The exam consists of 5 questions each of several parts. The exam is closed book. All of the questions can be correctly answered in a reasonable amount of time and space. If you need an excessive amount of time or computations to answer a problem, then you are doing something wrong. Show your work clearly.

1. **LINEAR PROGRAM (30 points):** Given the linear programming problem below.

\[
\begin{align*}
\text{min} & \quad 2x_1 + 3x_2 \\
\text{such that} & \quad 3x_1 + x_2 \geq 8 \\
& \quad x_1 - x_2 = 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

a) Solve this problem graphically. Identify the solution and all the extreme points.
b) Now formulate this problem in standard form and select an extreme point, which is not optimal. For this point:
   - identify the basic variables, \( x_B \), non-basic variables, \( x_N \), and the basis \( B \).
   - show that the test of optimality fails for this extreme point.

The feasible region is shown with two extreme points: \((0, 8)\) and \((2.5, 0.5)\). The objective function \( x_1 - x_2 \leq 2 \) is also shown.

**Standard form**

\[
\begin{align*}
\text{max} \quad & -2x_1 - 3x_2 \\
\text{with} \quad & 3x_1 + x_2 - x_3 = 8 \\
& x_1 - x_2 + x_4 = 2 \\
& x_1 \geq 0
\end{align*}
\]

**Optimality test**

\[
\begin{align*}
2 \sum_{j=1}^{3} a_{ij} x_j &= \left[ C_0 - \sum_{j=1}^{3} c_j B_j \right] \quad \text{for } i = 1, 2, 3 \\
\Rightarrow 2 \begin{bmatrix} 2 \end{bmatrix} x_1 &= 7 \quad \text{fails}
\end{align*}
\]

**Non-optimal extreme point**

\[
\begin{align*}
\bar{x} &= [0 \ 8 \ 0 \ 0]^T \\
\bar{x} - \bar{B} &= [2 \ 10 \ 10 \ 7] \quad \text{and} \quad R = [1 \ 0 \ 7] \quad \text{and} \quad N = [3 \ -1] \quad \text{and} \quad C_0 = [3]
\end{align*}
\]
Can also check that \( \partial^2 f / \partial x_3 \) will satisfy the test but must be true for all \( x_3 \in \mathbb{R} \).

\[
2 \partial^2 f / \partial x_3 = - \left( \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} - 0 \right) = -3 \neq 0
\]
2. INTEGER PROGRAM (10 points): Given the linear programming problem from problem 1, repeated below, suppose that \( x_2 \) is now restricted to be integer. Starting with the solution found in question (1a), solve using the branch-and-bound algorithm. (Note the solution at each branch should be obvious and a solution is found quickly for this problem so if you need to do a large number of branches you are making a mistake).

\[
\begin{align*}
\text{min } & \quad 2x_1 + 3x_2 \\
\text{such that } & \quad 3x_1 + x_2 \geq 8 \\
& \quad x_1 - x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0 \quad x_2 \text{ integer}
\end{align*}
\]

Solution from 1
\( x = (2.5, 0.5) \)

\( x_2 \leq 0 \) (Clearly) infeasible

\( x_2 \geq 1 \)

\( x_2 = 1 \) \( (2 \frac{2}{3}, 1) \)

Done, only feasible and optimal found
4. **GRAPH SEARCH (25 points):** Consider the graph below with possible routes to Spokane leaving from Pullman. The paths are directional as indicated by the arrows. The times shown next to the arcs show the actual driving times. The numbers next to the cities show the cumulative remaining time to reach Spokane (the goal). Find the route that will reach Spokane the quickest using:

   a) Formulate a linear program for this problem (but do not solve).
   b) Find the shortest path using Dijkstra's search. Show the steps clearly.
   c) Find the shortest path using A* search. Show the steps clearly. Does the estimate of the remaining distance to the goal satisfy the A* requirement for finding an optimal?

![Graph Diagram]

**Solution:**

a) Minimize

\[ 20x_{12} + 10x_{13} + 75x_{25} + 7x_{2z} + 80x_{34} + 20x_{45} \]

Subject to:

- \[ x_{12} + x_{13} = 1 \]
- \[ -x_{12} + x_{32} + x_{41} = 0 \]
- \[ -x_{12} - x_{3z} - x_{2s} = 0 \]
- \[ -x_{39} + x_{45} = 0 \]
- \[ -x_{2s} - x_{45} = -1 \]

b) **Dijkstra's**

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ 0 \quad 20 \quad 10 \quad 90 \quad - \quad - \]

\[ 0 \quad 17 \quad 10 \quad 90 \quad 92 \]

Done

Dijkstra's Algorithm:

- Start at node 1.
- Visit node 2 (distance 20).
- Visit node 3 (distance 17).
- Visit node 4 (distance 10).
- Visit node 5 (distance 90).
- Visit node 6 (distance 92).

Checking:

- Pullman to Albion: 20 minutes
- Albion to Colfax: 70 minutes
- Colfax to Spokane: 75 minutes
- Albion to Spokane: 15 minutes
- Spokane to Pullman: 80 minutes

Total: 20 + 70 + 75 + 15 + 80 = 240 minutes

b) **A**

\[ (1,80) \quad (20,90) \quad (10,95) \quad - \quad - \quad - \]

\[ (0,80) \quad (17,87) \quad (10,85) \quad (90,165) \quad (42,97) \quad (92,92) \]

\[ (0,60) \quad (17,87) \quad (10,85) \quad (90,165) \quad (42,97) \quad (92,92) \]

None

Yes, A* criterion satisfied since all estimates less than actual distance.
3. **DUALITY (15 points):** Again using the linear programming problem in 1, repeated below for convenience.

\[
\begin{align*}
\text{min } & \ 2x_1 + 3x_2 \\
\text{such that } & \ 3x_1 + x_2 \geq 8 \\
& \ x_1 - x_2 \leq 2 \\
& \ x_1, x_2 \geq 0
\end{align*}
\]

a) Find a dual of this linear program.
b) Is the dual feasible? If feasible will the objective have the same value as the primal? Explain your answer.

\[
\begin{align*}
a) \ \text{Dual} \ & \max & y_1 + 2y_2 \\
& \text{such that } & 3y_1 + y_2 \leq 2 \\
& & y_1 - y_2 \leq 3 \\
& & y_1 \geq 0, y_2 \leq 0
\end{align*}
\]

b) The dual must be feasible since the primal has a solution (is bounded) and the optimal must be the same \((z^* = 6.5, y^* = (2.25, -1.75))\).
5. **Duality (5 points):** Consider the constrained optimization problem below.

\[
\begin{align*}
\text{min} & \ 2x_1^2 + 3x_2^2 \\
\text{such that} & \ 3x_1 + 2x_2 \geq 4 \\
x_1, x_2 & \geq 0
\end{align*}
\]

a) Find the dual of this non-linear program.

\[\begin{align*}
\text{max} & \ \omega(\lambda) = 4(\frac{3}{4}x_1^2 + \frac{1}{2}x_2^2)(4 - \frac{9}{4}x_1 - \frac{2}{3}x_2) = \\
& = 4\lambda - \frac{35}{24} \lambda^2, \quad \lambda \geq 0
\end{align*}\]
6. SHORT ANSWER (10 points): Be sure to explain your answers clearly.

• Again considering the $A^*$ search. Assume you have two different evaluation functions, say $H_1$ and $H_2$, and both of these evaluation functions are "consistent." State the condition that allows $H_1$ to be more "informed" than $H_2$. What is known about the relative search efficiencies between using $H_1$ or $H_2$?

$$H_1 \geq H_2 \text{ everywhere then } H_1 \text{ is more efficient}$$

(assumes $H_i$ satisfies $A^*$ heuristic)

• In terms of computational complexity for the shortest path problem, is an LP solution using the Simplex Method or Dijkstra's shortest path algorithm more efficient? Is $A^*$ or Dijkstra's algorithm more efficient? Explain your answer.

- Dijkstra is more efficient than Simplex. Simplex in worst case is NP-complete, Dijkstra is polynomial $O(n^2)$

- $A^*$ is more efficient than Dijkstra; reduces to Dijkstra with $H=0$