Properties of the Frequency-Amplitude Curve

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***Abstract*—The Frequency-Amplitude (F-A) curve has been proposed in ref. [1] to characterize the electromechanical oscillation frequency of a single-machine-infinite-bus system considering nonlinearity of the swing equation. For a multi-machine system, an F-A curve regarding one oscillation mode is a projection of the system trajectory between the stable equilibrium point and the stability boundary onto the F-A plane. This letter provides rigorous proofs of six general properties of an F-A curve.**

***Index Terms*—F-A curve, electromechanical oscillation, nonlinear oscillation, oscillation frequency**

I. Introduction

APPROXIMATE analytical and numerical studies have been performed to reveal the frequency-amplitude relationship for nonlinear oscillators in both power engineering and other fields [1]-[5]. In power engineering, oscillation frequency (OF) was analytically formulated as a nonlinear function of oscillation amplitude (OA), called frequency-amplitude (F-A) curve, in [1] on a single-machine-infinite-bus (SMIB) system considering the nonlinearity inherent in the swing equation. Paper [1] also demonstrated that such an F-A curve exists for each mode of a multi-machine system. This letter rigorously proves six properties of an F-A curve, which are useful to uncover the nonlinearity of electromechanical oscillation modes.

II. Six Properties of the F-A Curve

The OF of a SMIB system can be analytically formulated by (1)-(3), where δ_{max} and δ_{min} are the positive and negative maximum angle deviations; δ_s is the angle of the rotor at the stable equilibrium; β is a parameter determined by the inertia of the generator, the synchronous frequency and the maximum power transfer [1].

$$T_u(\delta_{max}) = \sqrt{\frac{2}{\beta}}\int_0^{\delta_{max}} \frac{d\delta}{\sqrt{\cos(\delta_s+\delta)-\cos(\delta_s+\delta_{max})+(\delta-\delta_{max})\sin\delta_s}} \quad (1)$$

$$T_l(\delta_{min}) = \sqrt{\frac{2}{\beta}}\int_{\delta_{min}}^{0} \frac{d\delta}{\sqrt{\cos(\delta_s+\delta)-\cos(\delta_s+\delta_{min})+(\delta-\delta_{min})\sin\delta_s}} \quad (2)$$

$$f(\delta_{max},\delta_{min}) = \frac{1}{T_u+T_l} \quad (3)$$

Note that only one of δ_{max} and δ_{min} is independent since the other can be determined by (4) based on the conservation of energy law under the initial condition depending on δ_0 and $\dot{\delta}_0$.

$$\dot{\delta}_0^2 + \beta[\cos(\delta_{ep}+\delta_s)-\cos(\delta_0+\delta_s)+(\delta_{ep}-\delta_0)\sin\delta_s] = 0 \quad (4)$$

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Let δ_{max} be the independent variable, i.e. $f=f(\delta_{max})$. In addition, one assumption used for all proofs is $0\le\delta_s\le \pi/2$, which means that the generator is producing power. Fig. 1 illustrates the F-A curve of an SMIB system, which has intercepts at the stable equilibrium point (SEP) and the nose point, respectively.

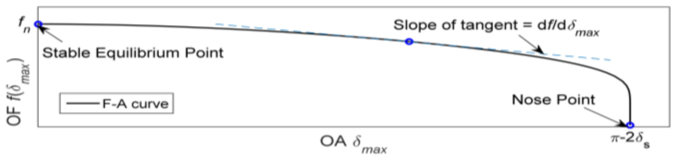

Fig.1. F-A curve of a SMIB system

***Property* 1: The domain of $f(\delta_{max})$ is $[0, \pi-2\delta_s]$**

Proof: By (1), the domain of $f(\delta_{max})$ is defined as the set of δ_{max} which leads to non-negative function $y(\delta)$, defined in (5), on $[0\ \delta_{max}]$. The idea is to show that if $\delta_{max}\le\pi-2\delta_s$, $y(\delta)$ is always non-negative and if $\delta_{max}>\pi-2\delta_s$, $y(\delta)$ will have negative values.

$$y(\delta) = \cos(\delta_s+\delta)-\cos(\delta_s+\delta_{max})+(\delta-\delta_{max})\sin\delta_s \quad (5)$$

$$y'(\delta) = -\sin(\delta_s+\delta)+\sin\delta_s \quad (6)$$

$$y''(\delta) = -\cos(\delta_s+\delta) \quad (7)$$

Since $y'(0)= y'(\pi-2\delta_s)=0$, $y''(0)=-\cos\delta_s<0$, $y''(\pi-2\delta_s)=\cos\delta_s>0$ and $y''(\delta)$ has only one zero at $\pi/2-\delta_s$, then $y'(\delta)$ is a non-positive function in $[0\ \pi-2\delta_s]$. Also note that $y(\delta_{max})=0$.

If $\delta_{max} \le \pi-2\delta_s$, then $y'(\delta) \le 0$ such that $y(\delta)\ge y(\delta_{max})=0$ in $[0\ \delta_{max}]$. If $\pi-2\delta_s < \delta_{max} \le \pi$, then $y'(\delta) > y'(\pi-2\delta_s)=0$ in $(\pi-2\delta_s\ \ \delta_{max})$ such that $y(\delta)< y(\delta_{max})=0$. Then, $y<0$ on $(\pi-2\delta_s\ \delta_{max})$. □

***Property* 2: $f(\delta_{max})$ intersects with the frequency-axis at the SEP $(0, f_n)$, where $f_n=\sqrt{\beta\cos\delta_s}/(2\pi)$**

Proof: In order to find the limit of $f(\delta_{max})$ as $\delta_{max}\to0^+$, we will first calculate the limits of T_u and T_l as $\delta_{max}\to0^+$. Due to the similarity between T_u and T_l, only T_u will be derived in detail.

First, apply the change of integral variable in (8) to the integral in (1) to obtain (9), where q is defined in (10). The Taylor expansion of q indicates the infinitesimals in (11). Substitute (11) into (9) and calculate T_u as $\delta_{max}\to0^+$ to obtain (12). Similarity, there is (13) for T_l. Finally, f_n is calculated in (14). □

$$\delta = t\delta_{max} \quad (8)$$

$$T_u(\delta_{\max}) = \sqrt{\frac{2}{\beta}}\int_0^1 \frac{\delta_{max}dt}{\sqrt{q}} \quad (9)$$

$$q = \cos(\delta_s+t\delta_{max})-\cos(\delta_s+\delta_{max})+(t-1)\delta_{max}\sin\delta_s \quad (10)$$

$$\sqrt{q} \sim \delta_{max}\sqrt{(1-t^2)\cos\delta_s/2}\cdot \quad \text{as} \quad \delta_{max}\to0^+ \quad (11)$$

$$\lim_{\delta_{max}\to0^+} T_u = \frac{2}{\sqrt{\beta\cos\delta_s}}\int_0^1 \frac{1}{\sqrt{1-t^2}}dt = \frac{\pi}{\sqrt{\beta\cos\delta_s}} \quad (12)$$

$$\lim_{\delta_{min}\to0^+} T_l = \frac{\pi}{\sqrt{\beta\cos\delta_s}} \quad (13)$$

$$f_n = \lim_{\delta_{max}\to 0^+} f = \frac{1}{\lim\limits_{\delta_{max}\to 0^+} T_u + \lim\limits_{\delta_{min}\to 0^+} T_l} = \frac{\sqrt{\beta\cos\delta_s}}{2\pi} \tag{14}$$

***Property* 3: $f(\delta_{max})$ intersects with the amplitude-axis at the nose point (π-2δ_s, 0)**

Proof: To show $f(\delta_{max})\to 0$ as $\delta_{max}\to\pi-2\delta_s$, we will first show that T_u approaches the positive infinity. The following gives a proof by contradiction. First assume that T_u approaches a limited value, say C, as $\delta_{max}\to\pi-2\delta_s$, then we have (15) where $y(\delta)$ is defined in (5). The assumption in (15) indicates (16). The next step is to show the integral in (16) is non-zero, which contradicts the assumption. Using the mean-value theorem of integral, the integral in (16) becomes (17), where $\pi-2\delta_s-\varepsilon\le\tau\le\pi-2\delta_s$. The Taylor expansion of y at $\delta=\pi-2\delta_s$ indicates the equivalent infinitesimals in (18) when $\varepsilon\to 0^+$. Substitute (18) into (17) and note that $|\tau-(\pi-2\delta_s)|\le\varepsilon$ and obtain the contradiction in (19). Thus, the assumption does not hold such that T_u approaches the positive infinity as $\delta_{max}\to\pi-2\delta_s$. Finally, (20) plus the non-negativity of $f(\delta_{max})$ finishes the proof. □

$$T_u(\pi-2\delta_s) = \int_0^{\pi-2\delta_s}\frac{2d\delta}{\sqrt{2\beta y(\delta)}} = C < +\infty \tag{15}$$

$$\lim_{\varepsilon\to 0^+}\int_{\pi-2\delta_s-\varepsilon}^{\pi-2\delta_s}\frac{2d\delta}{\sqrt{2\beta y(\delta)}} = 0 \tag{16}$$

$$\lim_{\varepsilon\to 0^+}\int_{\pi-2\delta_s-\varepsilon}^{\pi-2\delta_s}\frac{2d\delta}{\sqrt{2\beta y(\delta)}} = \sqrt{\frac{2}{\beta}}\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\sqrt{y(\tau)}} \tag{17}$$

$$\begin{aligned} & y(\delta) \sim \frac{\cos\delta_s}{2}\left(\delta-(\pi-2\delta_s)\right)^2 \quad \text{as } \delta\to\pi-2\delta_s \\ & \Rightarrow \sqrt{y(\delta)} \sim \sqrt{\frac{\cos\delta_s}{2}}\cdot\left|\delta-(\pi-2\delta_s)\right| \quad \text{as } \delta\to\pi-2\delta_s \end{aligned} \tag{18}$$

$$\lim_{\varepsilon\to 0^+}\int_{\pi-2\delta_s-\varepsilon}^{\pi-2\delta_s}\frac{2d\delta}{\sqrt{2\beta y(\delta)}} \ge \sqrt{\frac{2}{\beta}}\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\varepsilon\sqrt{\cos\delta_s/2}} = \frac{2}{\sqrt{\beta\cos\delta_s}} > 0 \tag{19}$$

$$f(\pi-2\delta_s) = \frac{1}{\lim\limits_{\delta_{max}\to(\pi-2\delta_s)^-}\left(T_u+T_l\right)} \le \frac{1}{\lim\limits_{\delta_{max}\to(\pi-2\delta_s)^-} T_u} = \frac{1}{+\infty} = 0 \tag{20}$$

***Property* 4: $f(\delta_{max})$ is a decreasing function on (0 π-2δ_s)**

Proof: The monotonicity of $f(\delta_{max})$ will be proved by the sign of its derivative. Based on (21), $f(\delta_{max})$ is decreasing if both T_u and T_l are increasing with δ_{max}. Due to the similarity, the following will only consider T_u in detail. The definition in (1) indicates that the integrand is singular at the upper limit δ_{max}. Thus unfortunately, we cannot directly take the derivative of T_u w.r.t. δ_{max}. To overcome this hurdle, the change of integral variable in (8)-(10) is used again. Then, the derivative of T_u w.r.t. δ_{max} is shown in (22), where $z(t)$ is defined in (23). The next step is to show $z(t)$ is a non-negative function. Since $z(1)=0$, we only need to show that the $z'(t)$ is non-positive on [0 1]. This is true since $z'(0)=0$ and $z''(t)$ is non-positive on [0 1] as shown in (25). Thus, (21) is non-positive. □

$$\left.\frac{df}{d\delta_{max}}\right|_{\delta_{max}=a} = \frac{-1}{\left(T_u+T_l\right)^2}\left.\left(\frac{dT_u}{d\delta_{max}}+\frac{dT_l}{d\delta_{max}}\right)\right|_{\delta_{max}=a} \tag{21}$$

$$\frac{dT_u}{d\delta_{max}} = \frac{1}{\sqrt{2\beta}}\int_0^1\frac{z(t)}{q^{3/2}}dt \tag{22}$$

$$\begin{aligned} z = & -\delta_{max}(1-t)\left(\sin\delta_s+\sin(t\delta_{max}+\delta_s)\right)+2\cos(t\delta_{max}+\delta_s) \\ & -2\cos(\delta_{max}+\delta_s) \end{aligned} \tag{23}$$

$$z'(t) = \delta_{max}\left(t\delta_{max}\cos(t\delta_{max}+\delta_s)+\sin\delta_s-\sin(t\delta_{max}+\delta_s)\right) \tag{24}$$

$$z''(t) = -t\delta_{max}^3\sin(t\delta_{max}+\delta_s)\le 0 \tag{25}$$

***Property* 5: $f(\delta_{max})$ has a slope of $-\sqrt{\beta}\sin\delta_s/(3\pi^2\sqrt{\cos\delta_s})$ at the SEP**

Proof: This proof starts from (22) to first calculate the derivative of T_u at δ_{max}=0. Rewrite (22) as (26) at δ_{max}=0. Note that both of z and q are infinitesimals as $\delta_{max}\to 0^+$. To calculate the limit, we need to find their orders by using their Taylor expansions at δ_{max}=0 as shown in (27) and (28), respectively. Substitute them into (26), calculate the definite integral and obtain (29). Similarly, we can get (30) for T_l. Finally, with (14) and (15), f's derivative at δ_{max}=0 is obtained by (31). □

$$\left.\frac{dT_u}{d\delta_{max}}\right|_{\delta_{max}=0} = \lim_{\delta_{max}\to 0}\frac{1}{\sqrt{2\beta}}\int_0^1\sqrt{\frac{z^2(t)}{q^3}}dt = \frac{1}{\sqrt{2\beta}}\int_0^1\left(\lim_{\delta_{max}\to 0}\sqrt{\frac{z^2(t)}{q^3}}\right)dt \tag{26}$$

$$z^2 \sim \frac{(1-t^3)^2\sin^2\delta_s}{36}\delta_{max}^6 \quad \text{as } \delta_{max}\to 0^+ \tag{27}$$

$$q^3 \sim \frac{(1-t^2)^3\cos^3\delta_s}{8}\delta_{max}^6 \quad \text{as } \delta_{max}\to 0^+ \tag{28}$$

$$\left.\frac{dT_u}{d\delta_{max}}\right|_{\delta_{max}=0} = \frac{\sin\delta_s}{3\sqrt{\beta}\cos^{3/2}\delta_s}\int_0^1\frac{1+t+t^2}{\sqrt{(1-t)(1+t)^3}}dt = \frac{2\sin\delta_s}{3\sqrt{\beta}\cos^{3/2}\delta_s} \tag{29}$$

$$\left.\frac{dT_l}{d\delta_{max}}\right|_{\delta_{max}=0} = \frac{2\sin\delta_s}{3\sqrt{\beta}\cos^{3/2}\delta_s} \tag{30}$$

$$\left.\frac{df}{d\delta_{max}}\right|_{\delta_{max}=0} = \frac{-1}{\left(T_u+T_l\right)^2}\left.\left(\frac{dT_u}{d\delta_{max}}+\frac{dT_l}{d\delta_{max}}\right)\right|_{\delta_{max}=0} = -\frac{\sqrt{\beta}\sin\delta_s}{3\pi^2\sqrt{\cos\delta_s}} \tag{31}$$

***Property* 6: $f(\delta_{max})$ has a slope of -∞ at the nose point**

Proof: This proof uses (22) and the idea is similar to that of the property 3 which uses (15). □

III. CONCLUSION

This letter provides six properties and their rigorous proofs for the F-A curve of the SMIB system. As shown in [1], for a multi-machine power system, an F-A curve exists for each of the electromechanical modes. Thus, the proofs in this letter lay a foundation for applications of the F-A curve concept in analyzing nonlinear oscillation and associated angular stability of a power system: for each oscillation mode, (i) OA cannot be too large if oscillation is stable; (ii) when OA is zero, OF equals natural frequency of the mode and has a slope equal to a small negative value; (iii) OF approaches zero at the largest allowable OA, which corresponds to the boundary of stability.

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