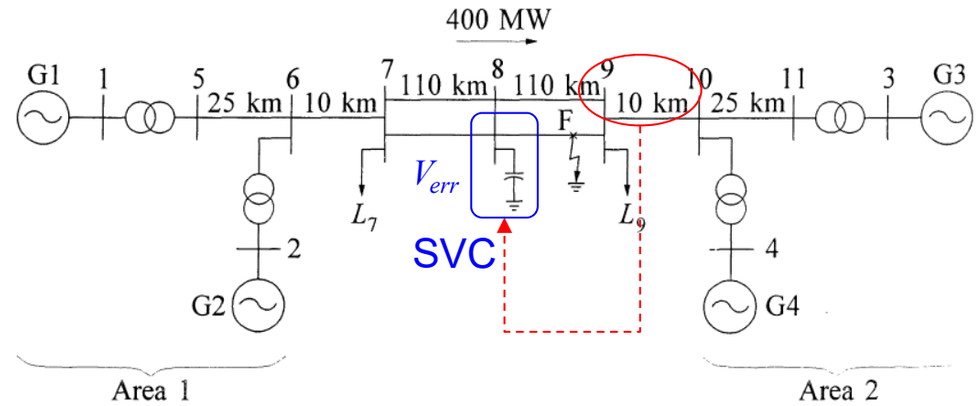


Figure 17.15 A simple two-area system

Table 17.7 Residues and observability factors

Signal	Prefault		Postfault	
	Residue	Observability	Residue	Observability
Δω of G1	−0.2680 − j0.1156	0.8738	−0.9522 − j0.3651	0.5471
Δω of G2	−0.2121 − j0.1004	0.7025	−0.7733 − j0.3347	0.4510
Δω of G3	0.4588 + j0.1121	1.4140	2.6200 + j0.2945	1.4140
Δω of G4	0.4064 + j0.0947	1.2510	2.4380 + j0.2469	1.3140
ΔP, line 6-7	−0.2286 + j0.4914	1.6230	−0.6551 + j1.5160	0.8861
ΔP, line 10-9	0.2122 − j0.8560	2.6400	0.5305 − j4.3570	2.3550
ΔQ, line 6-7	−0.0310 + j0.2107	0.6375	0.1289 + j1.2020	0.6847
ΔQ, line 10-9	−0.0400 + j0.2346	0.7126	−0.0924 + j1.6290	0.8752
ΔI, line 6-7	−0.3157 + j0.8618	2.7980	−0.8919 + j3.4640	1.9120
ΔI, line 10-9	0.2615 − j0.8732	2.7290	0.6484 − j3.8810	2.1110

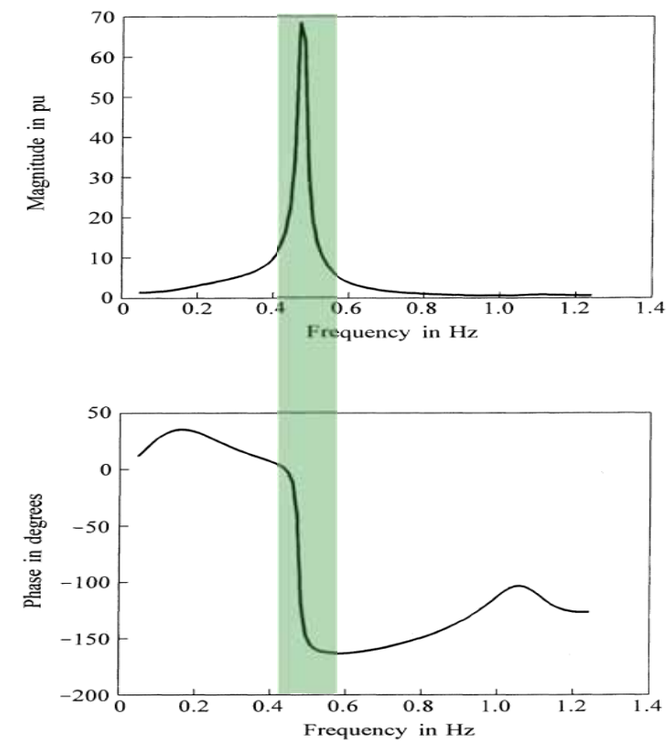
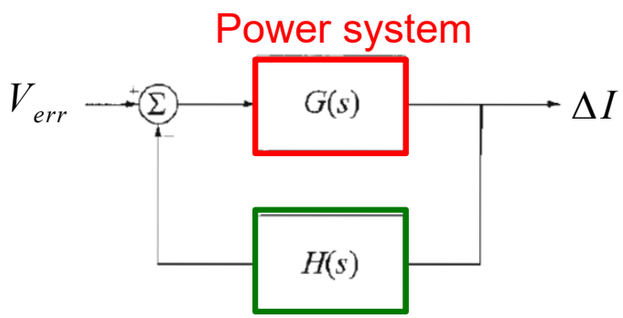


Figure 17.18 Frequency response of the transfer function between the SVC input and the current in line between buses 9 and 10

Designing the supplementary control



$$G_c(s) = \frac{\Delta I}{V_{err}} = \frac{G(s)}{1 + G(s)H(s)}$$

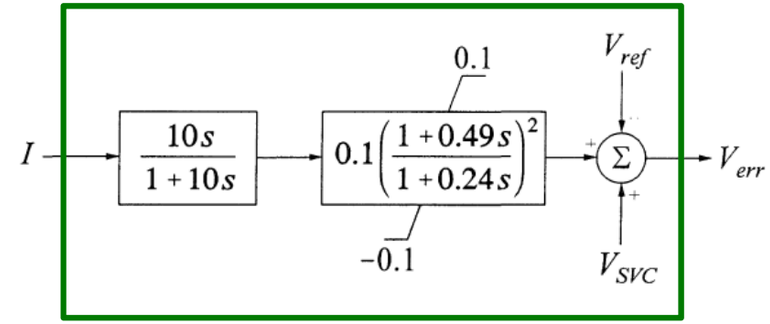
Design $\lambda_0 = \sigma + j\omega$

$$H(\lambda_0) = -\frac{1}{G(\lambda_0)} = \frac{1}{|G(\lambda_0)|}\angle 180° - \arg(G(\lambda_0))$$

Table 17.8 The desired location of the eigenvalues and the value of the system transfer function

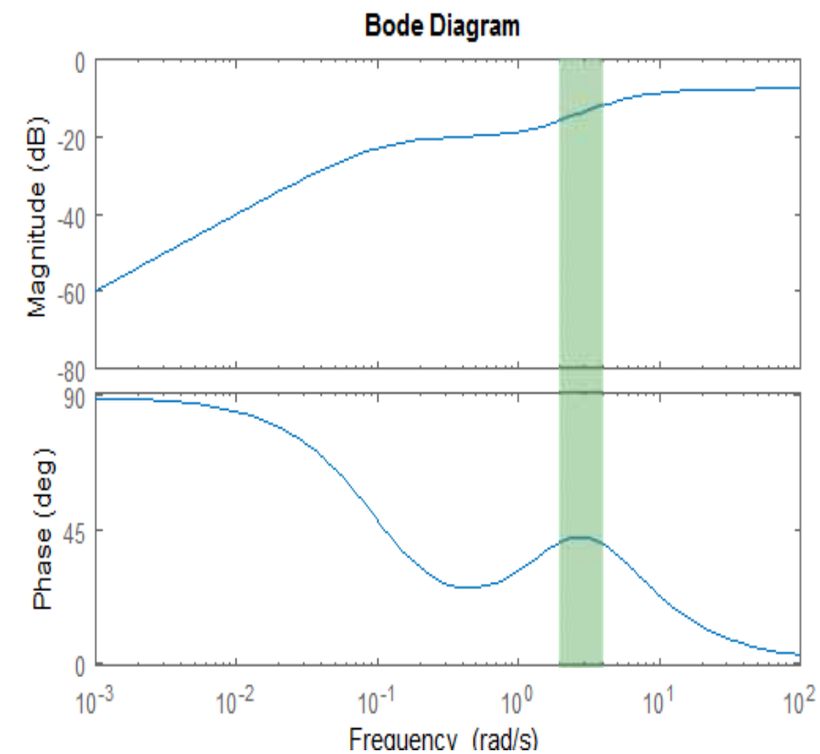
Prefault System		Postfault System	
New Eigenvalue Location	Value of Transfer Function	New Eigenvalue Location	Value of Transfer Function
$-0.127 \pm j3.544$	$4.262\angle 137.8°$	$-0.576 \pm j3.498$	$4.177\angle 132.3°$

$$H(\lambda_0^{pre}) = 0.235\angle 42.2° \qquad H(\lambda_0^{post}) = 0.239\angle 47.7°$$

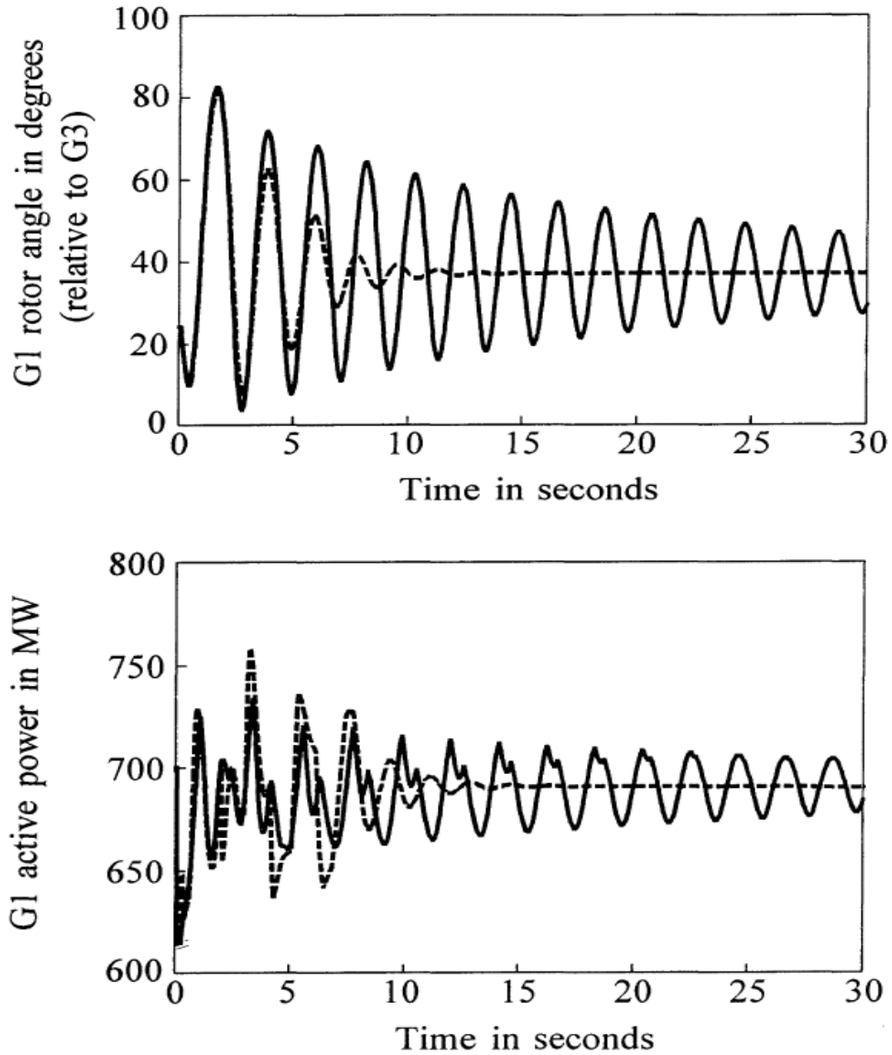


I = magnitude of current in line between buses 9 and 10

Figure 17.20 Supplementary control block diagram



Effect of SVC Supplementary Control



Prefault: f=0.564 Hz ζ=0.036
Postfault: f=0.557 Hz ζ=0.163

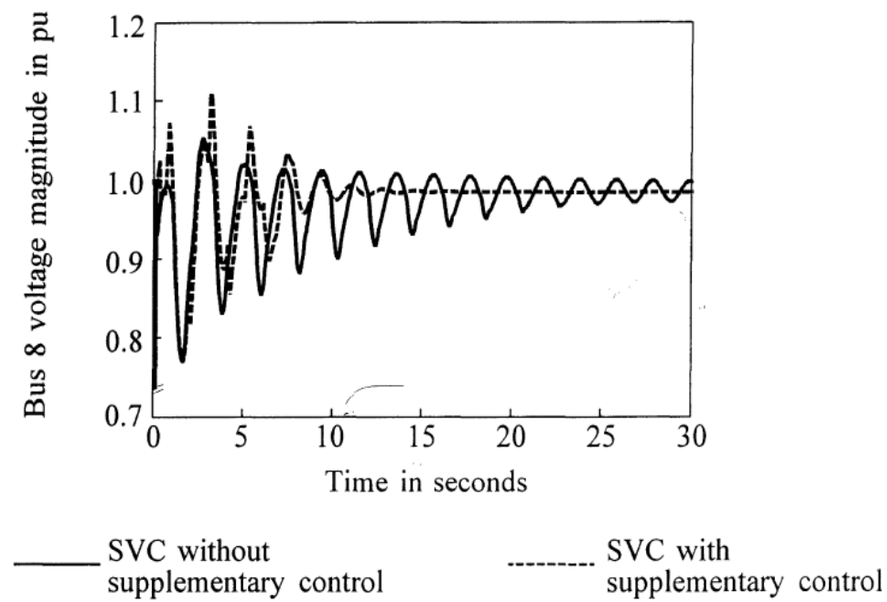


Figure 17.21 System response to a severe disturbance with and without supplementary control of an SVC at bus 8

Model-based methods vs. Measurement-based methods

- Model-based methods:
  Linearized model

Input:

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{A}\Delta\mathbf{x}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\Delta\mathbf{x}(t)$$

Eigen-analysis
to find $\lambda_i = \sigma_i \pm j\omega_i$,
ϕ_{ik} , ψ_{ik} , p_{ik}

Output:

$$y_j(t) = \mathbf{c}_j^T \Delta\mathbf{x}(t)$$

$$= \sum_{i=1}^{n} \mathbf{c}_j^T \boldsymbol{\phi}_i z_i(0) e^{\lambda_i t}$$

- Measurement-based methods
  (e.g. Prony analysis):

Measurements on **y**(t)

Signal processing and decomposition to find a number of damped sinusoids of σ_i, ω_i and φ_i whose weighted sum matches $\mathbf{y}(t)$

$\lambda_i = \sigma_i \pm j\omega_i \quad B_{ji} = |B_{ji}| \angle \varphi_i$

$$\sum_i |B_{ji}| e^{\sigma_i t} \sin(\omega_i t + \varphi_i) \quad \sim \quad y_j(t) = \sum_{i=1}^{n} B_{ji} e^{\lambda_i t}$$

A Measurement-based Method: Prony Analysis

- Given the data of $y(t)$ at the j-th location over a time window T sampled at intervals of $T_s=T/N$: $y[0], y[2], \ldots, y[N\text{-}1]$.
- Consider p modes: $y(t)=\sum_{i=1}^{p} B_i e^{\lambda_i t} = \sum_{i=1}^{p} B_i e^{\sigma_i t}\sin(\omega_i t+\varphi_i) \Rightarrow y[k]=\sum_{i=1}^{p} B_i z_i^{\,k}$ where $z_i^k = e^{\lambda_i (kT_s)}$ and $z_i = e^{\lambda_i T_s}$
- The problem is to find the estimates for all complex numbers B_i and z_i. Since the number of equations is N and the number of real unknowns is $4p$, so $N \geq 4p$ should be guaranteed.
  - **Step 1: Solve modes $z_1,\ldots, z_p$** by constructing a p-th order polynomial equation based on Autoregressive model AR(p):

$$z^p - a_1 z^{p-1} - \cdots - a_{p-1} z - a_p = 0$$

$$\lambda_i = (\ln z_i)/T_s$$

$$\begin{bmatrix} y[p] \\ y[p+1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} y[p-1] & y[p-2] & \cdots & y[0] \\ y[p] & y[p-1] & \cdots & y[1] \\ \vdots & \vdots & \ddots & \vdots \\ y[N-2] & y[N-3] & \cdots & y[N-p-1] \end{bmatrix}_{(N-p)\times p} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

$$\mathbf{y} = \mathbf{T}\mathbf{x} \Rightarrow$$

$$\mathbf{x} = \left(\mathbf{T}^{\mathrm{T}}\mathbf{T}\right)^{-1}\mathbf{T}^{\mathrm{T}}\mathbf{y}$$

Using Matlab pseudo-inverse function:

x = pinv(T)***y**

  - **Step 2: Solve $B_1, \ldots, B_p$**

$$y[k]=\sum_{i=1}^{p} B_i z_i^{\,k} = B_1 z_1^{\,k} + B_2 z_2^{\,k} + \cdots B_p z_p^{\,k} \Rightarrow \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_p \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_p^{N-1} \end{bmatrix}_{N\times p} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix}$$

$$\Delta x_k(t) = \sum_{i=1}^{n} \phi_{ki} z_i(t)$$

  - **Step 3: Repeat Steps 1-2 at data from multiple locations**. Values of B_i at these locations give the mode shape of mode i.

Ref: Hauer, J.F.; et al, "Initial results in Prony analysis of power system response signals," *IEEE Transactions on Power Systems*, vol.5, no.1, pp.80-89, Feb 1990