HW due next Tuesday

Apply contraction mapping theorem to last example.

For Gauss, with $N = -2$, then

$$\phi(x) = \frac{1}{2} x^2 - 0.625x + \frac{1}{2}$$

if $|\phi'(x)|$ is less than 1, then a contraction

$$\phi'(x) = x - 0.625 < 1$$

if $x \in (-0.375, 1.625)$

But it may not be necessary (outside it might converge as well).
Convergence characteristics

Mean Value Theorem

If $f()$ is differentiable and continuous on $(x^0, x^*)$ then $\exists a$ in such that

$$f''(a)(x^*-x^0) = f(x^*) - f(x^0) - f'(x^0)(x^*-x^0)$$

in $\mathbb{E}(x^0, x^*)$

Also, the extension to 2nd derivative

$$f''(a) \frac{(x^*-x^0)^2}{2} = f(x^*) - f(x^0) - f'(x^0)(x^*-x^0)$$

(this one way to derive Taylor series)

Let $e(x^*) = f(x^*) - f(x^0) - a(x^*-x^0)$

$$= (f(x^*) - y) - (f(x^0) - y) - a(x^*-x^0)$$

modify error term

$$\alpha^* e(x^*) = x^0 - \alpha^*[f(x^0) - y] - x^* = x^*-x^i = -\alpha^*e(x^*)$$

(And $f(x)$ be the function we want to find a solution $x^*$ with $f(x^*) = y$)
Now apply MVT to $e(x^*)$, substitute

$$f'(m)(x^*-x^0) = f(x^*) - f(x^0)$$

then

$$e(x^*) = f'(m)(x^*-x^0) - a(x^*-x^0)$$

assume $f'(m)$ is bounded and $|f''(m)| - a! \leq M^{(2)}$

$$|e(x^*)| \leq M^{(2)} |x^*-x^0|$$

and so

$$|x^*-x'| \leq \frac{M^{(2)}}{a} |x^*-x^0|$$

or

$$\frac{|x^*-x'|}{|x^*-x^0|} \leq \frac{M^{(2)}}{a} \Rightarrow \text{linear convergence (divergence)}$$

How about Newton-Raphson? $a'$ becomes $f''(x^0)$ and

$$e(x^*) = f''(m) \frac{(x^*-x^0)^2}{2}$$

(from MVT)

Substituting

$$-f'(x^0)(x^*-x^0) = f''(m) \left( \frac{(x^*-x^0)^2}{2} \right)$$
and $f''(m)$ bounded $|f''(m)| \leq M^{(2)}$ then

$$|x^* - x'| \leq \frac{|M^{(2)}|}{|f'(x^0)|} \left(\frac{x^* - x^0}{2}\right)^2$$

or

$$\frac{|x^* - x'|}{|x^* - x^0|^2} \leq \frac{1}{2} \left|\frac{M^{(2)}}{f'(x^0)}\right| \Rightarrow \text{Quadratic convergence (or divergence)}$$
Gauss-Seidel Method for LFE

\[ S_i = U_i \overline{I_i}^* = U_i \sum_{j=1}^{n} Y_{ij} \overline{V_j}^* \]

Take conjugate

\[ \overline{S_i}^* = \overline{V_i}^* \overline{I_i}^* = \overline{V_i}^* \sum_{j=1}^{n} Y_{ij} \overline{V_j} = \overline{V_i}^* \left( \sum_{j=1}^{n} Y_{ij} \overline{V_j} + Y_{ii} \overline{V_i} \right) \]

Rearrange \( i \) solve for \( V_i \)

\[ V_i^{(k+1)} = \frac{1}{Y_{ii}} \left( \frac{\overline{S_i}^*}{\overline{V_i}^{(k)}} - \sum_{j=1}^{n} Y_{ij} \overline{V_j}^{(k)} \right) \]

Relate to my general iterative format \( + / - \overline{V_i} \)

\[ V_i^{(k+1)} = V_i^{(k)} + \frac{1}{Y_{ii}} \left( \overline{S_i}^* - \sum_{j=1}^{n} Y_{ij} \overline{V_j}^{(k)} \right) \]

Algorithm

Conjugate of error

Diagonal matrix

Gauss-Seidel update voltages one by one use most recent updates (essentially a lower diagonal matrix for the algorithm)
Note:

1) At the slack bus, $V_i$ is known so nothing to do. (Can calculate $S_i$ directly).

2) At PV buses, use $S_i^{(k)} = V_i^{(k)} \sum_{j} Y_{ij} V_j^{*(k)}$ to find $Q_i$ then update $S_i$ only.

3) At PQ buses, the form I just presented.

4) We can add an accelerator term

\[
\frac{1}{Y_{ii} V_i} \Rightarrow \frac{\alpha}{Y_{ii} V_i^{* \alpha}} \quad 1 \leq \alpha \leq 2
\]
Newton-Raphson form for LFE

Write LFE

\[ P_i = f_i (V, S) \]
\[ Q_i = g_i (V, S) \]

General iterative form

\[
\begin{bmatrix}
S^{(k+1)}_i \\
V^{(k+1)}_i
\end{bmatrix} = \begin{bmatrix}
S^{(k)}_i \\
V^{(k)}_i
\end{bmatrix} - J^{-1} \begin{bmatrix}
f(V^{(k)}_i, S^{(k)}_i) - P \\
g(V^{(k)}_i, S^{(k)}_i) - Q
\end{bmatrix}
\]

A little closer at \( J \)

\[
J = \begin{bmatrix}
a f/2S & a f/2V \\
-2g/2S & -2g/2V
\end{bmatrix}
\]

(Spacing and handwriting indicate Jacobian ("algorithm") and bus mismatch (error terms).)
As a result

\[ 2f_i \partial \delta_k = -V_i V_k B_{ik} \quad 2f_i \partial \delta_i = V_i \sum_{j=1}^{n} V_{ij} B_{ij} \]

\[ 2f \partial \delta = 0 \]

\[ 2g_e \partial \delta = 0 \]

\[ 2g_i \partial \delta_i = V_i \delta_i - V_i B_{ii} + g_i \frac{V_i (V_i, \delta)}{V_i} \]

\[ 2g_i \partial \delta_{ik} = e V_i \left( B_{ik} \right) \]

\[ \Rightarrow \text{Real and reactive calculations decouple} \]
For implementation:

1) At slack bus, no $S_i$ specified so nothing to do

2) At PV bus, only $P_i$ is known so only have an error term for $P_i$ and only update $S_i$

3) At PQ bus, both $P_i$, $Q_i$ specified and update both $V_i$, $S_i$

This so far is a full Newton-Raphson. But at each step I need to invert a large matrix $((n-1+n_PQ) \times (n-1+n_PQ))$

Why not approximate it?

Observations:

1) Voltages are usually close to 1
2) Conductances $G_{ij}$ are usually small or zero (lossless)
3) Line angles $(\theta_i - \theta_j)$ are small so $\sin (\theta_i - \theta_j) \approx (\theta_i - \theta_j)$ and $\cos (\theta_i - \theta_j) \approx 1$