

Application of the Adomian Decomposition Method for Semi-Analytic Solutions of Power System Differential Algebraic Equations

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Abstract— This paper explores an alternative, semi-analytical approach to solution of the initial value problem of differential-algebraic equations modeling a power system. Different from the traditional numerical integration based approach, this new approach applies the Adomian Decomposition Method to derive an approximate solution, called a semi-analytic solution (SAS), as a closed-form explicit function of time, the initial state and parameters on the system condition. Such a solution directly gives the power system's dynamic trajectory starting from an initial state that is accurate over a certain time window. Then, a multi-stage scheme evaluating the same SAS repeatedly for sequential time windows is able to give the system's trajectory for a desired simulation period without iterative computations as numerical integration does. The new approach is tested for power system transient stability simulation on a 3-machine, 9-bus power system and the IEEE 10-machine, 39-bus system, and its accuracy and time performance are compared with the 4th order Runge–Kutta method.

Index Terms—Adomian Decomposition; differential-algebraic equations; initial value problem; power system simulation; Runge–Kutta method; semi-analytic solution; transient stability

I. INTRODUCTION

Power system time-domain simulation needs to solve the initial value problem (IVP) of a bunch of nonlinear differential-algebraic equations (DAEs) about the dynamic model, operating condition and contingency of a power system. It is traditionally solved by numerical integration methods in commercialized power system simulation software. The numerical methods, either explicit or implicit, compute the system's time-series trajectory by iterative computation for every time step until values of all state variables are convergent. A small integration time step, e.g. few milliseconds, is usually required. Thus, for a typical simulation period of tens of seconds, simulation may take lots of iterations and time in computing. Other than time performance, numerical instability is another concern more specifically with explicit methods like the Runge–Kutta (R-K) method, which is the dominating method applied in

today's simulation software. Implicit methods like the Trapezoidal method overcome numerical instability by employing implicit nonlinear algebraic equations, which however need to be solved also through iterative computation by numerical methods like the Newton-Raphson method. Thus, the computational complexity grows significantly.

Intuitively, if the analytical solution of the IVP of power system DAEs could be found as a set of explicit algebraic functions of symbolic variables including time, the initial state and parameters on the system operating condition, it would directly give the values of state variables at any time without conducting time-consuming iterations as numerical integration does. However, such an analytical solution being accurate for a long simulation time period does not exist in theory for power system DAEs. A compromise could be to find an approximate analytic solution, named a Semi-Analytic Solution (SAS), which keeps accuracy for a certain time window, and repeats using the same SAS until making up a desired simulation period. If a SAS is derived beforehand, solving the IVP becomes simply evaluation of the SAS, i.e. plugging in values of symbolic variables, and may be faster than numerical integration.

The true solution of power system DAEs may be approached by summing infinite terms of some series expansion. A SAS can be defined as the sum of finite terms that is accurate over a time window. Such a series expansion can be derived by using the Adomian Decomposition Method (ADM). Compare to other decomposition methods like Taylor's Series Expansion, The ADM is able to keep nonlinearity in the system model [1]. In this paper, we study the power system DAE model with a classical model for each generator, and then present the derivation of a SAS for a multi-machine power system. The approach would be easily extended to apply to other power system DAE models with detailed generator models. In the rest of the paper, section II first briefly introduces the ADM algorithms of the proposed approach and then presents the derivation of SASs for both a single-machine-infinite-bus (SMIB) system and a multi-machine power system, section III tests the proposed

approach on a 3-machine, 9-bus system and IEEE 10-machine, 39-bus system, and section IV draws conclusions and suggests future work.

II. DERIVATION OF SASS USING THE ADM

A. Algorithms of the ADM

Consider the classical 2nd order DAE model for a SMIB power system, which can be described by a 2nd order differential equation in the form of (1)

$$\ddot{x}(t) + D\dot{x}(t) = f(x(t)) \quad (1)$$

where f is a nonlinear differentiable function and D is a coefficient related to the system's oscillation damping. To solve its IVP, the first step of ADM is to apply Laplace transform $L[\cdot]$ to (1) to transform it into an algebraic equation about complex frequency s , and then solve $L[x]$ to get (2). Assume that the solution $x(t)$ can be decomposed as (3). Decompose $f(x)$ as the sum of Adomian polynomials calculated from (5) by introducing a Lagrange multiplier λ .

$$L[x] = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} - \frac{DL[\dot{x}]}{s^2} + \frac{L[f(x)]}{s^2} \quad (2)$$

$$x(t) = \sum_{n=0}^{\infty} x_n(t) \quad (3)$$

$$f(x) = \sum_{n=0}^{\infty} A_n(x_0, x_1, \dots, x_n) \quad (4)$$

$$A_n = \frac{1}{n!} \left[\frac{\partial^n}{\partial \lambda^n} f \left(\sum_{i=0}^n x_i \lambda^i \right) \right]_{\lambda=0} \quad (5)$$

After decomposing both $x(t)$ and $f(x)$, compare their terms to easily derive recursive formulas (6) and (7) for $L[x_n]$.

$$L[x_0] = x(0)/s + \dot{x}(0)/s^2 \quad (6)$$

$$L[x_{n+1}] = DL[\dot{x}_n]/s^2 + L[A_n]/s^2, n \geq 0 \quad (7)$$

By applying inverse Laplace transform to their both sides, x_n for any n can be obtained. Consequently, a SAS of (1) is yielded by summing first N terms of x_n :

$$x^{SAS}(t) = \sum_{n=0}^{N-1} x_n(t) \quad (8)$$

When x_n is plugged into (7) recursively to calculate A_n , some computational burden is caused since in (5), x_0 always appears in f and its derivatives. A Modified ADM (M-ADM for short) proposed by [2] calculates the following (9)-(11) instead of (6) and (7) to reduce the computational burden. Thus, the SAS becomes a power series expansion with only polynomial nonlinearity.

$$L[x_0] = x(0)/s \quad (9)$$

$$L[x_1] = \dot{x}(0)/s^2 + DL[\dot{x}_0]/s^2 + L[A_0]/s^2 \quad (10)$$

$$L[x_{n+2}] = DL[\dot{x}_{n+1}]/s^2 + L[A_{n+1}]/s^2, n \geq 0 \quad (11)$$

Similarly, for a K -machine power system, the DAE model in (12) is solved by the ADM in the same manner. The state variables and nonlinear functions are decomposed by (13) and (14), respectively. Adomian polynomials are calculated by (16). Then, the SAS for each state variable is yielded.

$$\begin{aligned} \ddot{\mathbf{x}}(t) + \mathbf{D}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) \\ \mathbf{x}(t) &= [x_1(t) \quad x_2(t) \quad \dots \quad x_K(t)]^T \\ \mathbf{f}(\cdot) &= [f_1(\cdot) \quad f_2(\cdot) \quad \dots \quad f_K(\cdot)]^T \\ \mathbf{D} &= \begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & D_K \end{bmatrix} \end{aligned} \quad (12)$$

$$\mathbf{x}(t) = \sum_{n=0}^{\infty} \mathbf{x}_n(t), \quad \mathbf{x}_n = [x_{1,n} \quad x_{2,n} \quad \dots \quad x_{K,n}]^T \quad (13)$$

$$f_i(\mathbf{x}) = \sum_{n=0}^{\infty} A_{i,n}(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n), \quad i = 1 \dots K \quad (14)$$

$$A_{i,n} = \frac{1}{n!} \left[\frac{\partial^n}{\partial \lambda^n} f_i \left(\sum_{i=0}^n \mathbf{x}_i \lambda^i \right) \right]_{\lambda=0} \quad (15)$$

Since the SAS of the IVP is accurate only for a limited time window, denoted by T , a multi-stage scheme is adopted to extend accuracy to an expected period [3] by two steps: 1) partition the period into sequential windows of T each able to keep an acceptable accuracy of the SAS; 2) evaluate the SAS for the first T using a given initial state and starting from the second T , evaluate the SAS by taking the final state of the previous T as the initial state.

In the following, the derivation of a SAS is illustrated on a SMIB system and then a 3-machine, 9-bus system.

B. SMIB System

Consider the 2nd order SMIB system model in (16) about the rotor angle of the generator.

$$\begin{aligned} 2H\ddot{\delta} + D\dot{\delta} &= P_m - P_e \\ &= P_m - P_{\max} \sin \delta \end{aligned} \quad (16)$$

Use the formulas (2)–(8) for the original ADM or (2)-(5) plus (8)-(11) for the M-ADM to derive the SAS about δ with a specific number of terms. Assume that every variable or parameter in (16) is symbolized for the largest generality of use under any possible condition of the system. Let $a = \delta(0)$, $b = \dot{\delta}(0)$, $c = bt + a$ and $d = bt + 2a$. The SAS with 3 terms from the

original ADM is given by (17) and (18).

$$\delta^{SAS}(t) = \delta_0(t) + \delta_1(t) + \delta_2(t) \quad (17)$$

$$\delta_0 = bt + a$$

$$\delta_1 = \frac{-Db + P_m t^2 - \frac{P_{\max} \cos a}{2Hb} t + \frac{P_{\max} [-\sin a + \sin(bt + a)]}{2Hb^2}}{4H}$$

$$\delta_2 = \frac{D^2b - DP_m t^3 + \frac{Db(\cos a - \cos c) + P_m \cos c}{8H^2b^2} P_{\max} t^2}{24H^2} + \frac{P_{\max} [8Db(\sin a + \sin c) - 4P_m(\sin a + 2\sin c) - P_{\max}(2\cos d + 2\cos bt + \cos 2a + 4)]}{16H^2b^3} t$$

$$\frac{P_{\max} [32Db(\cos a - \cos c) - 24P_m(\cos a - 2\cos c) - P_{\max}(4\sin d + \sin 2c + 12\sin bt - 5\sin 2a)]}{32H^2b^4} \quad (18)$$

The SAS from the M-ADM is given by (17) and (19), which is more compact than that from the original ADM since equations in (18) involve sine and cosine functions of time while equations in (19) are only polynomials of time.

$$\delta_0 = a$$

$$\delta_1 = bt + \frac{P_m - P_{\max} \sin a}{4H} t^2$$

$$\delta_2 = \frac{P_{\max} \cos a (P_{\max} \sin a - P_m)}{96H^2} t^4 + \left[\frac{D(P_{\max} \sin a - P_m)}{24H^2} - \frac{bP_{\max} \cos a}{12H} \right] t^3 - \frac{Db}{4H} t^2 \quad (19)$$

C. Multi-machine Power Systems

The 9-bus system has the same parameters as that in [4] except that three generators are all represented by 2nd order classic models. The SAS expression with all parameters symbolized are much longer. For the illustration purpose, if we only symbolize time, initial values of state variables and generators' mechanical powers, which represent the solution with the initial system state under a specific loading condition, then the 3-term SAS from the M-ADM about the first generator angle is given by (20) and (21) as an example, where $a_i = \delta_i(0)$, $b_i = \dot{\delta}_i(0)$, $a_{ij} = a_i - a_j$ and $b_{ij} = b_i - b_j$. The SAS is a polynomial of time.

The SAS from the original ADM is even longer, so we only show the 3-term SAS with a set of specific values for P_{mi} and initial state variables listed in TABLE I. The SAS for the rotor angle of the first generator is given by (20) and (22), which contains sine and cosine functions of time.

TABLE I
PARAMETERS OF THE 9-BUS SYSTEM

i	$\delta_i(0)$ (rad)	$\dot{\delta}_i(0)$ (rad/s)	P_{mi} (pu)
1	-0.0022	-0.0802	0.716
2	0.1667	4.1631	1.63
3	0.1261	2.9868	0.85

$$\delta_1^{SAS}(t) = \delta_{1,0} + \delta_{1,1}(t) + \delta_{1,2}(t) \quad (20)$$

$$\delta_{1,0} = a_1$$

$$\delta_{1,1} = (3.9869P_{m1} - 5.0679 - 0.5709 \cos a_{12} - 3.1247 \sin a_{12} - 0.7813 \cos a_{13} - 4.5570 \sin a_{13}) t^2 + b_1 t$$

$$\delta_{1,2} = [15.7434 \sin(a_{13} + a_{23}) + 0.1874 \cos(a_{13} + a_{23}) + 2.2989 \sin(a_{12} + a_{13}) + 0.8404 \cos(a_{12} + a_{13}) + 5.0860 \sin(a_{12} + a_{32}) + 0.1160 \cos(a_{12} + a_{32}) + (0.5191P_{m1} - 4.0773P_{m2} - 4.3236) \sin a_{13} - (3.0280P_{m1} + 23.7815P_{m3} + 4.4600) \cos a_{13} + (0.3793P_{m1} - 1.4011P_{m2} - 14.8103) \sin a_{12} - (2.0763P_{m1} - 7.6694P_{m2} + 5.5952) \cos a_{12} + 15.6701 \sin 2a_{13} + 0.5934 \cos 2a_{13} + 3.8923 \sin 2a_{12} + 0.2973 \cos 2a_{12} - 5.7584] t^4 + [(1.0416b_{21} + 0.0040) \cos a_{12} + (0.1903b_{12} + 0.0220) \sin a_{12} + (1.5190b_{31} + 0.0055) \cos a_{13} + (0.2604b_{13} + 0.0321) \sin a_{13} - 0.0281P_{m1} + 0.0357] t^3 - 0.0106b_1 t^2 \quad (21)$$

$$\delta_{1,0} = -0.0022 - 0.0802t$$

$$\delta_{1,1} = -0.0449 + 4.5094t - 2.2133t^2 + 0.0042 \cos(4.2433t) - 0.3528 \sin(4.2433t) + 0.0408 \cos(3.0670t) - 0.9822 \sin(3.0670t)$$

$$\delta_{1,2} = [-2.1739 \cos(3.0670t) - 0.0903 \sin(3.0670t) - 0.7808 \cos(4.2433t) - 0.0092 \sin(4.2433t)] t^2 + [6.2159 + 4.3114 \cos(3.0670t) + 3.0190 \sin(3.0670t) + 1.5823 \cos(4.2433t) + 0.7548 \sin(4.2433t) - 1.7627 - 0.1204 \sin(6.1340t) + 0.0100 \cos(6.1340t) + 1.0773 \sin(1.1763t) + 0.0320 \cos(1.1763t) - 0.7470 \sin(4.2433t) + 0.2532 \cos(4.2433t) - 0.0156 \sin(8.4866t) + 0.0004 \cos(8.4866t) - 0.0888 \sin(7.3103t) + 0.0047 \cos(7.3103t) - 2.8325 \sin(3.0670t) + 1.4624 \cos(3.0670t)] t^4 \quad (22)$$

It should be pointed out that although SAS expressions for multi-machine systems could be long, to evaluate a SAS is to simply plug in values of all symbolic variables, which is fast since it is only a one-time algebraic computation and does not involve any iterative computation as numerical integration does. For a SAS from the M-ADM, it is a polynomial and hence can be computed easier than a SAS from the original ADM. Also, its expression is more compact in size. However, it is usually less accurate than the SAS from the original ADM and hence is valid for a shorter time window.

III. VALIDATIONS

A. 3-machine, 9-bus System

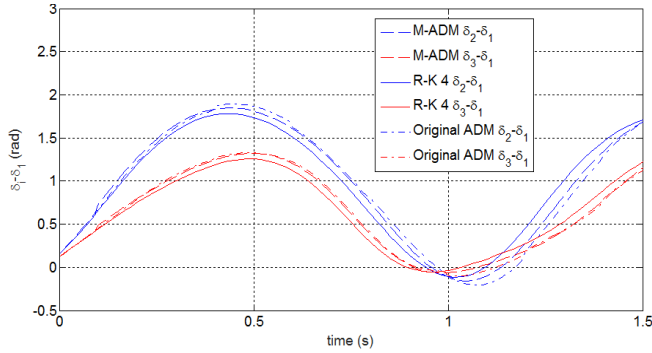


Figure 1. Comparison of the original and Modified ADMs and RK4

To create angular instability, the initial system state, i.e. $\delta_i(0)$ and $\dot{\delta}_i(0)$, is moved further away from the equilibrium. The result from the 5-term SAS of the M-ADM with time window $T=0.1s$ also agrees with the result from R-K4 as shown in Figure 2. It validates the M-ADM's ability of simulating an unstable system.

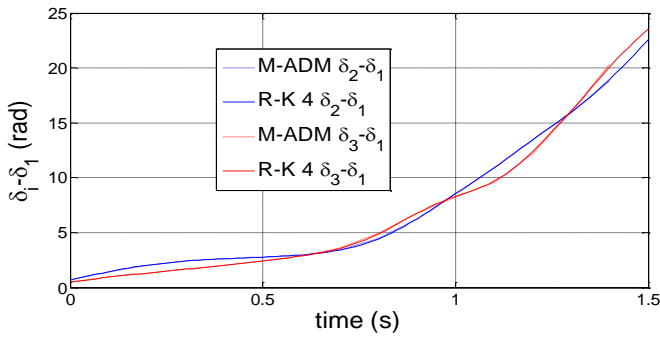


Figure 2. Results from M-ADM and RK4 for an unstable system condition

The numerical stability with the M-ADM's 5-term SAS is examined. As shown in Figure 3, the result from that SAS repeatedly evaluated every 0.23s start diverging from the numerical solution of the R-K4 after 1.6s. In this case, to guarantee accuracy of that SAS for 2s, the repeating time window should be less than 0.2s.

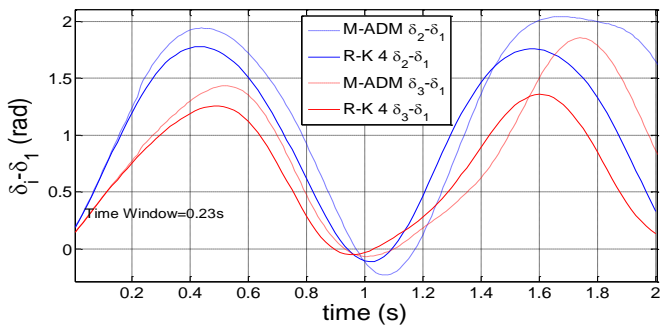


Figure 3. Divergence of the M-ADM's 5-term SAS with $T=0.23s$

B. IEEE 39-bus System

To test the SAS-based multi-stage approach on simulating a more real-world power system model, the IEEE 10-generator, 39-bus system shown in Figure 4 is adopted, which represents the backbone transmission system of the New England power grid.

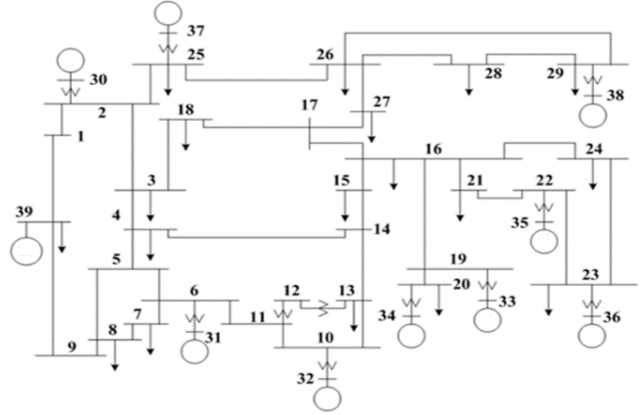


Figure 4. IEEE 10-generator 39-bus system

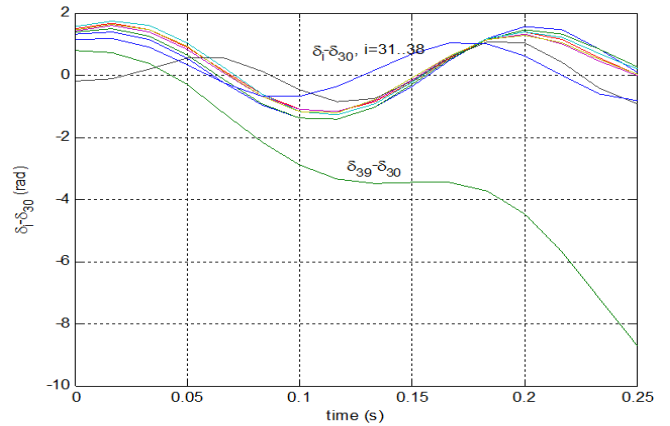


Figure 5. Rotor angles from RK-4 with 1 cycle integration step

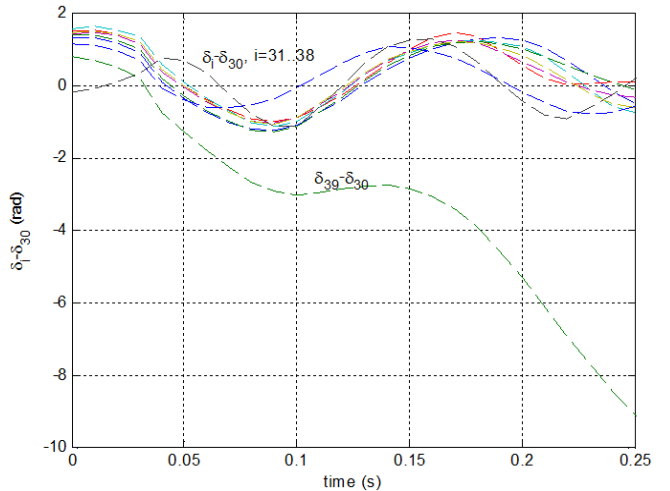


Figure 6. Rotor angles from the SAS (M-ADM) with $T=0.04s$

The SAS with 3 terms is derived by the M-ADM for state variables of each generator. The largest time window leading to a converging SAS is found to be $T=0.06s$. A three-phase fault at bus 2 cleared by tripping line 2-25 after 5 cycles is considered. First, the R-K4 method is performed for the 4-cycle fault-on period to determine the post-fault initial system state. Then, the 3-term SAS is evaluated repeatedly for sequential time windows of $T=0.04s$ to estimate the post-fault system response. Figures 5 and 6 give rotor angles relative to generator 30 calculated respectively from the SAS and R-K4 method, which match well.

To compare their results in more detail, the angles of three selected generators are shown in Figures 7-9. Here, the R-K4 method is performed using a variable integration step ≥ 1 cycle, and the SAS is evaluated 4 times per T , i.e. every 0.01s. The SAS gives smoother results than the R-K4.

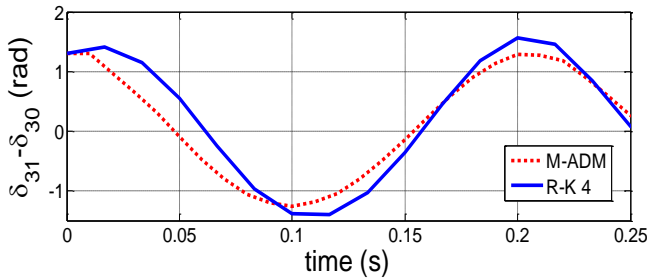


Figure 7. $\delta_{31} - \delta_{30}$ Comparison between M-ADM and Runge-Kutta 4

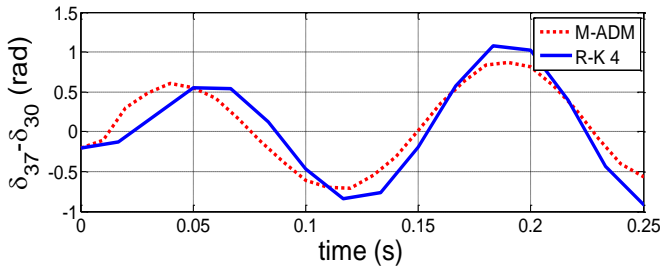


Figure 8. $\delta_{37} - \delta_{30}$ Comparison between M-ADM and Runge-Kutta 4

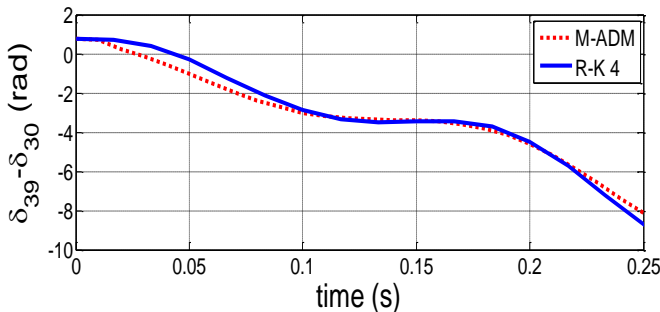


Figure 9. $\delta_{39} - \delta_{30}$ Comparison between M-ADM and Runge-Kutta 4

To demonstrate the time performance the proposed SAS-based approach, the following three cases are tested in the environment of MATLAB and MAPLE:

Case-A: only symbolizing time t , the initial values of state variables $\delta_i(0)$ and $\dot{\delta}_i(0)$, i.e. for one specific simulation.

Case-B: also symbolizing the reduced admittance matrix about 10 generator EMFs, i.e. for simulating different faults under one specific loading condition. The magnitude and angle of each element in reduced admittance matrix will be symbolized separately. Therefore, there will be two symmetric symbolic 10×10 matrices for magnitudes and angles.

Case-C: also symbolizing generators' mechanical powers to make the SAS be also good for simulating different faults and loading conditions with each load modeled by a constant impedance.

Application of the SASs for power system simulation may be implemented by a two-stage approach: the derivation of SASs is performed offline and then the evaluation of a SAS is performed online. Those offline and online stages are respectively performed in MAPLE and MATLAB in our test. Table II lists the test results on time spent in the online and offline stages. For the online stage, a minimum of 3 time points of the SAS need to be evaluated in each window $T=0.04s$: one in the middle of T and two near the end to estimate the initial state for the next T . However, evaluations of those three points are independent and hence can be done in parallel. Thus, the online SAS evaluation time τ in the table is for evaluating one point, i.e. 3 points by three parallel computers. If only one computer is used, the evaluation time cost becomes 3τ per T .

TABLE II
TIME PERFORMANCE OF THE SAS BASED APPROACH

	<i>Case-A</i>	<i>Case-B</i>	<i>Case-C</i>
SAS derivation time	211.66 s	392.79 s	395.64 s
SAS evaluation time τ	0.0015 s	0.0047 s	0.0047 s
T/τ	26.7	8.5	8.5

The time performance is tested on a desktop computer with Intel Core i7-3770 3.40GHz CPU and 8GB RAM. From the comparison between *Case-A* and *Case-B*, symbolizing the admittance matrix makes the offline and online computations take 80% and 210% longer, respectively, because 110 additional symbolic variables for the admittance matrix are added for *Case-B*. However, after adding more symbolic variables on the system operating condition, *Case-C* takes almost the same offline and online times as *Case-B*. The reason is that the symbolic variables on generations and loads are only involved in operations with addition, neither multiplication nor any sine or cosine function. The ratio T/τ in the table indicates how many times the SAS-based online simulation can be faster than clock time. For comparison purposes, we also tested the time performance of R-K4 with variable integration step sizes (usually faster than integration with fixed step sizes) at 1, 2 and 6 cycles. The R-K4 method

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takes 0.019s, 0.018s and 0.012s respectively to simulate for 0.04s, which are longer than τ .

CONCLUSIONS

This paper proposed an alternative, semi-analytical approach to solution of power system DAEs. The new approach may be applied to simulate a power system under a contingency. Since it does not need iterations and partitions computational burdens into an offline stage (deriving a SAS) and an online stage (evaluating the SAS), it may have big potentials in fast power system simulation. The next step will be validation of the new approach on large power systems with detailed models and systematical investigation on numerical stability with this new approach.

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