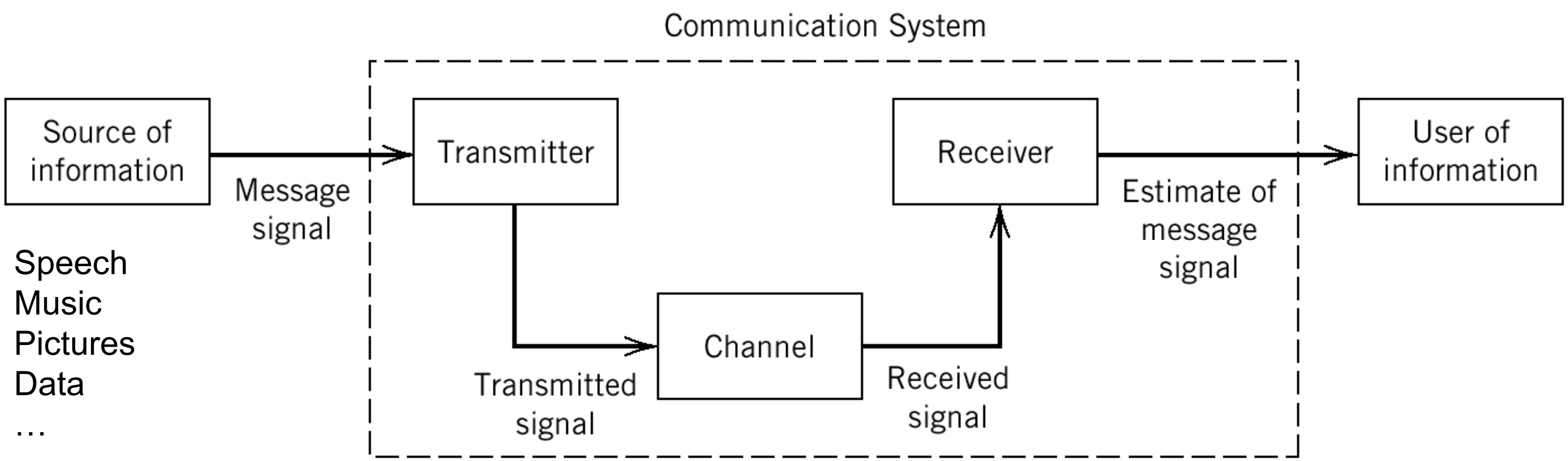
What's Communications?

- Communication involves the transfer of information from one point to another.
- Three basic elements
  - **Transmitter**: converts message into a form suitable for transmission
  - **Channel**: the physical medium, introduces distortion, noise, interference
  - **Receiver**: reconstruct a recognizable form of the message



Speech
Music
Pictures
Data
…

Noise in Communications

- Unavoidable presence of noise in the channel
  - Noise refers to unwanted waves that disturb communications
  - Signal is contaminated by noise along the path.
- External noise: interference from nearby channels, human-made noise, natural noise...
- Internal noise: thermal noise, random emission... in electronic devices
- Noise is one of the basic factors that set limits on communications.
- A widely used metric is the signal-to-noise (power) ratio (SNR)

$$\text{SNR} = \frac{\text{signal power}}{\text{noise power}}$$

Transmitter and Receiver

- The transmitter modifies the message signal into a form suitable for transmission over the channel
- This modification often involves **modulation**
  - Moving the signal to a high-frequency carrier (up-conversion) and varying some parameter of the carrier wave
  - Analog: AM, FM, PM
  - Digital: ASK, FSK, PSK (SK: shift keying)
- The receiver recreates the original message by **demodulation**
  - Recovery is not exact due to noise/distortion
  - The resulting degradation is influenced by the type of modulation
- Design of analog communication is conceptually simple
- Digital communication is more efficient and reliable; design is more sophisticated

Objectives of System Design

- Two primary resources in communications
  - Transmitted **power** (should be green)
  - Channel **bandwidth** (very expensive in the commercial market)
- In certain scenarios, one resource may be more important than the other
  - Power limited (e.g. deep-space communication)
  - Bandwidth limited (e.g. telephone circuit)
- Objectives of a communication system design
  - The message is delivered both efficiently and reliably, subject to certain design constraints: power, bandwidth, and cost.
  - **Efficiency** is usually measured by the amount of messages sent in unit power, unit time and unit bandwidth.
  - **Reliability** is expressed in terms of SNR or probability of error.

Why Probability/Random Process?

- Probability is the core mathematical tool for communication theory.
- The stochastic model is widely used in the study of communication systems.
- Consider a radio communication system where the received signal is a random process in nature:
  - Message is random. No randomness, no information.
  - Interference is random.
  - Noise is a random process.
  - And many more (delay, phase, fading, ...)
- Other real-world applications of probability and random processes include
  - Stock market modelling, gambling (Brown motion as shown in the previous slide, random walk)…

Probabilistic Concepts

- What is a random variable (RV)?
  - It is a variable that takes its values from the outputs of a random experiment.
- What is a random experiment?
  - It is an experiment the outcome of which cannot be predicted precisely.
  - All possible identifiable outcomes of a random experiment constitute its ***sample space*** Ω.
  - An ***event*** is a collection of possible outcomes of the random experiment.
- Example
  - For tossing a coin, $\Omega = \{ H, T \}$
  - For rolling a die, $\Omega = \{ 1, 2, \ldots, 6 \}$

Probability Properties

- $P_X(x_i)$: the *probability* of the random variable X taking on the value x_i
- The probability of an event to happen is a non-negative number, with the following properties:
  - The probability of the event that includes all possible outcomes of the experiment is 1.
  - The probability of two events that do not have any common outcome is the sum of the probabilities of the two events separately.
- Example
  - Roll a die: $P_X(x = k) = 1/6$ for $k = 1, 2, \ldots, 6$

CDF and PDF

- The **(cumulative) distribution function (cdf)** of a random variable X is defined as the probability of X taking a value less than the argument x:

$$F_X(x) = P(X \leq x)$$

- Properties

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

$$F_X(x_1) \leq F_X(x_2) \quad \text{if } x_1 \leq x_2$$

- The **probability density function (pdf)** is defined as the derivative of the distribution function:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)dy$$

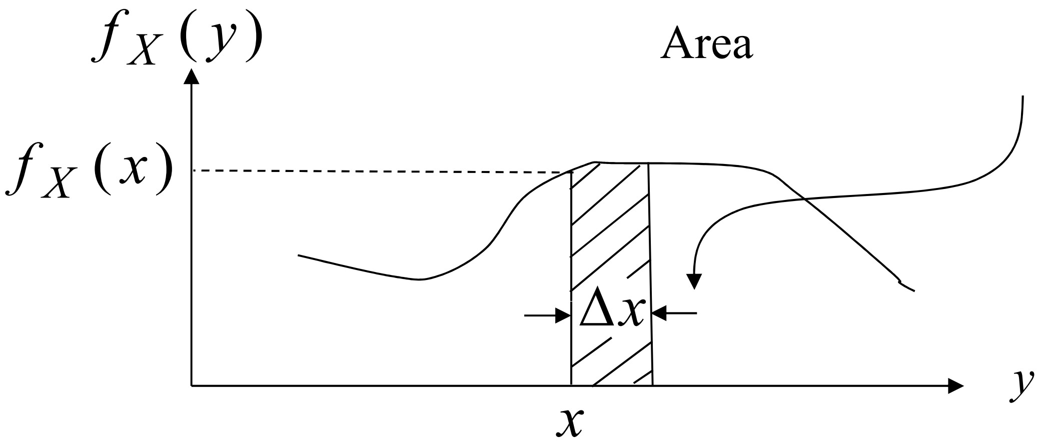
$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_{a}^{b} f_X(y)dy$$

$$f_X(x) = \frac{dF_X(x)}{dx} \geq 0 \ \text{ since } F_X(x) \text{ is non-decreasing}$$

Mean and Variance

- If Δx is sufficiently small,

$$P(x < X \leq x + \Delta x) = \int_{x}^{x+\Delta x} f_X(y)dy \cong \underline{f_X(x)\Delta x}$$



- Mean (or expected value ⇔ DC level):

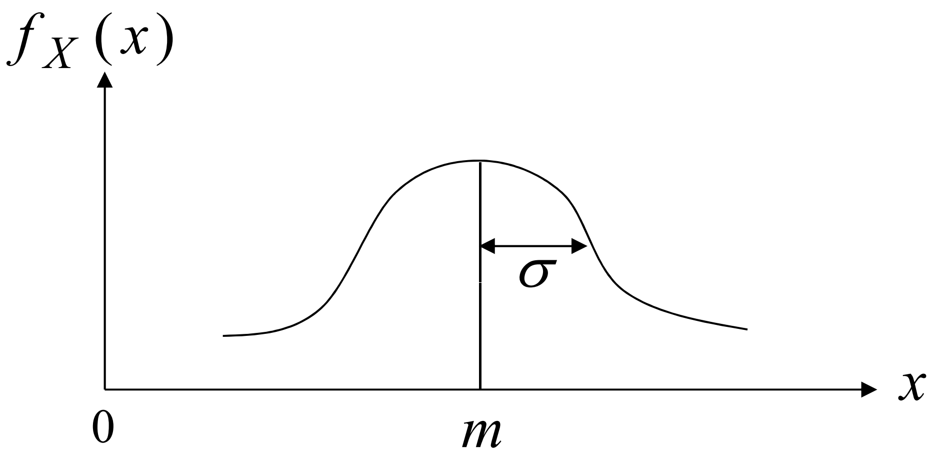
$$E[X] = \mu_X = \int_{-\infty}^{\infty} x\, f_X(x)dx$$

$E[\ ]$: expectation operator

- Variance (⇔ power for zero-mean signals):

$$\sigma_X^2 = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx = E[X^2] - \mu_X^2$$

Normal (Gaussian) Distribution



$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \quad \text{for } -\infty < x < \infty$$

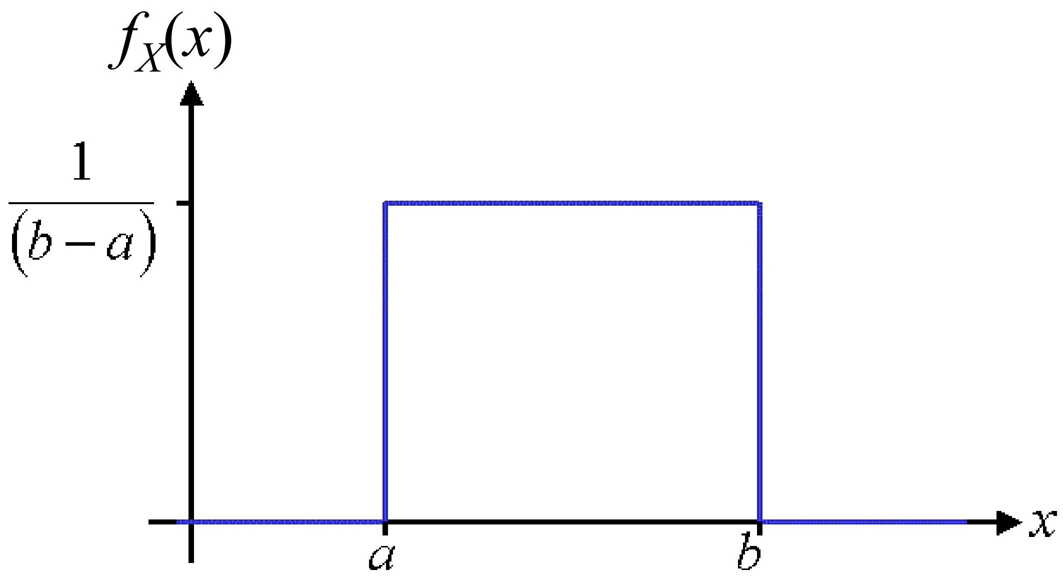
$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{(y-m)^2}{2\sigma^2}} dy$$

$E[X] = m$

$\sigma_X^{\ 2} = \sigma^2$

σ : rms value

Uniform Distribution



$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$E[X] = \frac{a+b}{2}$$

$$\sigma_X{}^2 = \frac{(b-a)^2}{12}$$

Joint Distribution

- Joint distribution function for two random variables X and Y

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

- Joint probability density function

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

- Properties

1) $F_{XY}(\infty, \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u, v)\,du\,dv = 1$

2) $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)\,dy$

3) $f_Y(x) = \int_{x=-\infty}^{\infty} f_{XY}(x, y)\,dx$

4) X, Y are **independent** $\Leftrightarrow f_{XY}(x, y) = f_X(x) f_Y(y)$

5) X, Y are **uncorrelated** $\Leftrightarrow E[XY] = E[X]E[Y]$

Independent vs. Uncorrelated

- Independent implies Uncorrelated
- Uncorrelated does not imply Independence
- For normal RVs (jointly Gaussian), Uncorrelated implies Independent (this is the only exceptional case!)
- An example of uncorrelated but dependent RV's

Let θ be uniformly distributed in $[0,2\pi]$

$$f_\theta(x) = \frac{1}{2\pi} \quad \text{for } 0 \le x \le 2\pi$$

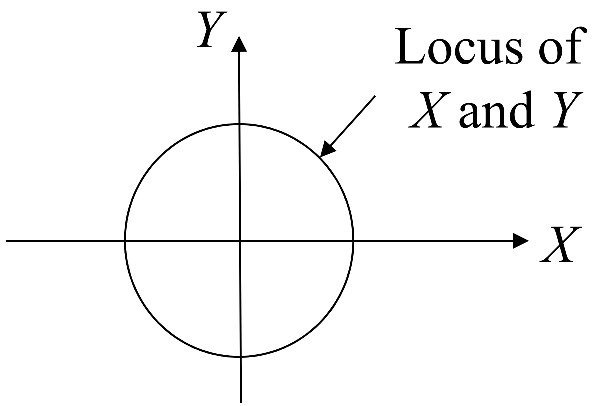
Define RV's X and Y as

$$X = \cos\theta \quad Y = \sin\theta$$

Clearly, X and Y are not independent.

But X and Y are uncorrelated:

$$E[XY] = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \sin\theta d\theta = 0!$$



Joint Distribution of *n* RVs

- Joint cdf

$$F_{X_1X_2\ldots X_n}(x_1, x_2, \ldots x_n) \equiv P(X_1 \le x_1, X_2 \le x_2, \ldots X_n \le x_n)$$

- Joint pdf

$$f_{X_1X_2\ldots X_n}(x_1, x_2, \ldots x_n) \equiv \frac{\partial^n F_{X_1X_2\ldots X_n}(x_1, x_2, \ldots x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

- **Independent**

$$F_{X_1X_2\ldots X_n}(x_1, x_2, \ldots x_n) = F_{X_1}(x_1)F_{X_2}(x_2)\ldots F_{X_n}(x_n)$$

$$f_{X_1X_2\ldots X_n}(x_1, x_2, \ldots x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\ldots f_{X_n}(x_n)$$

- **i.i.d.** (independent, identically distributed)
  - The random variables are independent and have the same distribution.
  - Example: outcomes from repeatedly flipping a coin.

Central Limit Theorem

- For i.i.d. random variables,
  $$z = x_1 + x_2 + \cdots + x_n$$
  tends to Gaussian as n goes to infinity.
- Extremely useful in communications.
- That's why noise is usually Gaussian. We often say "**Gaussian noise**" or "Gaussian channel" in communications.

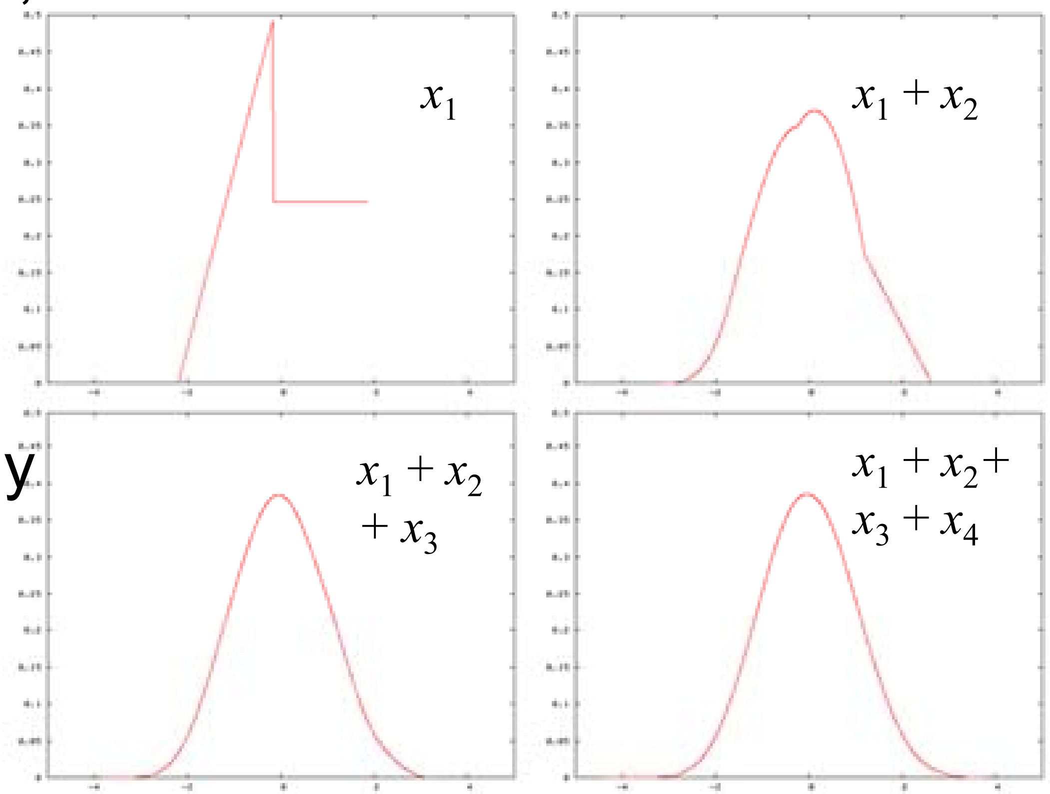


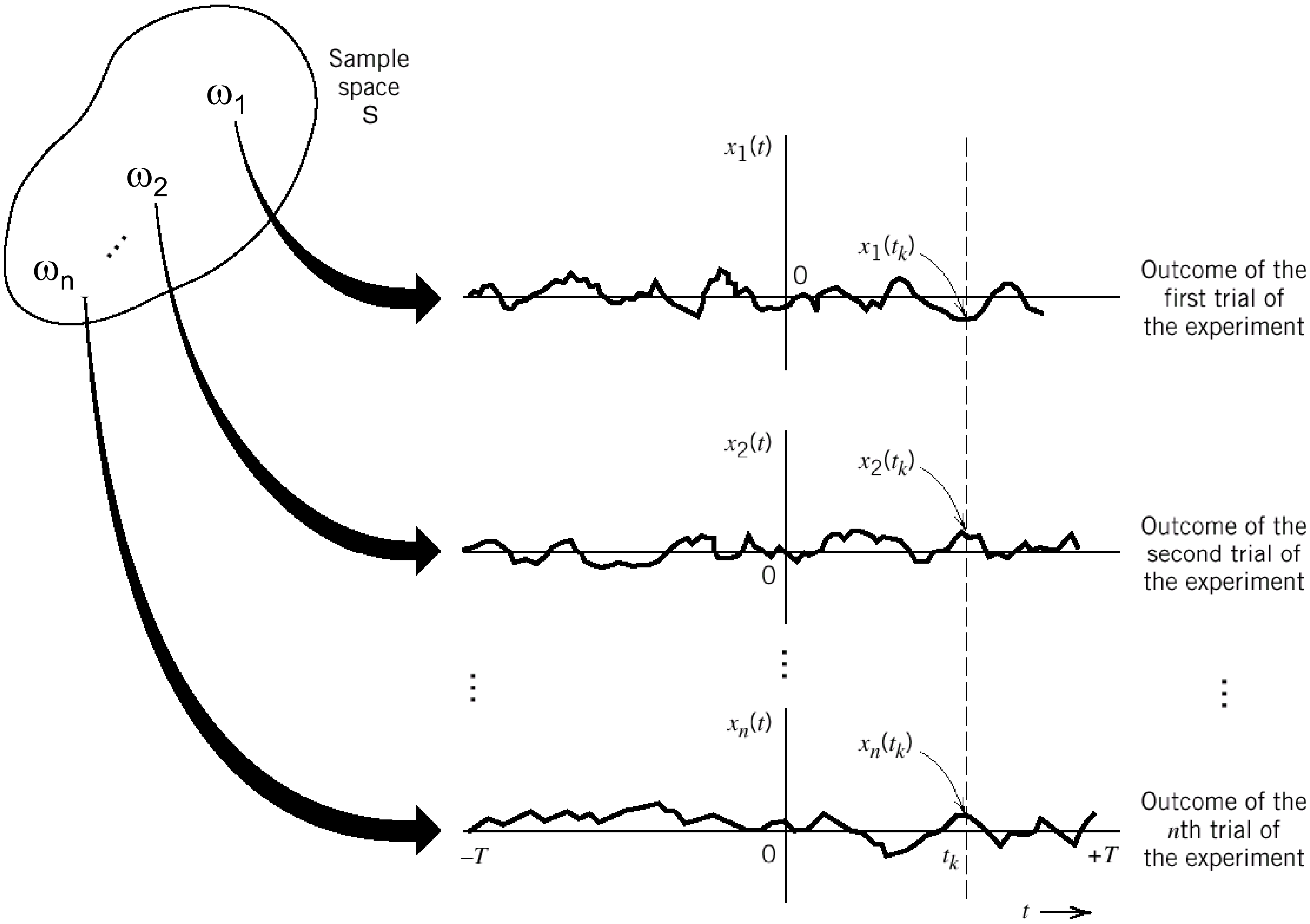
Illustration of convergence to Gaussian distribution

What is a Random Process?

- A random process is a time-varying function that assigns the outcome of a random experiment to each time instant: $X(t; \omega)$.
- For a fixed (sample path) ω: a random process is a time varying function, e.g., a signal.
- For fixed t: a random process is a random variable.
- If ω scans all possible outcomes of the underlying random experiment, we shall get an **ensemble** of signals.

- Noise can often be modelled as a **Gaussian random process**.

An Ensemble of Signals



Statistics of a Random Process

- For fixed *t*: the random process becomes a random variable, with mean

$$\mu_X(t) = E[X(t;\omega)] = \int_{-\infty}^{\infty} x\, f_X(x;t)dx$$

  - In general, the mean is a function of t.

- Autocorrelation function

$$R_X(t_1,t_2) = E[X(t_1;\omega)X(t_2;\omega)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x,y;t_1,t_2)dxdy$$

  - In general, the autocorrelation function is a two-variable function.

Stationary Random Processes

- A random process is (wide-sense) stationary if
  - Its mean does not depend on *t*

$$\mu_X(t) = \mu_X$$

  - Its autocorrelation function only depends on time difference

$$R_X(t, t+\tau) = R_X(\tau)$$

- In communications, noise and message signals can often be modelled as stationary random processes.

Power Spectral Density

- Power spectral density (PSD) is a function that measures the distribution of power of a random process with frequency.
- PSD is only defined for stationary processes.
- **Wiener-Khinchine relation:** The PSD is equal to the Fourier transform of its autocorrelation function:

$$\boxed{S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau}$$

  – A similar relation exists for deterministic signals
- Then the average power can be found as

$$P = R_X(0) = \int_{-\infty}^{\infty} S_X(f) df$$

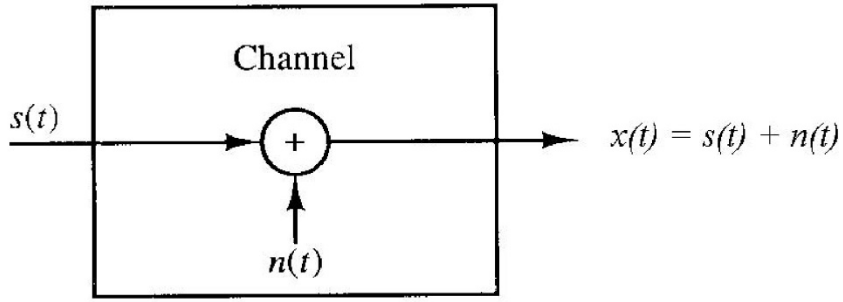
- The frequency content of a process depends on how rapidly the amplitude changes as a function of time.
  – This can be measured by the autocorrelation function.

Noise

- Noise is the unwanted and beyond our control waves that disturb the transmission of signals.
- Where does noise come from?
  – External sources: e.g., atmospheric, galactic noise, interference;
  – Internal sources: generated by communication devices themselves.
    - This type of noise represents a basic limitation on the performance of electronic communication systems.
    - **Shot noise:** the electrons are discrete and are not moving in a continuous steady flow, so the current is randomly fluctuating.
    - **Thermal noise:** caused by the rapid and random motion of electrons within a conductor due to thermal agitation.
- Both are often stationary and have a zero-mean **Gaussian distribution** (following from the central limit theorem).

White Noise

- The additive noise channel
  - $n(t)$ models all types of noise
  - zero mean



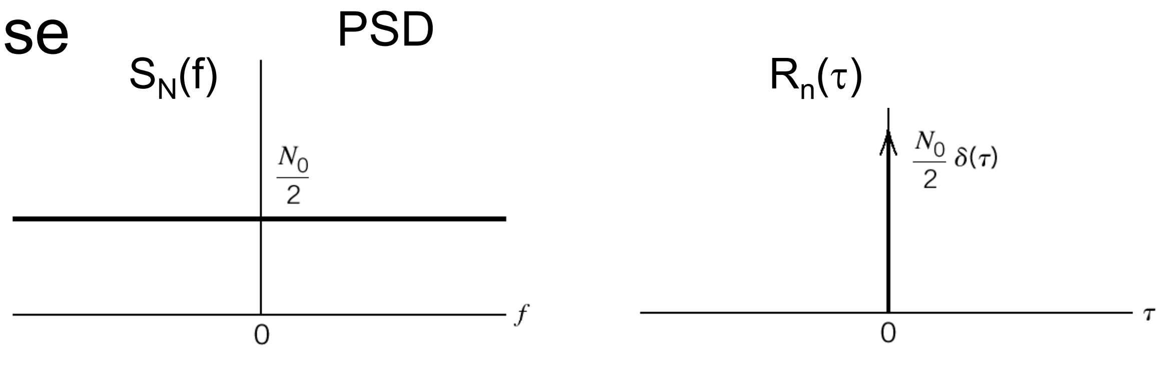
- White noise
  - Its power spectrum density (PSD) is constant over all frequencies, i.e.,

$$S_N(f) = \frac{N_0}{2}, \qquad -\infty < f < \infty$$

  - Factor 1/2 is included to indicate that half the power is associated with positive frequencies and half with negative.
  - The term ***white*** is analogous to white light which contains equal amounts of all frequencies (within the visible band of EM wave).
  - It's only defined for stationary noise.
- An infinite bandwidth is a purely theoretic assumption.

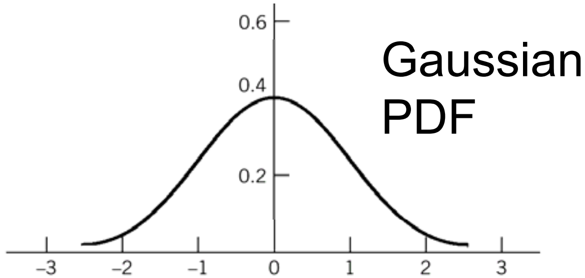
White vs. Gaussian Noise

- White noise



(a) (b)

  - Autocorrelation function of $n(t): R_n(\tau) = \frac{N_0}{2}\delta(\tau)$
  - Samples at different time instants are uncorrelated.
- Gaussian noise: the distribution at any time instant is Gaussian
  - Gaussian noise can be colored
- White noise $\neq$ Gaussian noise
  - White noise can be non-Gaussian



- Nonetheless, in communications, it is typically additive white Gaussian noise (AWGN).

Ideal Low-Pass White Noise

- Suppose white noise is applied to an ideal low-pass filter of bandwidth B such that

$$S_N(f) = \begin{cases} \dfrac{N_0}{2}, & |f| \le B \\ 0, & \text{otherwise} \end{cases}$$

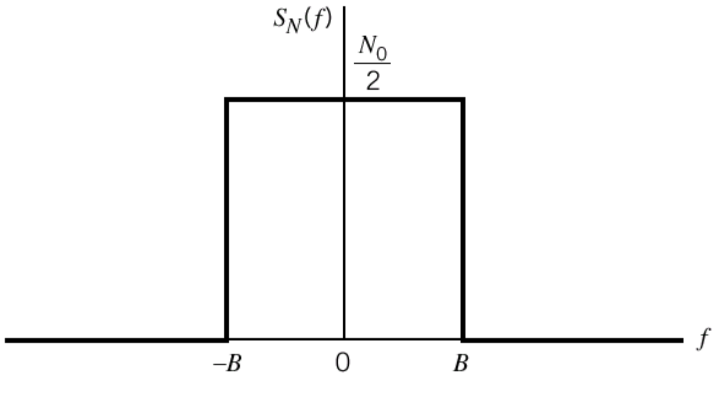
- By Wiener-Khinchine relation, autocorrelation function

$$R_n(\tau) = E[n(t)n(t+\tau)] = N_0 B \,\text{sinc}(2B\tau) \qquad (3.1)$$

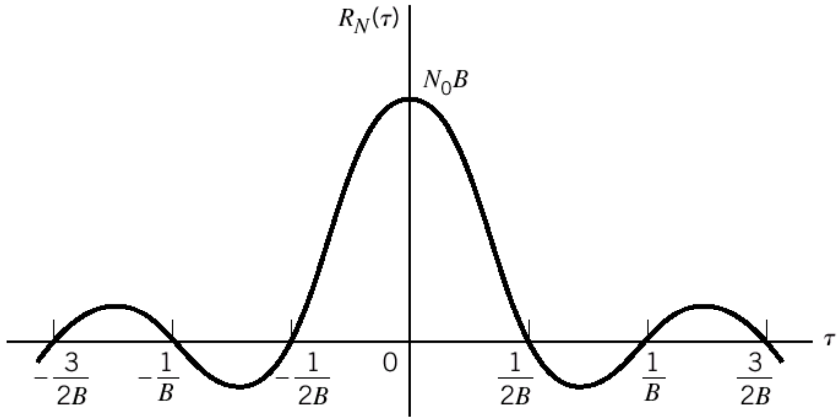
where $\text{sinc}(x) = \sin(\pi x)/\pi x$.

- Samples at Nyquist frequency $2B$ are uncorrelated

$$R_n(\tau) = 0, \qquad \tau = k/(2B),\ k = 1, 2, \ldots$$



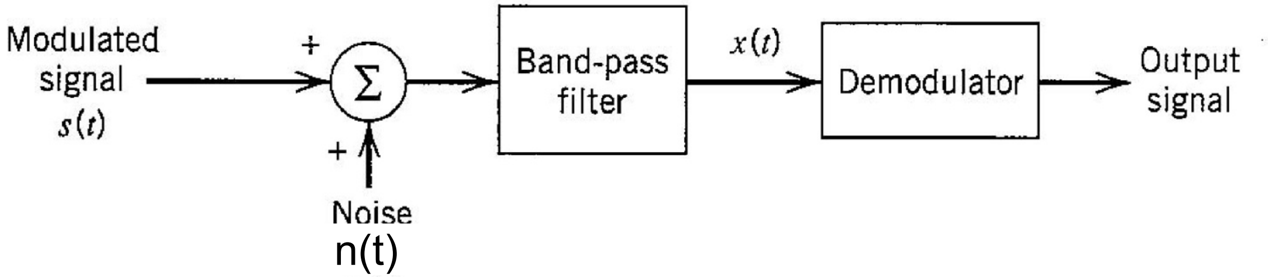
(a)



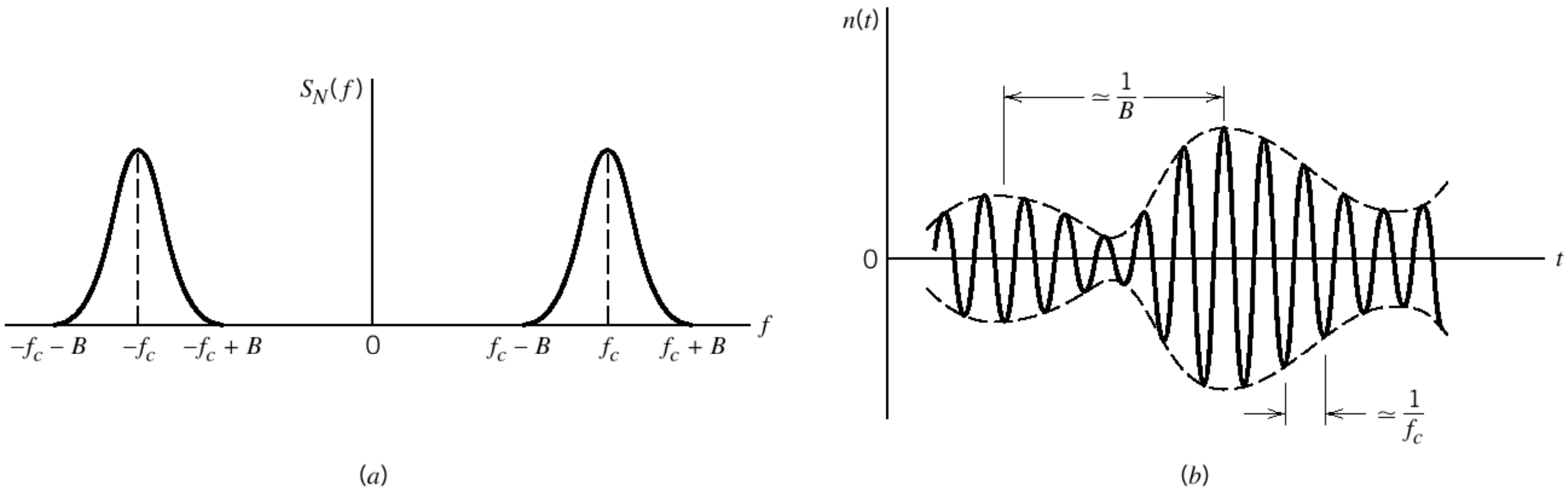
(b)

Bandpass Noise

- Any communication system that uses carrier modulation will typically have a bandpass filter of bandwidth B at the front-end of the receiver.



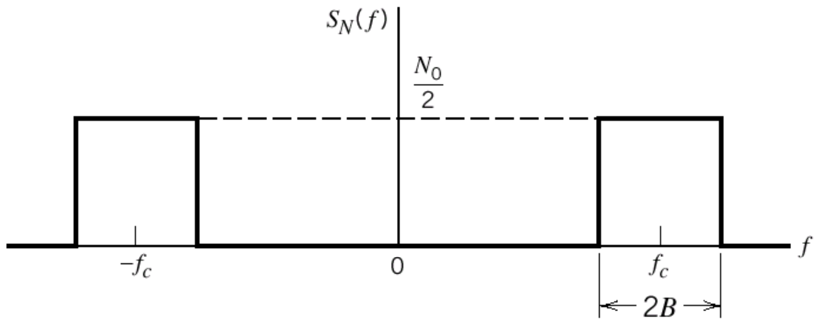
- Any noise that enters the receiver will therefore be bandpass in nature: its spectral magnitude is non-zero only for some band concentrated around the carrier frequency f_c (sometimes called **narrowband noise**).



Example

- If white noise with PSD of $N_0/2$ is passed through an ideal bandpass filter, then the PSD of the noise that enters the receiver is given by

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & |f - f_C| \le B \\ 0, & \text{otherwise} \end{cases}$$



- Autocorrelation function

$$R_n(\tau) = 2N_0 B \text{sinc}(2B\tau)\cos(2\pi f_c \tau)$$

  – which follows from (3.1) by applying the frequency-shift property of the Fourier transform

$$\boxed{\begin{array}{c} g(t) \Leftrightarrow G(\omega) \\ g(t)\cdot 2\cos\omega_0 t \Leftrightarrow [G(\omega-\omega_0)+G(\omega+\omega_0)] \end{array}}$$

- Samples taken at frequency $2B$ are still uncorrelated.

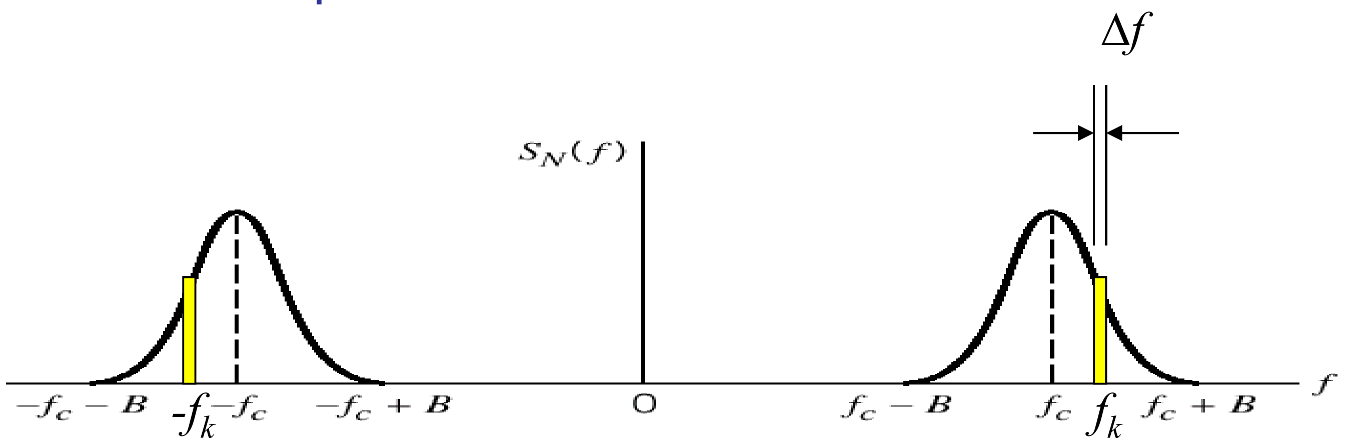
$$R_n(\tau) = 0, \quad \tau = k/(2B),\ k = 1, 2, \ldots$$

Decomposition of Bandpass Noise

- Consider bandpass noise within $|f - f_C| \le B$ with any PSD (i.e., not necessarily white as in the previous example)
- Consider a frequency slice Δf at frequencies f_k and $-f_k$.
- For Δf small:

$$n_k(t) = a_k \cos(2\pi f_k t + \theta_k)$$

  - θ_k: a random phase assumed independent and uniformly distributed in the range [0, 2π)
  - a_k: a random amplitude.



Representation of Bandpass Noise

- The complete bandpass noise waveform $n(t)$ can be constructed by summing up such sinusoids over the entire band, i.e.,

$$n(t) = \sum_k n_k(t) = \sum_k a_k \cos(2\pi f_k t + \theta_k) \qquad f_k = f_c + k\Delta f \qquad (3.2)$$

- Now, let $f_k = (f_k - f_c) + f_c$, and using $\cos(A + B) = \cos A \cos B - \sin A \sin B$ we obtain the **canonical form of bandpass noise**

$$\boxed{n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)}$$

where

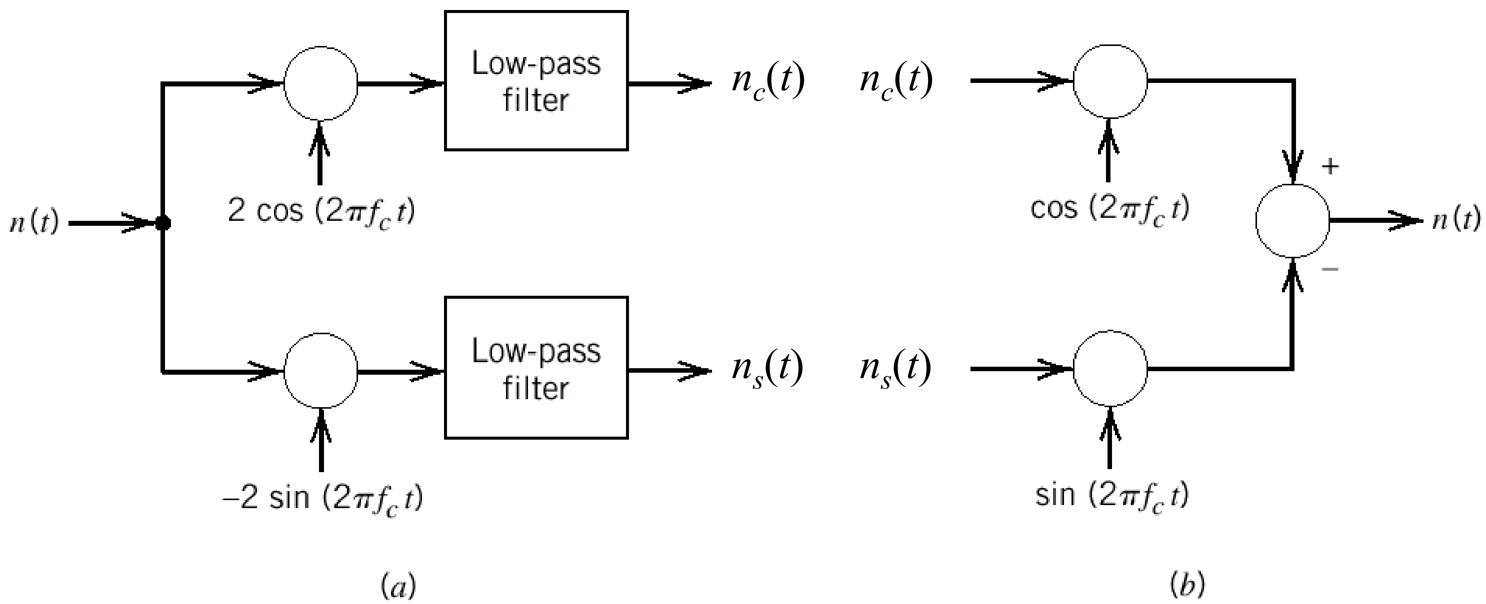
$$n_c(t) = \sum_k a_k \cos(2\pi(f_k - f_c)t + \theta_k)$$

$$n_s(t) = \sum_k a_k \sin(2\pi(f_k - f_c)t + \theta_k) \qquad (3.3)$$

  – $n_c(t)$ and $n_s(t)$ are ***baseband*** signals, termed the in-phase and quadrature component, respectively.

Extraction and Generation

- $n_c(t)$ and $n_s(t)$ are fully representative of bandpass noise.
  - (a) Given bandpass noise, one may extract its in-phase and quadrature components (using LPF of bandwith B). This is extremely useful in analysis of noise in communication receivers.
  - (b) Given the two components, one may generate bandpass noise. This is useful in computer simulation.



Properties of Lowpass Noise

- If the noise $n(t)$ has zero mean, then $n_c(t)$ and $n_s(t)$ have zero mean.
- If the noise $n(t)$ is Gaussian, then $n_c(t)$ and $n_s(t)$ are Gaussian.
- If the noise $n(t)$ is stationary, then $n_c(t)$ and $n_s(t)$ are stationary.
- If the noise $n(t)$ is Gaussian and its power spectral density $S(f)$ is symmetric with respect to the central frequency f_c, then $n_c(t)$ and $n_s(t)$ are statistical independent.
- The components $n_c(t)$ and $n_s(t)$ have the same variance (= power) as $n(t)$.

Power Spectral Density

- Further, each baseband noise waveform will have the same PSD:

$$\boxed{S_c(f) = S_s(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & |f| \leq B \\ 0, & \text{otherwise} \end{cases}} \tag{3.4}$$

- This is analogous to

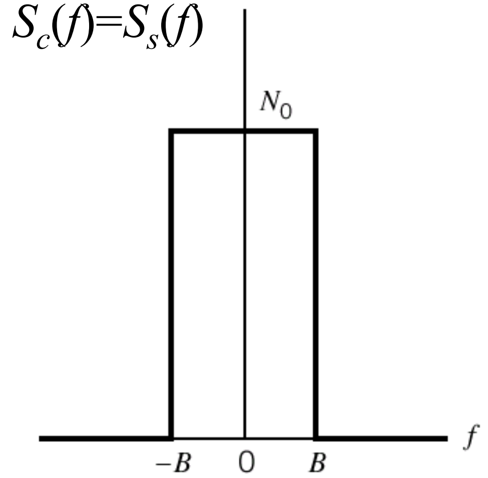
$$\boxed{\begin{aligned} g(t) &\Leftrightarrow G(\omega) \\ g(t) \bullet 2\cos\omega_0 t &\Leftrightarrow [G(\omega - \omega_0) + G(\omega + \omega_0)] \end{aligned}}$$

  - A rigorous proof can be found in A. Papoulis, *Probability, Random Variables, and Stochastic Processes,* McGraw-Hill.
  - The PSD can also be seen from the expressions (3.2) and (3.3) where each of $n_c(t)$ and $n_s(t)$ consists of a sum of closely spaced base-band sinusoids.

Noise Power

- For **ideally filtered narrowband noise**, the PSD of $n_c(t)$ and $n_s(t)$ is therefore given by

$$\boxed{S_c(f) = S_s(f) = \begin{cases} N_0, & |f| \le B \\ 0, & \text{otherwise} \end{cases}} \qquad (3.5)$$



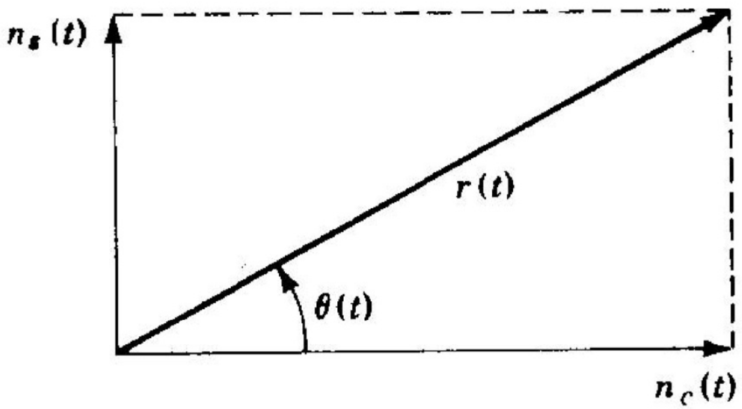
- Corollary: The average power in *each* of the baseband waveforms $n_c(t)$ and $n_s(t)$ is ***identical*** to the average power in the bandpass noise waveform $n(t)$.
- For ideally filtered narrowband noise, the variance of $n_c(t)$ and $n_s(t)$ is $2N_0B$ each.

Phasor Representation

- We may write bandpass noise in the alternative form:

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$
$$= r(t)\cos[2\pi f_c t + \phi(t)]$$

  - $r(t) = \sqrt{n_c(t)^2 + n_s(t)^2}$: the envelop of the noise
  - $\phi(t) = \tan^{-1}\left(\frac{n_s(t)}{n_c(t)}\right)$: the phase of the noise



$$\theta(t) \equiv 2\pi f_c t + \phi(t)$$

Distribution of Envelop and Phase

- It can be shown that if $n_c(t)$ and $n_s(t)$ are Gaussian-distributed, then the magnitude $r(t)$ has a **Rayleigh** distribution, and the phase $\phi(t)$ is **uniformly** distributed.

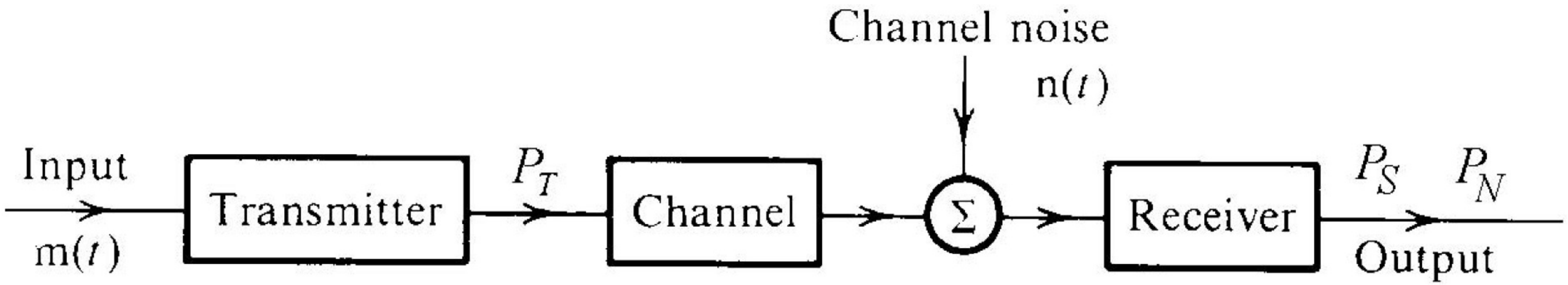
- What if a sinusoid $A\cos(2\pi f_c t)$ is mixed with noise?
- Then the magnitude will have a **Rice** distribution.

Summary

- White noise: PSD is constant over an infinite bandwidth.
- Gaussian noise: PDF is Gaussian.
- Bandpass noise
  - In-phase and quadrature compoments $n_c(t)$ and $n_s(t)$ are low-pass random processes.
  - $n_c(t)$ and $n_s(t)$ have the same PSD.
  - $n_c(t)$ and $n_s(t)$ have the same variance as the band-pass noise $n(t)$.
  - Such properties will be pivotal to the performance analysis of bandpass communication systems.
- The in-phase/quadrature representation and phasor representation are not only basic to the characterization of bandpass noise itself, but also to the analysis of bandpass communication systems.

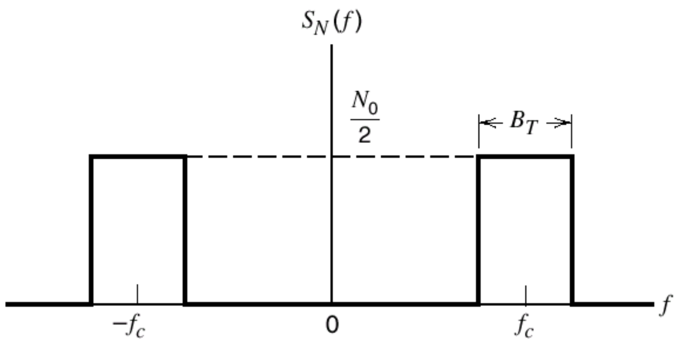
Noise in Analog Communication Systems

- How do various analog modulation schemes perform in the presence of noise?
- Which scheme performs best?
- How can we measure its performance?



Model of an analog communication system

Noise PSD: B_T is the bandwidth, $N_0/2$ is the double-sided noise PSD



SNR

- We must find a way to quantify (= to measure) the performance of a modulation scheme.
- We use the signal-to-noise ratio (SNR) at the output of the receiver:

$$SNR_{\mathrm{o}} \equiv \frac{\text{average power of message signal at the receiver output}}{\text{average power of noise at the receiver output}} = \frac{P_S}{P_N}$$

  - Normally expressed in decibels (dB)
  - SNR (dB) = 10 $\log_{10}$(SNR)
  - This is to manage the wide range of power levels in communication systems
  - In honour of Alexander Bell
  - Example:
    - ratio of 2 → 3 dB; 4 → 6 dB; 10 → 10dB

dB
If x is power,
X (dB) = 10 $\log_{10}$(x)

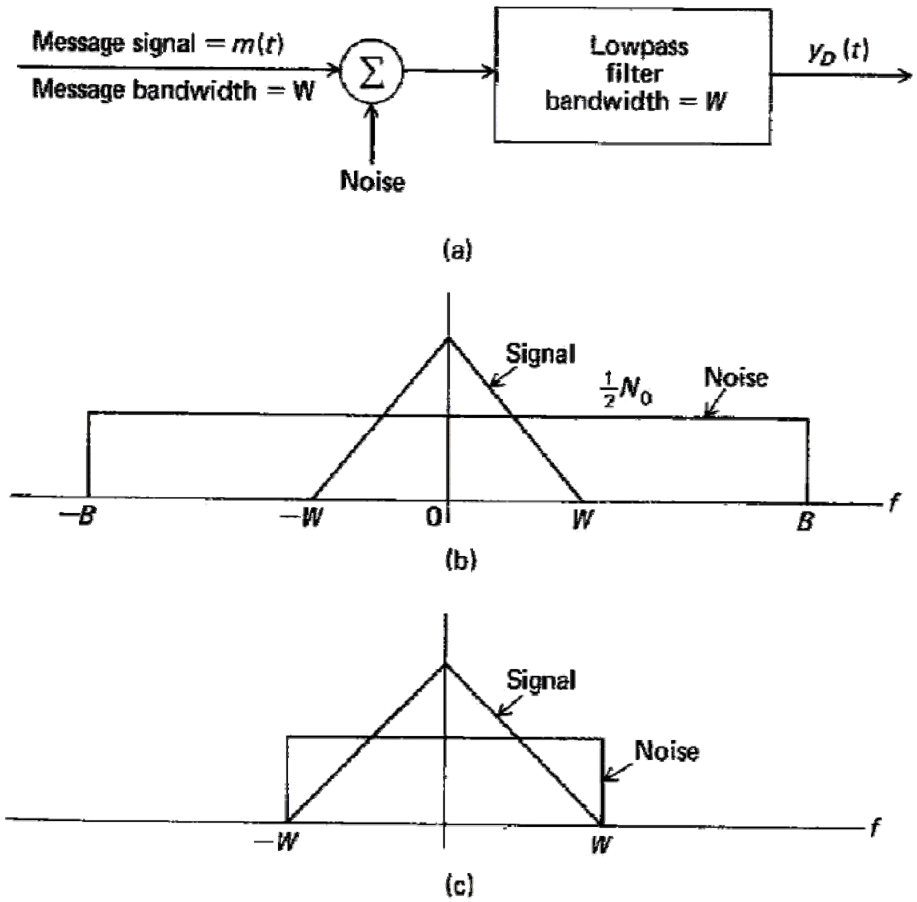
If x is amplitude,
X (dB) = 20 $\log_{10}$(x)

Transmitted Power

- P_T: The transmitted power
- Limited by: equipment capability, battery life, cost, government restrictions, interference with other channels, green communications etc
- The higher it is, the more the received power (P_S), the higher the SNR
- For a fair comparison between different modulation schemes:
  - P_T should be the same for all
- We use the **baseband** signal to noise ratio SNR_{baseband} to calibrate the *SNR* values we obtain

A Baseband Communication System

- It does not use modulation
- It is suitable for transmission over wires
- The power it transmits is identical to the message power: $P_T = P$
- No attenuation: $P_S = P_T = P$
- The results can be extended to band-pass systems



Output SNR

- Average signal (= message) power
  P = the area under the triangular curve
- Assume: Additive, white noise with power spectral density PSD = $N_0/2$
- Average noise power at the receiver
  P_N = area under the straight line = $2W \times N_0/2 = WN_0$
- SNR at the receiver output:

$$\boxed{SNR_{\text{baseband}} = \frac{P_T}{N_0 W}}$$

  – Note: Assume no propagation loss
- Improve the SNR by:
  – increasing the transmitted power ($P_T \uparrow$),
  – restricting the message bandwidth ($W \downarrow$),
  – making the channel/receiver less noisy ($N_0 \downarrow$).

Revision: AM

- General form of an AM signal:

$$s(t)_{AM} = [A + m(t)]\cos(2\pi f_c t)$$

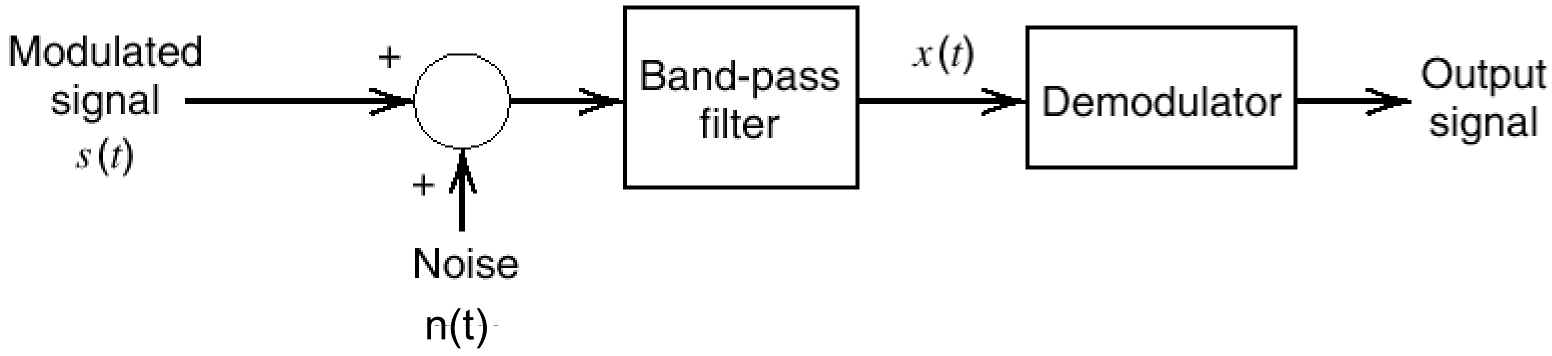
  - A: the amplitude of the carrier
  - f_c: the carrier frequency
  - $m(t)$: the message signal

- Modulation index:

$$\mu = \frac{m_p}{A}$$

  - m_p: the peak amplitude of $m(t)$, i.e., $m_p = \max |m(t)|$

Signal Recovery



Receiver model

1) $\mu \leq 1 \Rightarrow A \geq m_p$: use an envelope detector.
   This is the case in almost all commercial AM radio receivers.
   Simple circuit to make radio receivers cheap.
2) Otherwise: use synchronous detection = product detection = coherent detection

The terms detection and demodulation are used interchangeably.

Synchronous Detection

- Multiply the waveform at the receiver with a local carrier of the same frequency (and phase) as the carrier used at the transmitter:

$$\begin{aligned} 2\cos(2\pi f_c t)s(t)_{AM} &= [A+m(t)]2\cos^2(2\pi f_c t) \\ &= [A+m(t)][1+\cos(4\pi f_c t)] \\ &= A+m(t)+\cdots \end{aligned}$$

- Use a LPF to recover $A+m(t)$ and finally $m(t)$
- Remark: At the receiver you need a signal perfectly synchronized with the transmitted carrier

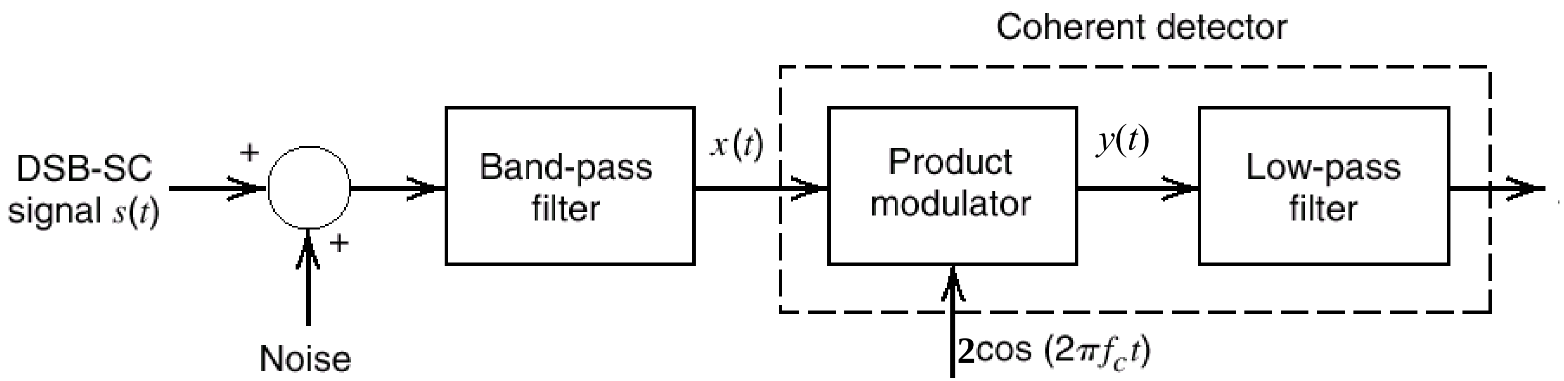
DSB-SC

- Double-sideband suppressed carrier (DSB-SC)

$$s(t)_{DSB-SC} = Am(t)\cos(2\pi f_c t)$$

- Signal recovery: with synchronous detection only
- The received noisy signal is

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= s(t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \\ &= Am(t)\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \\ &= [Am(t) + n_c(t)]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \end{aligned}$$



Synchronous Detection

- Multiply with $2\cos(2f_c t)$:

$$y(t) = 2\cos(2\pi f_c t)x(t)$$
$$= Am(t)2\cos^2(2\pi f_c t) + n_c(t)2\cos^2(2\pi f_c t) - n_s(t)\sin(4\pi f_c t)$$
$$= Am(t)[1+\cos(4\pi f_c t)] + n_c(t)[1+\cos(4\pi f_c t)] - n_s(t)\sin(4\pi f_c t)$$

- Use a LPF to keep

$$\tilde{y} = Am(t) + n_c(t)$$

- Signal power at the receiver output:

$$P_S = E\{A^2 m^2(t)\} = A^2 E\{m^2(t)\} = A^2 P$$

- Power of the noise $n_c(t)$ (recall (3.5)):

$$P_N = \int_{-W}^{W} N_0 df = 2N_0 W$$

Comparison

- *SNR* at the receiver output:

$$\boxed{SNR_o = \frac{A^2 P}{2 N_0 W}}$$

- To which **transmitted** power does this correspond?

$$P_T = E\{A^2 m(t)^2 \cos^2(2\pi f_c t)\} = \frac{A^2 P}{2}$$

- So

$$SNR_o = \frac{P_T}{N_0 W} = SNR_{DSB-SC}$$

- Comparison with

$$SNR_{baseband} = \frac{P_T}{N_0 W} \quad \Rightarrow \quad SNR_{DSB-SC} = SNR_{baseband}$$

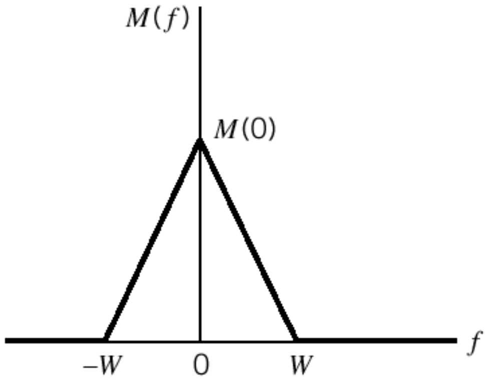
- **Conclusion:** DSB-SC system has the same SNR performance as a baseband system.

SSB Modulation

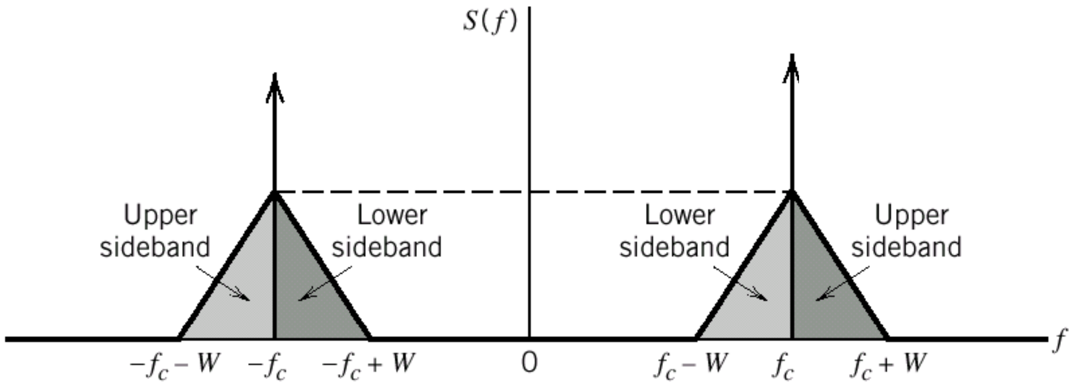
- Consider single (lower) sideband AM:

$$s(t)_{SSB} = \frac{A}{2} m(t) \cos 2\pi f_c t + \frac{A}{2} \hat{m}(t) \sin 2\pi f_c t$$

  where $\hat{m}(t)$ is the Hilbert transform of $m(t)$.
- $\hat{m}(t)$ is obtained by passing $m(t)$ through a linear filter with transfer function $-j\mathrm{sgn}(f)$.
- $\hat{m}(t)$ and $m(t)$ have the same power P.
- The average power is $A^2P/4$.



(a)



(b)

Noise in SSB

- Receiver signal $x(t) = s(t) + n(t)$.
- Apply a band-pass filter on the lower sideband.
- Still denote by $n_c(t)$ the lower-sideband noise (different from the double-sideband noise in DSB).
- Using coherent detection:

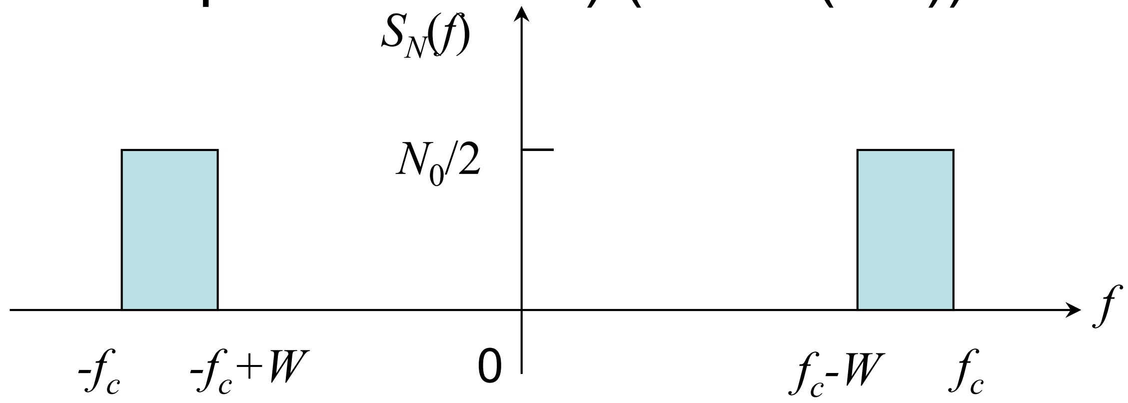
$$
\begin{aligned}
y(t) &= x(t) \times 2\cos(2\pi f_c t) \\
&= \left(\frac{A}{2} m(t) + n_c(t)\right) + \left(\frac{A}{2} m(t) + n_c(t)\right)\cos(4\pi f_c t) \\
&\quad + \left(\frac{A}{2} \hat{m}(t) - n_s(t)\right)\sin(4\pi f_c t)
\end{aligned}
$$

- After low-pass filtering,

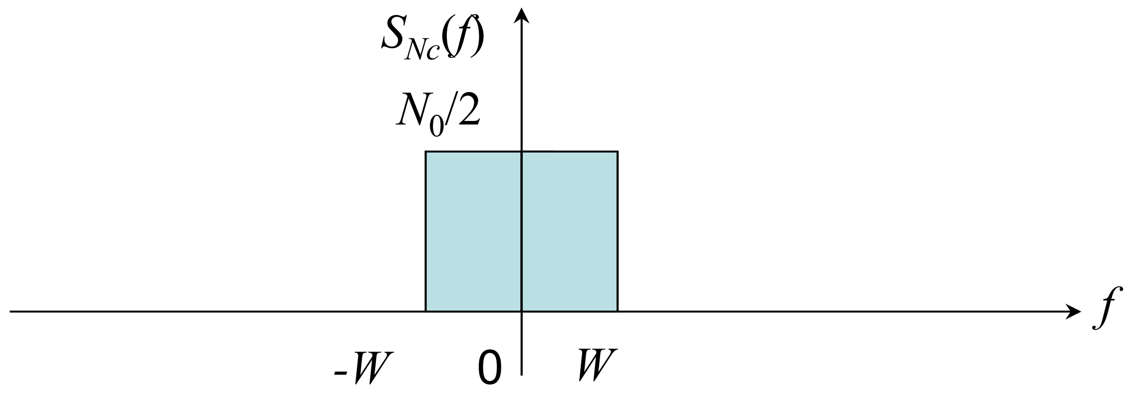
$$
y(t) = \left(\frac{A}{2} m(t) + n_c(t)\right)
$$

Noise Power

- Noise power for $n_c(t)$ = that for band-pass noise = N_0W (halved compared to DSB) (recall (3.4))



Lower-sideband noise



Baseband noise

Output SNR

- Signal power $A^2P/4$

- SNR at output

$$\boxed{SNR_{SSB} = \frac{A^2 P}{4N_0 W}}$$

- For a baseband system with the *same* transmitted power $A^2P/4$

$$SNR_{baseband} = \frac{A^2 P}{4N_0 W}$$

- **Conclusion:** SSB achieves the same SNR performance as DSB-SC (and the baseband model) but only requires half the band-width.

Standard AM: Synchronous Detection

- Pre-detection signal:

$$\begin{aligned} x(t) &= [A + m(t)]\cos(2\pi f_c t) + n(t) \\ &= [A + m(t)]\cos(2\pi f_c t) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \\ &= [A + m(t) + n_c(t)]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \end{aligned}$$

- Multiply with $2\cos(2\pi f_c t)$:

$$\begin{aligned} y(t) = [A + m(t) + n_c(t)][1 + \cos(4\pi f_c t)] \\ - n_s(t)\sin(4\pi f_c t) \end{aligned}$$

- LPF

$$\tilde{y} = A + m(t) + n_c(t)$$

Output SNR

- Signal power at the receiver output:

$$P_S = E\{m^2(t)\} = P$$

- Noise power:

$$P_N = 2N_0W$$

- SNR at the receiver output:

$$\boxed{SNR_o = \frac{P}{2N_0W} = SNR_{AM}}$$

- Transmitted power

$$P_T = \frac{A^2}{2} + \frac{P}{2} = \frac{A^2 + P}{2}$$

Comparison

- SNR of a baseband signal with the same transmitted power:

$$SNR_{baseband} = \frac{A^2 + P}{2N_0 W}$$

- Thus:

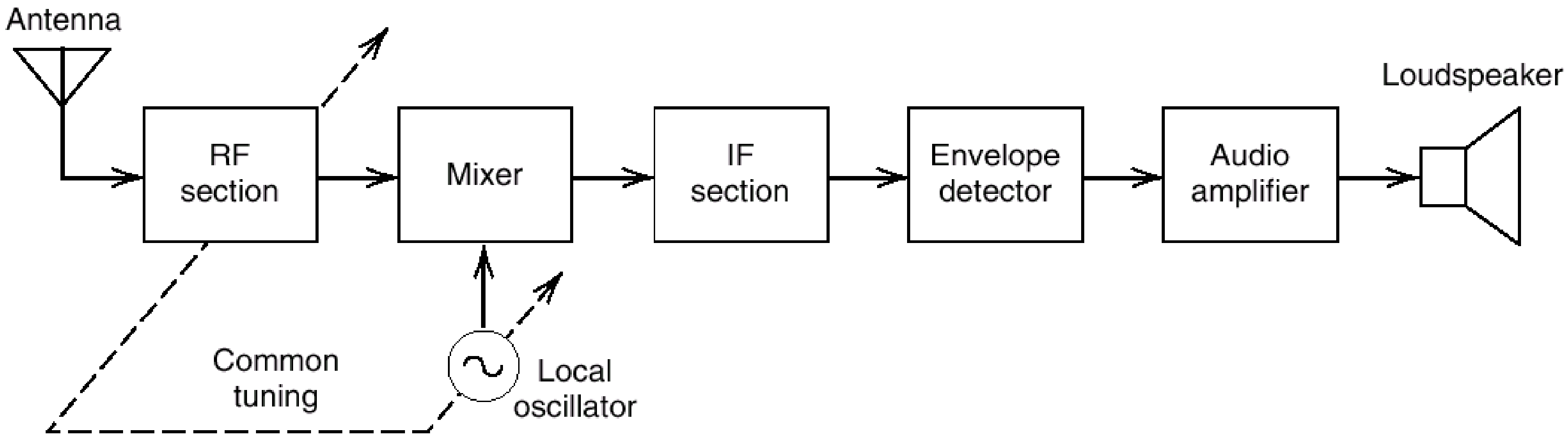
$$SNR_{AM} = \frac{P}{A^2 + P} SNR_{baseband}$$

- Note:

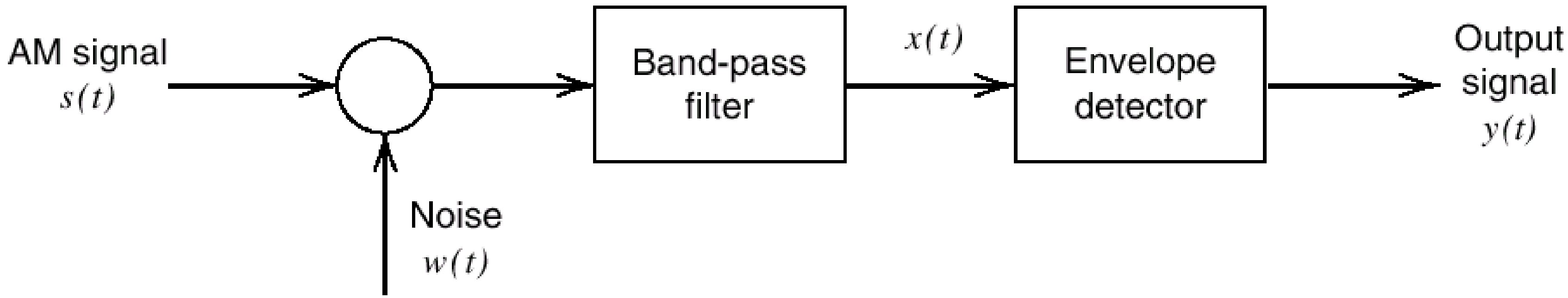
$$\frac{P}{A^2 + P} < 1$$

- **Conclusion:** the performance of standard AM with synchronous recovery is worse than that of a baseband system.

Model of AM Radio Receiver



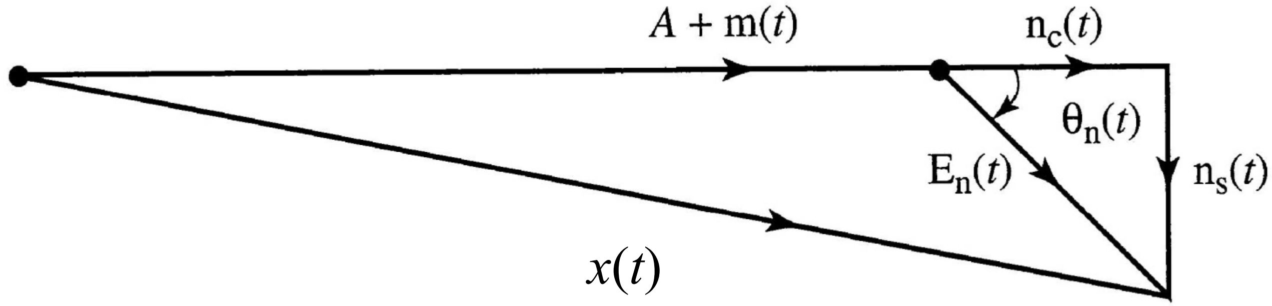
AM radio receiver of the superheterodyne type



Model of AM envelope detector

Envelope Detection for Standard AM

- Phasor diagram of the signals present at an AM receiver



- Envelope

$$y(t) = \text{envelope of } x(t)$$
$$= \sqrt{[A + m(t) + n_c(t)]^2 + n_s(t)^2}$$

- Equation is too complicated
- Must use limiting cases to put it in a form where noise and message are added

Small Noise Case

- **1st Approximation: (a)** **Small Noise Case**

$$n(t) << [A + m(t)]$$

- Then

$$n_s(t) << [A + m(t) + n_c(t)]$$

- Then

$$y(t) \approx [A + m(t) + n_c(t)]$$

Identical to the post-detection signal in the case of synchronous detection!

- Thus

$$SNR_o = \frac{P}{2N_0W} \approx SNR_{env}$$

- And in terms of baseband SNR:

$$SNR_{env} \approx \frac{P}{A^2 + P} SNR_{baseband}$$

- **Valid for small noise only!**

Large Noise Case

- **2nd Approximation: (b)** **Large Noise Case**

$$n(t) >> [A + m(t)]$$

- Isolate the small quantity:

$$y^2(t) = [A + m(t) + n_c(t)]^2 + n_s^2(t)$$
$$= (A + m(t))^2 + n_c^2(t) + 2(A + m(t))n_c(t) + n_s^2(t)$$
$$= [n_c^2(t) + n_s^2(t)]\left\{1 + \frac{(A + m(t))^2}{n_c^2(t) + n_s^2(t)^2} + \frac{2(A + m(t))n_c(t)}{n_c^2(t) + n_s^2(t)}\right\}$$

$$y^2(t) \approx [n_c^2(t) + n_s^2(t)]\left(1 + \frac{2[A + m(t)]n_c(t)}{n_c^2(t) + n_s^2(t)}\right)$$
$$= E_n^2(t)\left(1 + \frac{2[A + m(t)]n_c(t)}{E_n^2(t)}\right) \qquad E_n(t) \equiv \sqrt{n_c^2(t) + n_s^2(t)}$$

Large Noise Case: Threshold Effect

- From the phasor diagram: $n_c(t) = E_n(t)\cos\theta_n(t)$
- Then:

$$y(t) \approx E_n(t)\sqrt{1+\frac{2[A+m(t)]\cos\theta_n(t)}{E_n(t)}}$$

- Use $\sqrt{1+x} \approx 1+\frac{x}{2}$ for $x << 1$

$$y(t) \approx E_n(t)\left(1+\frac{[A+m(t)]\cos\theta_n(t)}{E_n(t)}\right)$$

$$= E_n(t) + [A+m(t)]\cos\theta_n(t)$$

- Noise is multiplicative here!
- No term proportional to the message!
- Result: a **threshold effect**, as below some carrier power level (very low *A*), the performance of the detector deteriorates very rapidly.

Summary

(De-) Modulation Format	Output SNR	Transmitted Power	Baseband Reference SNR	Figure of Merit (= Output SNR / Reference SNR)
AM Coherent Detection	$\frac{P}{2N_0W}$	$\frac{A^2+P}{2}$	$\frac{A^2+P}{2N_0W}$	$\frac{P}{A^2+P}<1$
DSB-SC Coherent Detection	$\frac{A^2P}{2N_0W}$	$\frac{A^2P}{2}$	$\frac{A^2P}{2N_0W}$	1
SSB Coherent Detection	$\frac{A^2P}{4N_0W}$	$\frac{A^2P}{4}$	$\frac{A^2P}{4N_0W}$	1
AM Envelope Detection (Small Noise)	$\frac{P}{2N_0W}$	$\frac{A^2+P}{2}$	$\frac{A^2+P}{2N_0W}$	$\frac{P}{A^2+P}<1$
AM Envelope Detection (Large Noise)	Poor	$\frac{A^2+P}{2}$	$\frac{A^2+P}{2N_0W}$	Poor

A: carrier amplitude, P: power of message signal, N_0: single-sided PSD of noise, W: message bandwidth.

Frequency Modulation

- Fundamental difference between AM and FM:

- AM: message information contained in the signal **amplitude** ⇒ Additive noise: corrupts directly the modulated signal.

- FM: message information contained in the signal **frequency** ⇒ the effect of noise on an FM signal is determined by the extent to which it changes the frequency of the modulated signal.

- Consequently, FM signals is less affected by noise than AM signals

Revision: FM

- A carrier waveform

$$s(t) = A\cos[\theta_i(t)]$$

  – where $\theta_i(t)$: the **instantaneous phase angle**.

- When

$$s(t) = A\cos(2\pi f\,t) \Rightarrow \theta_i(t) = 2\pi f\,t$$

  we may say that

$$\frac{d\theta}{dt} = 2\pi f \Rightarrow f = \frac{1}{2\pi}\frac{d\theta}{dt}$$

- Generalisation: **instantaneous frequency**:

$$f_i(t) \triangleq \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

FM

- In FM: the instantaneous frequency of the carrier varies linearly with the message:

$$f_i(t) = f_c + k_f m(t)$$

  - where k_f is the **frequency sensitivity** of the modulator.
- Hence (assuming $\theta_i(0)=0$):

$$\theta_i(t) = 2\pi \int_0^t f_i(\tau) d\tau = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau$$

- Modulated signal:

$$s(t) = A \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]$$

- Note:
  - (a) The envelope is constant
  - (b) Signal *s*(t) is a non-linear function of the message signal *m*(t).

Bandwidth of FM

- $m_p = \max|m(t)|$: peak message amplitude.
- $f_c - k_f m_p$ < instantaneous frequency < $f_c + k_f m_p$
- Define: **frequency deviation** = the deviation of the instantaneous frequency from the carrier frequency:

$$\Delta f = k_f m_p$$

- Define: **deviation ratio:**

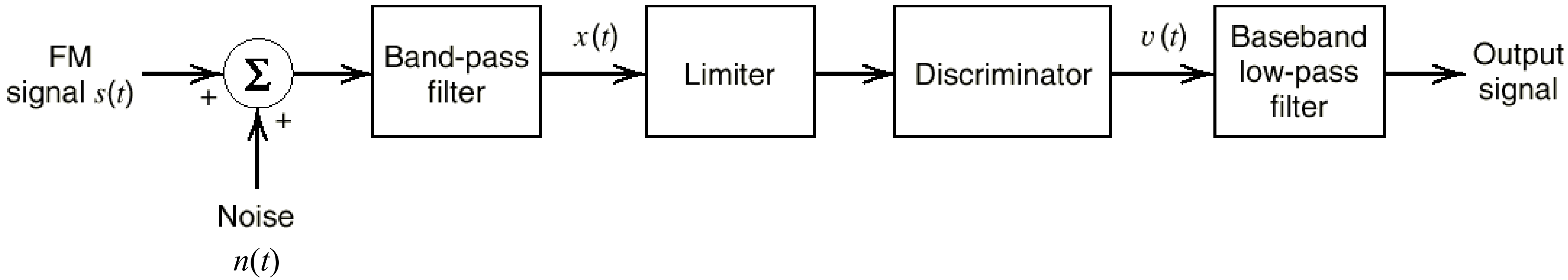
$$\beta = \Delta f / W$$

  - *W*: the message bandwidth.
  - Small β: FM bandwidth ≈ 2x message bandwidth (**narrow-band FM**)
  - Large β: FM bandwidth >> 2x message bandwidth (**wide-band FM**)
- **Carson's rule of thumb**:

$$B_T = 2W(\beta+1) = 2(\Delta f + W)$$

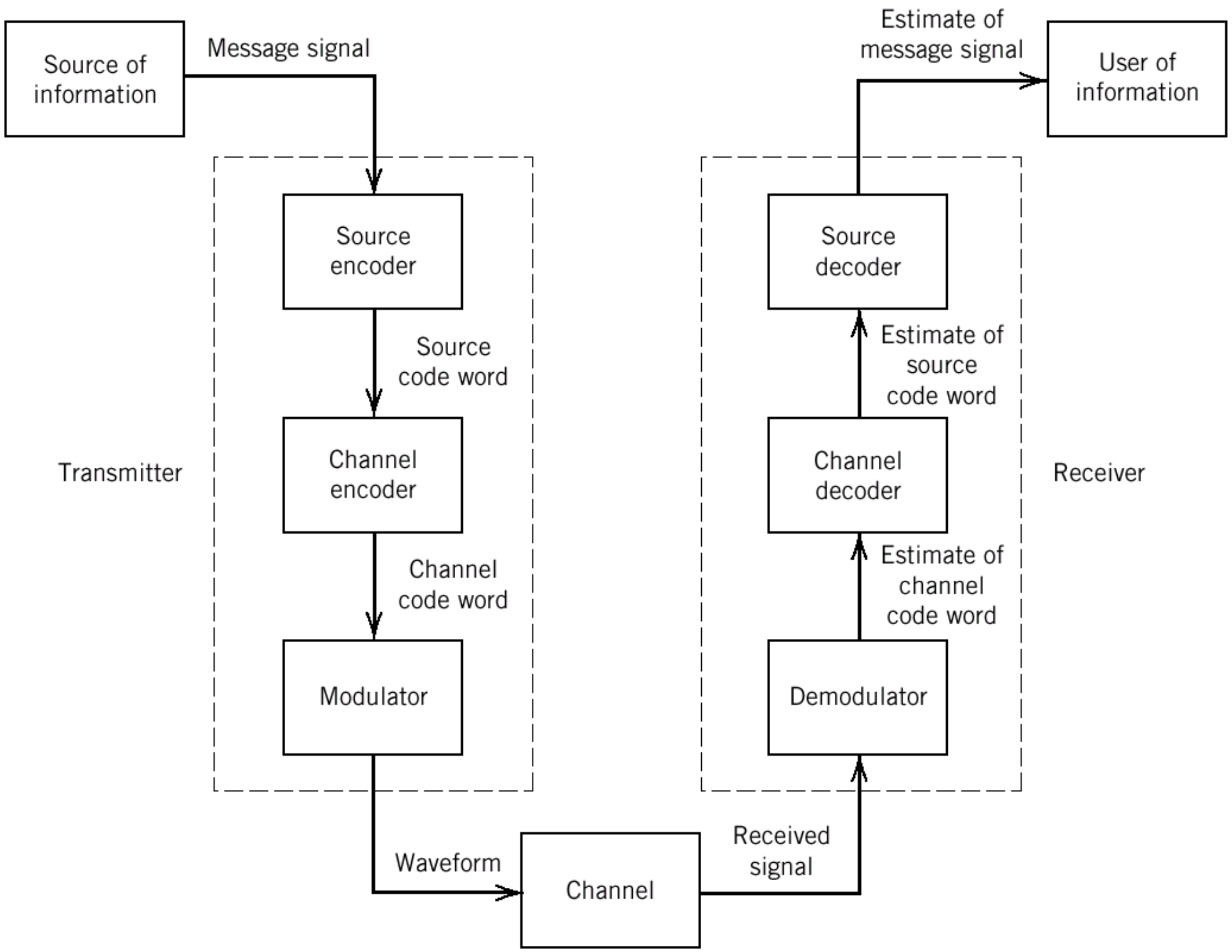
  - $\beta << 1 \Rightarrow B_T \approx 2W$ (as in AM)
  - $\beta >> 1 \Rightarrow B_T \approx 2\Delta f$

FM Receiver



- **Bandpass filter**: removes any signals outside the bandwidth of $fc \pm B_T/2 \Rightarrow$ the predetection noise at the receiver is bandpass with a bandwidth of B_T.
- FM signal has a constant envelope $\Rightarrow$ use a **limiter** to remove any amplitude variations
- **Discriminator**: a device with output proportional to the deviation in the instantaneous frequency $\Rightarrow$ it recovers the message signal
- **Final baseband low-pass filter**: has a bandwidth of $W \Rightarrow$ it passes the message signal and removes out-of-band noise.

Block Diagram of Digital Communication

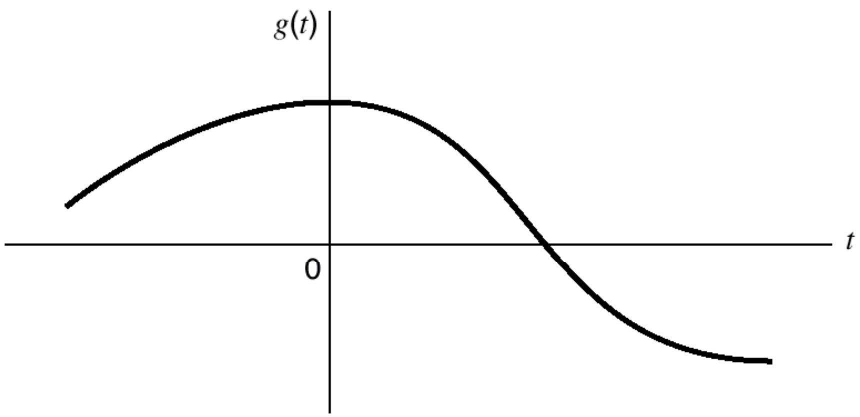


Why Digital?

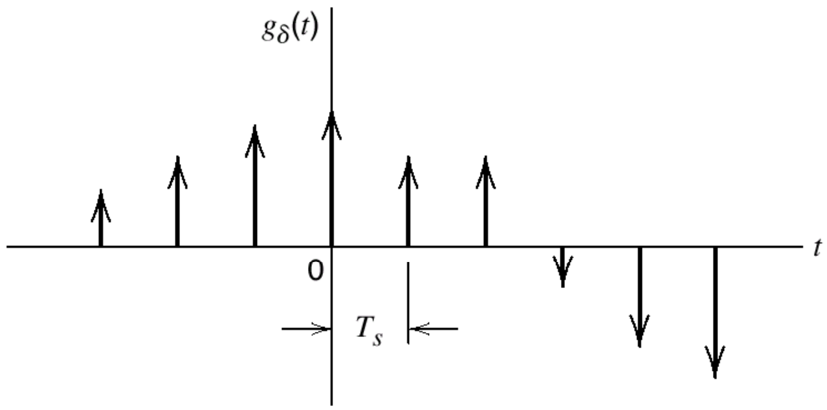
- Advantages:
  - Digital signals are more immune to channel noise by using channel coding (perfect decoding is possible!)
  - Repeaters along the transmission path can detect a digital signal and retransmit a new noise-free signal
  - Digital signals derived from all types of analog sources can be represented using a uniform format
  - Digital signals are easier to process by using microprocessors and VLSI (e.g., digital signal processors, FPGA)
  - Digital systems are flexible and allow for implementation of sophisticated functions and control
  - More and more things are digital…
- For digital communication: analog signals are converted to digital.

Sampling

- How densely should we sample an analog signal so that we can reproduce its form accurately?
- A signal the spectrum of which is band-limited to W Hz, can be reconstructed exactly from its samples, if they are taken uniformly at a rate of $R \geq 2W$ Hz.
- Nyquist frequency: $f_s = 2W$ Hz



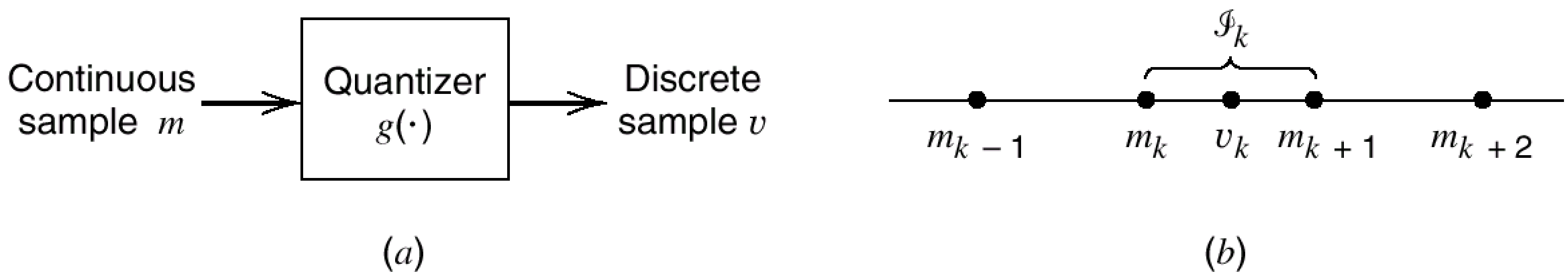
(a)



(b)

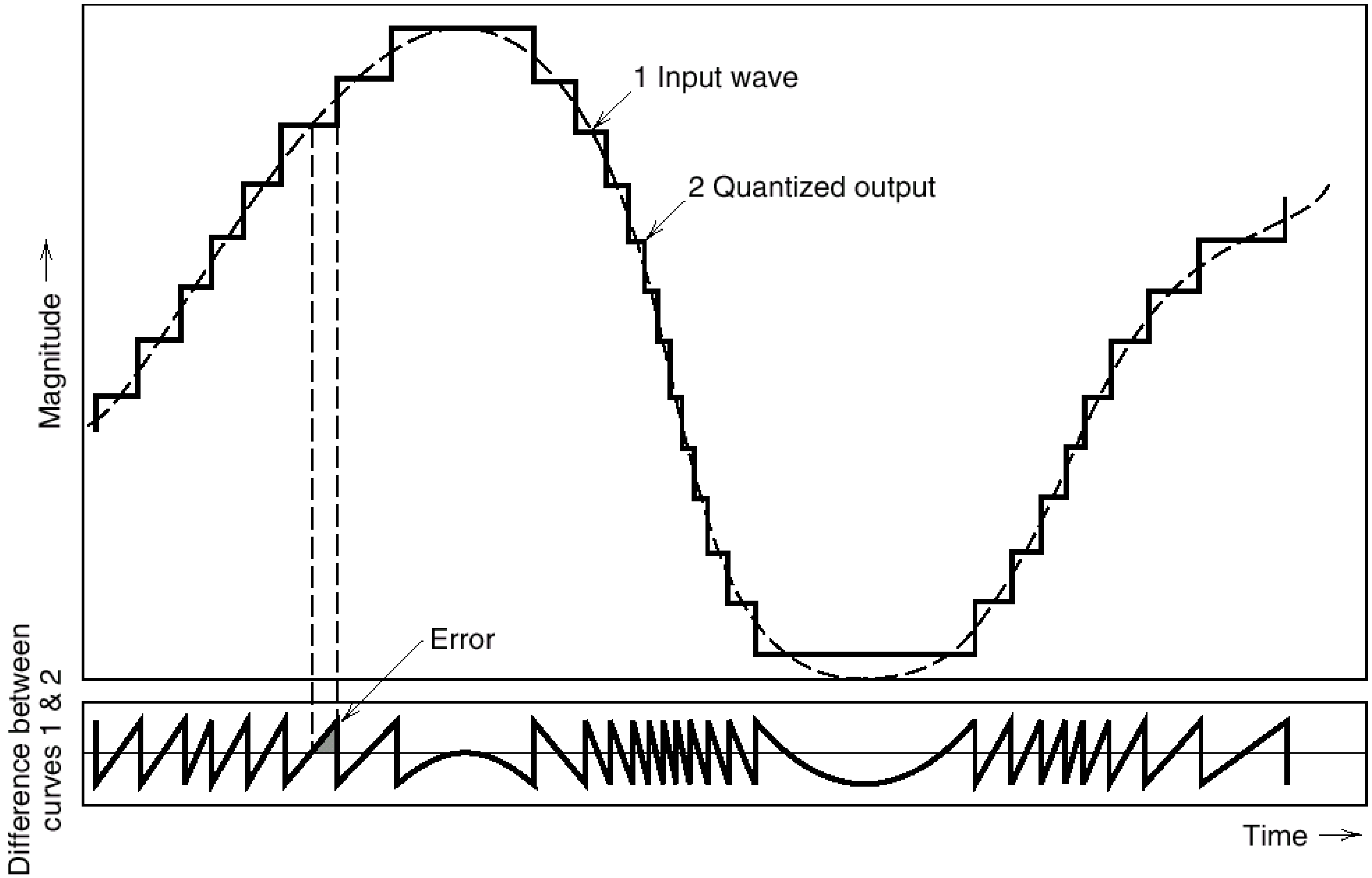
Quantization

- Quantization is the process of transforming the sample amplitude into a discrete amplitude taken from a finite set of possible amplitudes.
- The more levels, the better approximation.
- Don’t need too many levels (human sense can only detect finite differences).
- Quantizers can be of a uniform or nonuniform type.



Quantization Noise

- Quantization noise: the error between the input signal and the output signal



Variance of Quantization Noise

- Δ: gap between quantizing levels (of a uniform quantizer)
- q: Quantization error = a random variable in the range

$$-\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2}$$

- Assume that it is **uniformly distributed** over this range:

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \leq q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

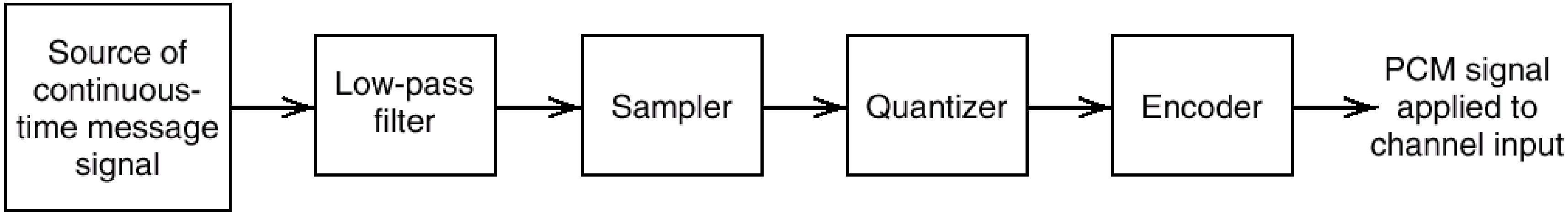
- **Noise variance**

$$P_N = E\{e^2\} = \int_{-\infty}^{\infty} q^2 f_Q(q)dq$$

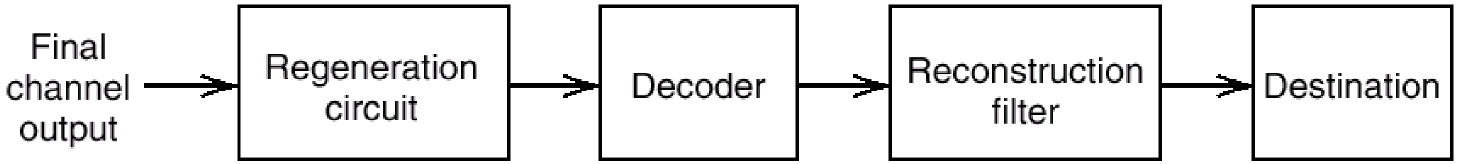
$$= \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} q^2 dq = \frac{1}{\Delta} \bullet \frac{q^3}{3}\bigg|_{-\Delta/2}^{\Delta/2} = \frac{1}{\Delta}\left[\frac{\Delta^3}{24} - \frac{(-\Delta)^3}{24}\right]$$

$$= \frac{\Delta^2}{12}$$

Pulse-Coded Modulation (PCM)



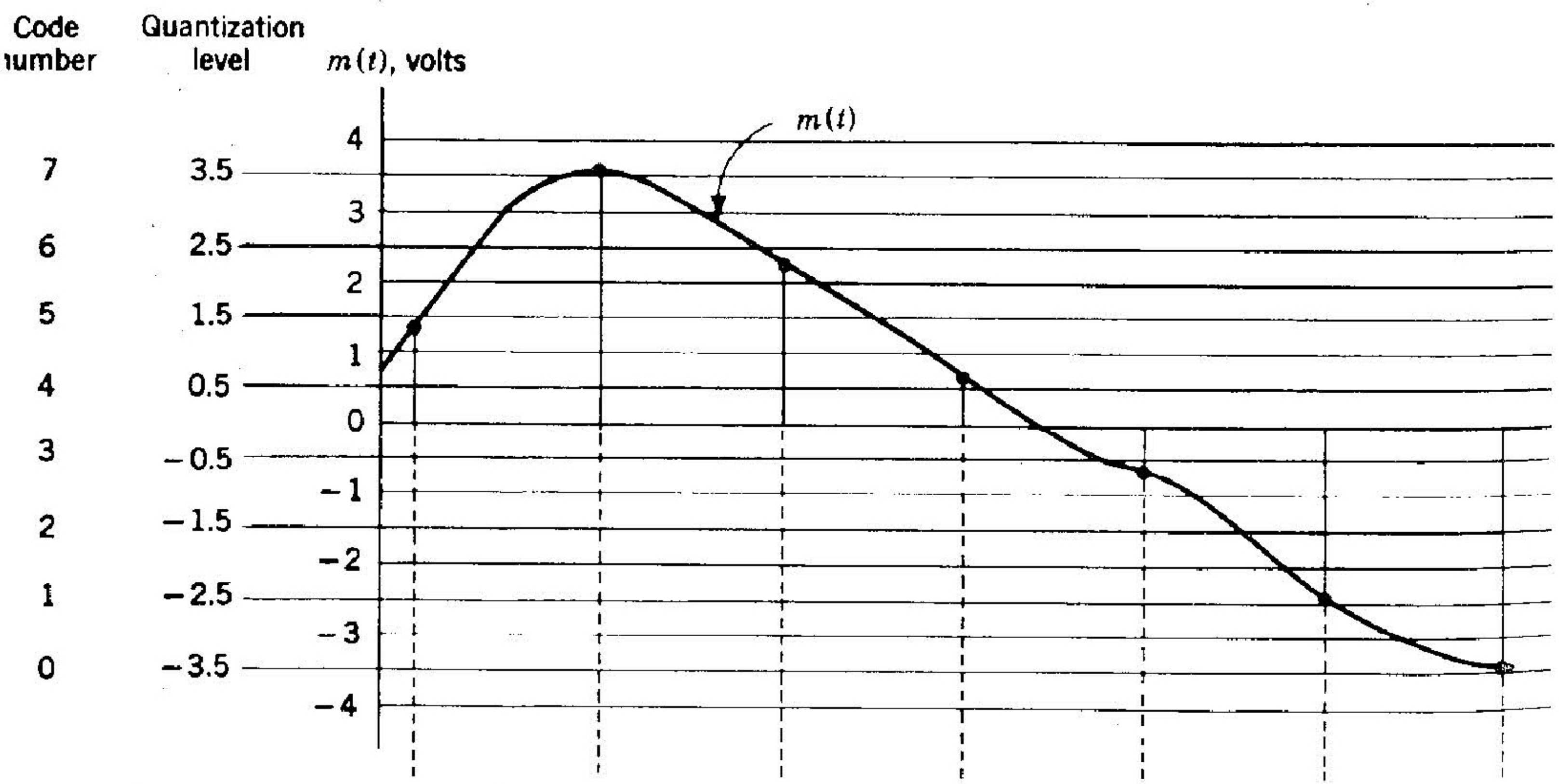
(*a*) Transmitter



(*c*) Receiver

- Sample the message signal above the Nyquist frequency
- Quantize the amplitude of each sample
- Encode the discrete amplitudes into a binary codeword
- *Caution:* PCM isn't modulation in the usual sense; it's a type of Analog-to-Digital Conversion.

The PCM Process



Sample value	1.3	3.6	2.3	0.7	−0.7	−2.4	−3.4
Nearest quantization level	1.5	3.5	2.5	0.5	−0.5	−2.5	−3.5
Code number	5	7	6	4	3	1	0
Binary representation	101	111	110	100	011	001	000

Problem With Uniform Quantization

- Problem: the output SNR is adversely affected by peak to average power ratio.
- Companding is the corresponding to pre-emphasis and de-emphasis scheme used for FM.
- Predistort a message signal in order to achieve better performance in the presence of noise, and then remove the distortion at the receiver.
- Typically small signal amplitudes occur more often than large signal amplitudes.
  - The signal does not use the entire range of quantization levels available with equal probabilities.
  - Small amplitudes are not represented as well as large amplitudes, as they are more susceptible to quantization noise.

Companding

- Solution: ***Nonuniform quantization*** that uses quantization levels of variable spacing, denser at small signal amplitudes, broader at large amplitudes.
- A practical solution to nonuniform quantization:
  - Compress the signal first
  - Quantize it
  - Transmit it
  - Expand it
- **Companding** = **Comp**ressing + Exp**anding**
- The exact SNR gain obtained with companding depends on the exact form of the compression used.
- With proper companding, the output SNR can be made insensitive to peak to average power ratio.

Summary

- Digitization of signals requires
  - Sampling: a signal of bandwidth W is sampled at the Nyquist frequency 2W.
  - Quantization: the link between analog waveforms and digital representation.
    - SNR

$$SNR_o(\text{dB}) = 6n + 10\log_{10}\left(\frac{3P}{m_p^2}\right) \text{ (dB)}$$

    - Companding can improve SNR.
- PCM is a common method of representing audio signals.
  - In a strict sense, “pulse coded modulation” is in fact a (crude) source coding technique (i.e, method of digitally representing analog information).
  - There are more advanced source coding (compression) techniques in information theory.

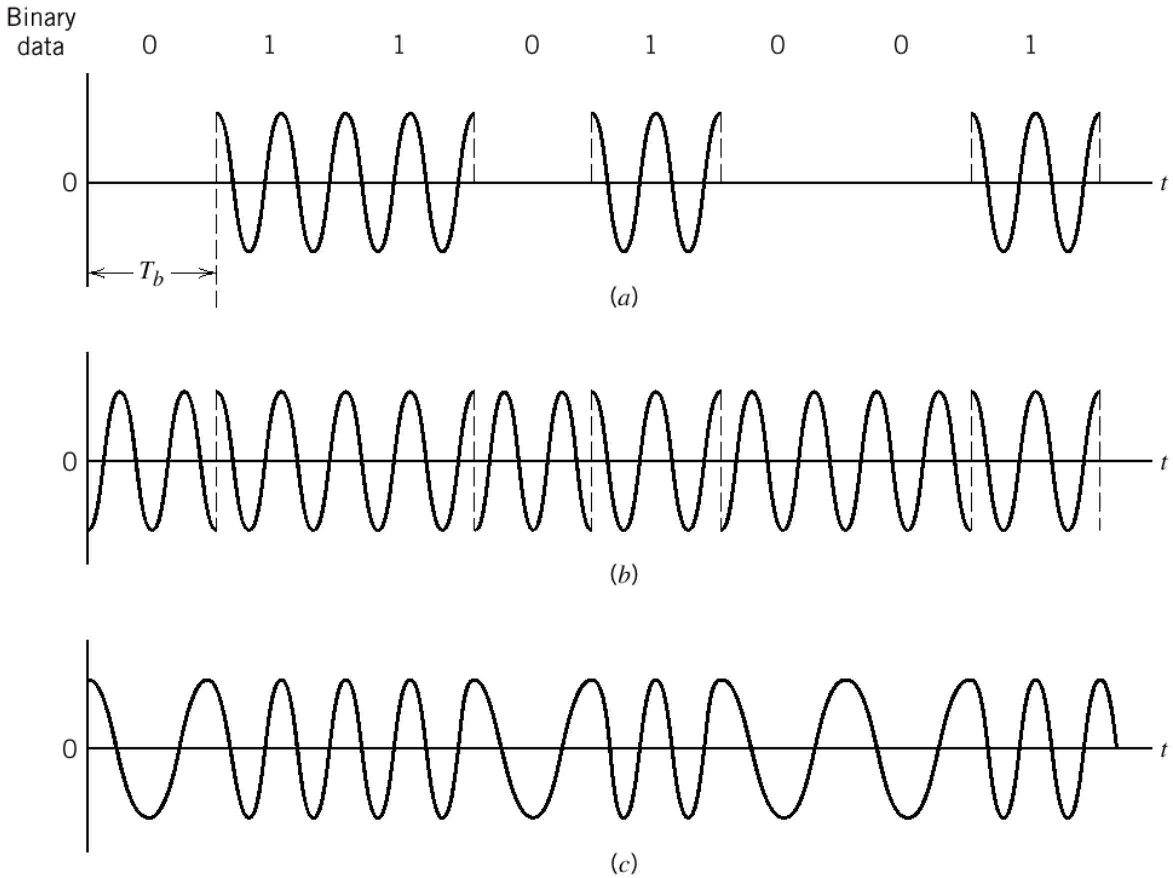
Digital Modulation

- Three Basic Forms of Signaling Binary Information

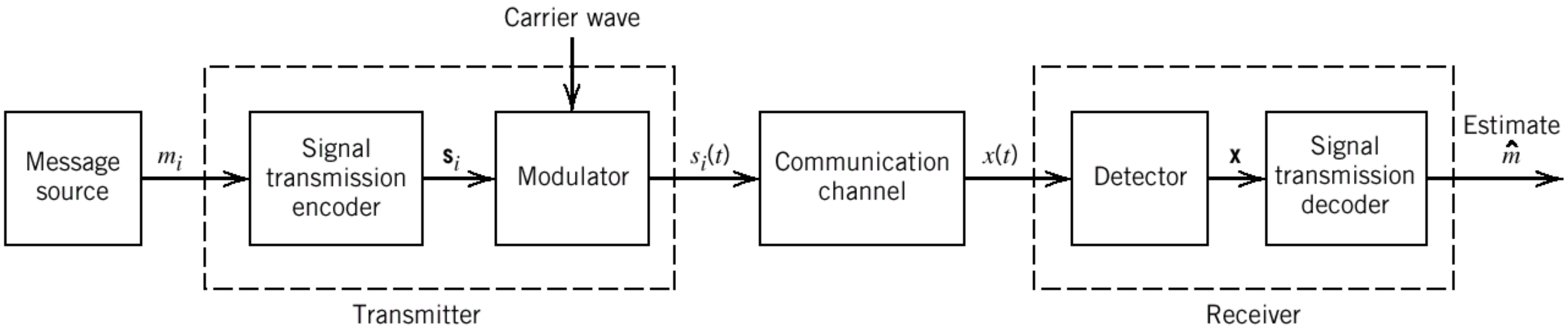
(a) Amplitude-shift keying (ASK).

(b) Phase-shift keying (PSK).

(c) Frequency-shift keying (FSK).



Demodulation



- Coherent (synchronous) demodulation/detection
  - Use a BPF to reject out-of-band noise
  - Multiply the incoming waveform with a cosine of the carrier frequency
  - Use a LPF
  - Requires carrier regeneration (both frequency and phase synchronization by using a phase-lock loop)
- Noncoherent demodulation (envelope detection etc.)
  - Makes no explicit efforts to estimate the phase

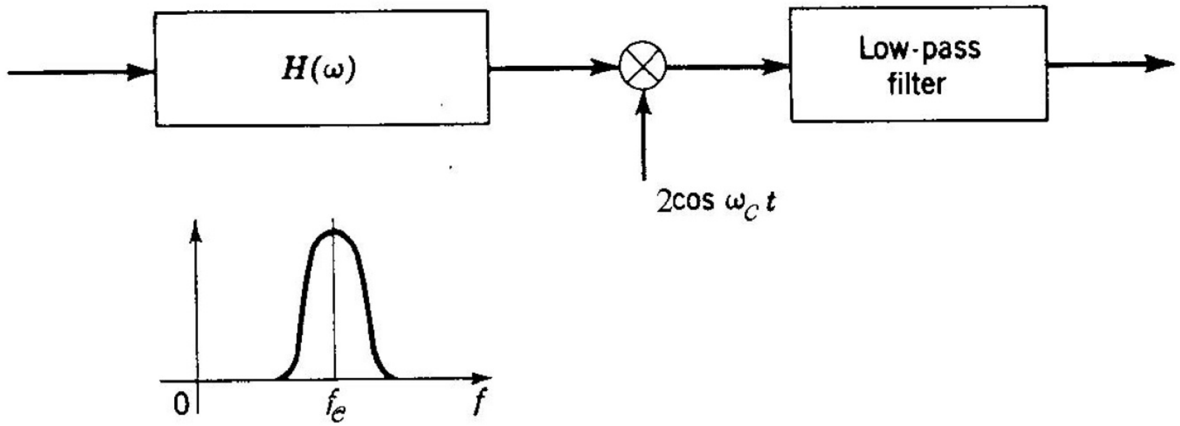
ASK

- Amplitude shift keying (ASK) = on-off keying (OOK)

$$s_0(t) = 0$$
$$s_1(t) = A\cos(2\pi f_c t)$$

or
$$s(t) = A(t)\cos(2\pi f_c t), \qquad A(t) \in \{0, A\}$$

- Coherent detection



- Assume an ideal band-pass filter with unit gain on $[f_c - W, f_c + W]$. For a practical band-pass filter, $2W$ should be interpreted as the equivalent bandwidth.

PSK

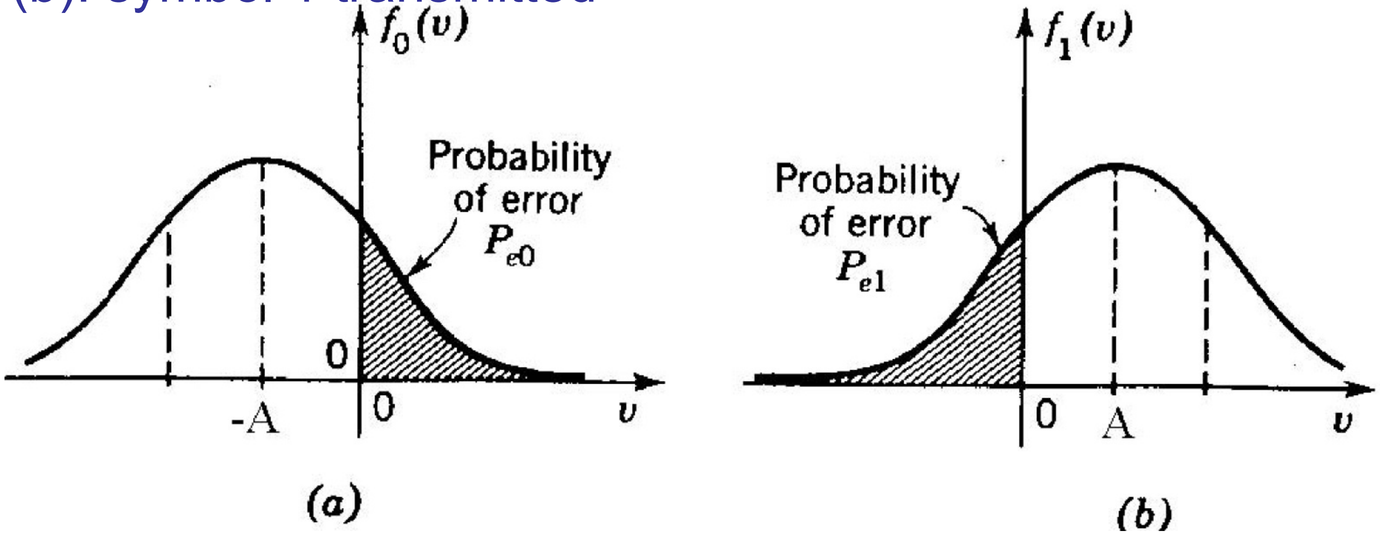
- Phase shift keying (PSK)

$$s(t) = A(t)\cos(2\pi f_c t), \qquad A(t) \in \{-A, A\}$$

- Use coherent detection again, to eventually get the detection signal:

$$\tilde{y}(t) = A(t) + n_c(t)$$

- Probability density functions for PSK for equiprobable 0s and 1s in noise (use threshold 0 for detection):
  - (a): symbol 0 transmitted
  - (b): symbol 1 transmitted



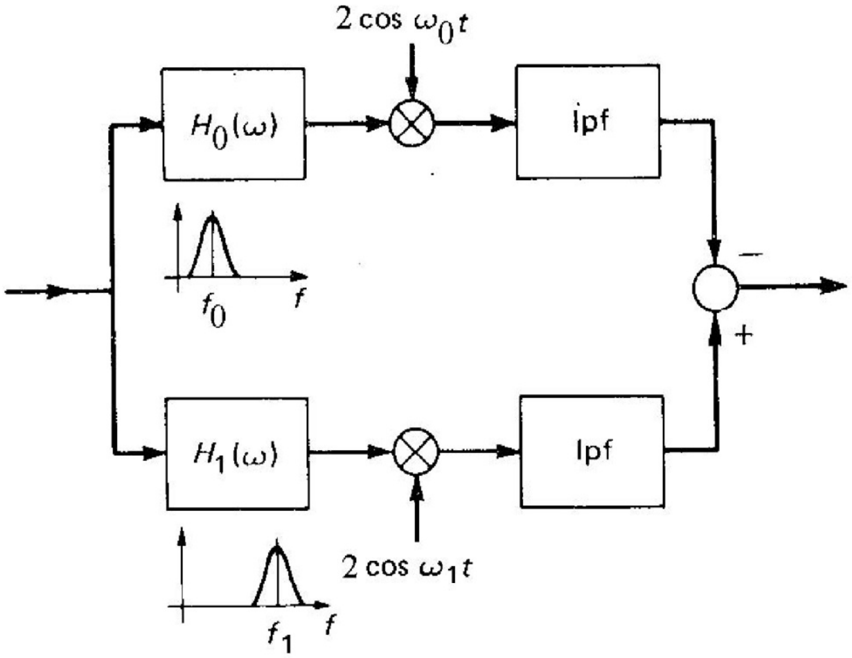
FSK

- Frequency Shift Keying (FSK)

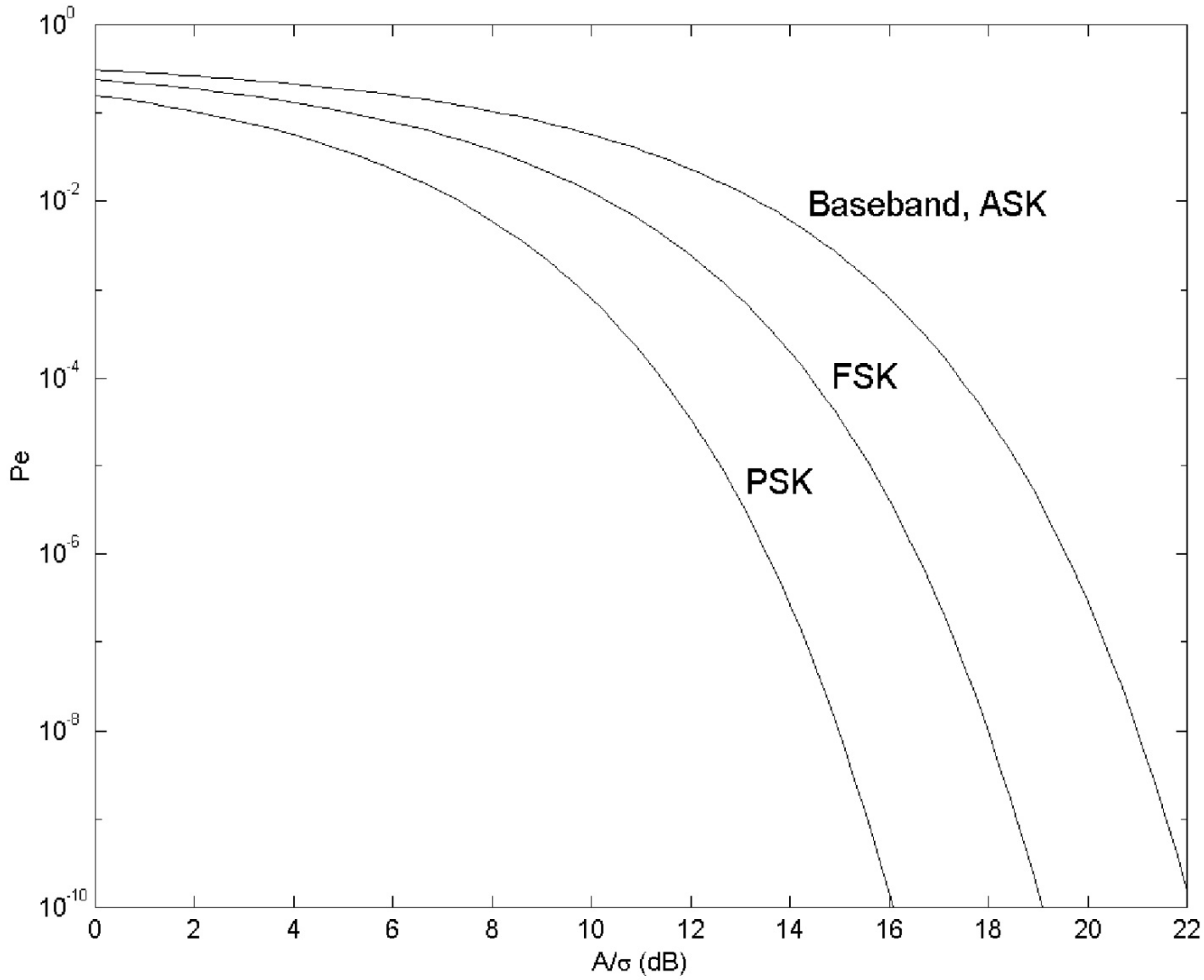
$$s_0(t) = A\cos(2\pi f_0 t), \quad \text{if symbol 0 is transmitted}$$
$$s_1(t) = A\cos(2\pi f_1 t), \quad \text{if symbol 1 is transmitted}$$

- Symbol recovery:
  - Use two sets of coherent detectors, one operating at a frequency f_0 and the other at f_1.

Coherent FSK demodulation. The two BPF's are non-overlapping in frequency spectrum



Comparison of Three Schemes



1. Show that if $n_c(t)$ and $n_s(t)$ are Gaussian distributed, then the magnitude $r(t)$ has a **Rayleigh** distribution, and the phase $\varphi(t)$ is **uniformly** distributed.